Unitary Evolution on a Discrete Phase Space

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We construct unitary evolution operators on a phase space with power of two discretization. These operators realize the metaplectic representation of the modular group $SL(2, \mathbb{Z}_{2^n})$. It acts in a natural way on the coordinates of the non-commutative 2-torus, $\mathbb{T}_{2^n}^2$ and thus is relevant for non-commutative field theories as well as theories of quantum space-time. The class of operators may also be useful for the efficient realization of new quantum algorithms.

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1. Introduction

Recent progress in M-theory indicates that spacetime itself becomes noncommutative at scales where D-branes play an important role [1, 2]. This noncommutativity comes about in a rather natural way because D-branes are charged, gravitational solitons, moving in backgrounds with magnetic flux and their worldvolume acquires non-commutative geometry. What happens is analogous to the Landau problem, where the noncommutativity of the two, real, space coordinates is brought about by the magnetic flux[3]. The strength of the flux provides a measure of non-commutativity. This is the first time where explicit dynamics on non-commutative spacetime has been throughly studied and it has led to a new understanding of Yang–Mills theories (with fluxes) in *commutative* spacetime as $U(1)$ gauge theories in non-commutative spacetime.

The new insight is the trading of spacetime non-commutativity with the non-commutativity of the gauge group. In general it has been established that Yang–Mills gauge theories (in commutative spacetime) have similar short-distance behavior but different *long–distance* behavior from *non-commutative* Yang–Mills theories.

The above facts led people to think again the old idea of discretization of spacetime with cellular structure, which could be described by non-commmutative coordinates. For Minkowski spacetimes care must be taken to use non-commutativity only in the *space* part, otherwise unitarity in perturbation theory is lost.

More drastic ideas, not coming from D-brane studies, have been advanced by 't Hooft[4] in an effort to reconcile gravity with quantum mechanics. The proposal is that, at scales where gravity becomes strong, it is no longer appropriate to work with continuous variables, but at the classical level, to use *discrete* labels for spacetime coordinates as well as the dynamical state of the system. Quantum mechanics arises from the superoposition of these classical dynamical states and the mapping of the configuration and momentum space in wavefunctions on a finite–dimensional (and discrete) Hilbert space. Locality in space is lost, since there is a minimal lengthscale and, because of causality, locality is lost in time as well. Hamiltonians are thus non-local and only unitary, one-time-step evolution operators have any meaning.

In this contribution we present the construction of unitary evolution operators on a toroidal phase space with power of two discretiztion. This case could not be included in the previous cases of prime or odd discretization[5] due to the impossibility of defining $1/2 \bmod 2^n$. The solution we present here relies in absorbing these factors and considering twice as many points per phase space direction; We then show that the restriction to half the points is consistent and present the transformation that groups them together in order to ensure a unitary evolution.

We study a model for a discrete and periodic one–dimensional space and, at the same time, discrete and periodic momentum, i.e. a discretized, toroidal, phase space, $\mathbb{Z}_N \times \mathbb{Z}_N$, on which classical, linear, maps, elements of $SL(2, \mathbb{Z}_N)$, which discretize continuous, $SL(2, \mathbb{R})$ maps on the 2-torus, of unit length, $\mathbb{T}^2$, for motions on the rational points, with denominator N. Taking $N \to \infty$ a subset of possible trajectories has a smooth limit and we recover $SL(2, \mathbb{R})$. This discretization has the nice property, that can be transferred to the quantum–mechanical level, of assigning to each $\mathsf{A} \in SL(2, \mathbb{Z}_N)$, a unique, $N \times N$, unitary map, $U(\mathsf{A})$, which represents faithfully sequences of classical maps that are irreducible representations of $SL(2, \mathbb{Z}_N)$. This can be done for any N odd using as elementary blocks the case $N = p^n$, p prime and, for the general case, $N = p_1^{n_1} \times \cdots p_k^{n_k} \cdots$,

we use the basic property of $SL(2,\mathbb{Z}_N)$,

$$SL(2,\mathbb{Z}_N) = \otimes SL(2,\mathbb{Z}_{p_k^{n_k}})$$

In the case $N = 2^n$, or, more generally, N even, there are constraints imposing periodicity on the unitary matrices $U(\mathsf{A})$, which restrict the group $SL(2,\mathbb{Z}_N)$ to its normal subgroup

$$NSL(2,\mathbb{Z}_N) = \begin{pmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{pmatrix}$$

There is an additional subset of $SL(2,\mathbb{Z}_N)$, consisting of elements

$$\mathsf{A} = \begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix}$$

which, together with the elements of $NSL(2,\mathbb{Z}_N)$, form a bigger subgroup, which satisfy the periodicity constraints.

In the next section we give the basic definitions for the metaplectic representation and we quantize translations and dilatations. Using these as building blocks, it is thus possible to quantize any classical action. We close with a brief discussion of direction of future inquiry.

