Symmetry and scaling in the Q-exact lattice (2,2) 2d Wess-Zumino model

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As a nonperturbative check on the Q-exact lattice formulation, we demonstrate that the continuum R-symmetries are recovered. We locate the critical domain of the lattice theory. Aspects of the continuum nonrenormalization theorems are found to be respected at finite lattice spacing. Preliminary attempts to extract critical exponents—another nonperturbative check—are discussed. All of our results are obtained from Monte Carlo simulations with dynamical fermions.

XXIIIrd International Symposium on Lattice Field Theory
25-30 July 2005
Trinity College, Dublin, Ireland

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The continuum (2,2) 2d Wess-Zumino (2dWZ) model (obtained from a dimensional reduction of the 4d Wess-Zumino model [1]) is supposed to provide a Landau-Ginzburg description of the minimal discrete series of $\mathcal{N} = 2$ superconformal field theories [2]. In our recent article [3], we have examined an important aspect of the simplest of these models—the one with a cubic superpotential—in the context of a class of lattice actions that have an exact lattice supersymmetry. These lattice actions were first formulated in [4, 5] using *Nicolai map* [6] methods, relying on earlier Hamiltonian [7] and continuum [8] studies that also utilized the Nicolai map. Detailed studies of the spacetime lattice system were performed in [9] by stochastic quantization methods and in [10] by the Monte Carlo simulation approach.

Once auxiliary fields are introduced, the lattice action takes a Q-exact form: $S = QX$, as was emphasized in the topological interpretation of [11] and the lattice superfield approach of [12]. Here Q is a lattice supercharge with derivatives realized through discrete difference operators; with respect to a discrete approximation of the continuum theory superalgebra, $Q^2 = 0$ is a nilpotent subalgebra. Because S is Q-exact, the action is trivially invariant with respect to this lattice super-symmetry: $QS = Q^2X = 0$.

It was shown in [3] that in the massive continuum theory there is an exact $Z_2(R)$ symmetry. It is an *R-symmetry,* meaning it does not commute with the supercharges. This symmetry is spon-taneously broken at infinite volume. In the massless case, i.e., in the critical domain, the classical R-symmetry is enlarged to $U(1)_R$. It cannot be spontaneously broken since it is a continuous sym-metry in 2d [13]. If the lattice theory has the correct continuum limit, it should reproduce these features. On the other hand, these R-symmetries are only approximate in the Q-exact lattice action; the symmetry is explicitly broken by the Wilson mass term that is used to lift doublers. [1]

It has been shown in [12] that the continuum limit of the lattice perturbation series is identical to that of the continuum theory, due to cancellations that follow from $Q^2 = 0$. Thus, the Q-exact spacetime lattice has behavior that is similar to what was found on the $Q, Q^\dagger$-preserving spatial lat-tice in [4]. However, it was also shown in [12] that the most