2. Unitary Evolution for Translations and Dilatations

We start with the observation that any matrix A, element of $SL(2,\mathbb{Z}_N)$, may be written as

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\mathsf{A}} = \underbrace{\begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix}}_{\mathsf{L}(bd^{-1})} \underbrace{\begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix}}_{\mathsf{D}(d)} \underbrace{\begin{pmatrix} 1 & 0 \\ cd^{-1} & 1 \end{pmatrix}}_{\mathsf{R}(cd^{-1})} \tag{2.1}$$

i.e. in terms of translations and dilatations in phase space. We note that obstructions to this construction may appear, if d is not invertible mod N. For $N = 2^n$ this means, in particular, that d should be odd. We shall try to construct the unitary operator, $U(\mathsf{A})$ from the corresponding unitary operators

$$U^{\mathsf{L}}(x) \equiv U(\mathsf{L}(x)) \ U^{\mathsf{D}}(d) \equiv U(\mathsf{D}) \ U^{\mathsf{R}}(y) \equiv U(\mathsf{R}(y))$$

For any discretization N it is possible to find operators $J_{r,s}$ that generate the Heisenberg–Weyl group. They are given in terms of the clock and shift operators Q and P by

$$J_{r,s} = \omega_N^{rs/2} P^r Q^s \tag{2.2}$$

where $\omega_N \equiv \exp(2\pi i/N)$. In the basis where the clock operator Q is diagonal the matrix elements are given by

$$P_{k,l} = \delta_{k-1,l} \ Q_{k,l} = \omega_N^k \delta_{k,l} \ [J_{r,s}]_{k,l} = \delta_{k-r,l} \omega_N^{s(k+l)/2} \tag{2.3}$$

where $k,l = 0,1,\ldots,2^n - 1$. We already notice the $1/2$ factors that need to be defined seperately, when $N = 2^n$. The unitary operators $U(\mathsf{A})$ need to satisfy two requirements:

- They realize a group representation: for any two operators, $A, B \in SL(2, \mathbb{Z}_{2^n})$ we have

$$U(A \cdot B) = U(A) \cdot U(B) \tag{2.4}$$

- They realize the *metaplectic* representation: for any point, (r,s), of the classical phase space

$$U(A)J_{r,s}U(A)^{-1} = J_{(r,s)A} \tag{2.5}$$

In this section we shall sketch the construction for $U^{\mathrm{L}}(x)$, $U^{\mathrm{R}}(y)$ and $U^{\mathrm{D}}(d)$. Details may be found in the paper[7].

We first consider a larger space, of $2^{n+1} - 1$ points, i.e. $SL(2, \mathbb{Z}_{2^{n+1}})$. In this space we define the matrix elements of $J_{r,s}$ as

$$[J_{r,s}]_{k,l} = \delta^{(n+1)}_{k-r,l} \omega^{s(k+l)}_{n+2} \tag{2.6}$$

where $\omega_n \equiv \exp(2\pi i/2^n)$ and the superscript on the Kronecker delta indicates that the operation is performed mod 2^{n+1}. Note that we have absorbed the $1/2$ factor in the order of the root of unity.

However the "physical" points correspond to the *even* values of the indices r, s, k, l and, thus the "physical" sub-space is, indeed, 2^n–dimensional.

It is possible to show that the following operators realize a group representation and the metaplectic reprsentation:

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$$\left[U^{\mathrm{L}}(x)\right]_{k,l} = \frac{1 + (-)^k}{2} \omega^{xk^2}_{n+2} \delta^{(n+1)}_{k,l}$$

-

$$\left[U^{\mathrm{D}}(d)\right]_{k,l} = \delta^{(n+1)}_{k,dl}$$

-

$$\left[U^{\mathrm{R}}(y)\right]_{k,l} = \left[F^{-1}U^{\mathrm{L}}(-y)F\right]_{k,l} = \frac{1}{2^{n+1}} \sum_{m=0}^{2^n-1} \frac{1 + (-)^{l+k}}{2} \omega^{-m^2x+m(l-k)}_n$$

where F is the Fourier Transform operator,

$$F_{k,l} = \frac{\omega^{kl}_{n+1}}{\sqrt{2^{n+1}}}$$

Here the indices $k, l = 0, 1, \ldots, 2^{n+1} - 1$-however the projector

$$\frac{1 + (-)^k}{2}$$

projects on the even values, $k, l = 0, 2, 4, \ldots, 2^{n+1} - 2$. Similarly, when checking the metaplectic representation, eq. (2.5), the indices $r, s = 0, 2, 4, \ldots, 2^{n+1} - 2$. That this is consistent may be deduced from the fact that the points of the classical phase space ($r = \text{even}, s = \text{even}$) are transformed among themselves by a matrix $A \in SL(2, \mathbb{Z}_{2^n})$. We thus may check that the evolution is unitary on this sub-lattice. However, since we have introduced a projector, unitarity is spoiled on the original lattice. Remains then to construct an operator that rearranges the sites in such a way as to render

unitarity manifest on the even sub-lattice. Such an operator is the *bit reversal* operator used in the Fast Fourier Transform[6, 7]!

We thus may conclude that the general classical action $A \in SL(2, \mathbb{Z}_{2^n})$, with the element d odd, may be consistently quantized through an embedding in a space of twice as many points. Details and proofs may be found in [7].

3. Conclusions and Outlook

We have presented the construction of unitary evolution operators that consistently quantize the classical action on phase spaces of power of two discretization. This completes the program of papers[5] and may open the way for new quantum algorithms based on operators that are more general than the Fourier Transform. Furthermore the dimensionality considered here may be useful in the description of systems with fermionic degrees of freedom.

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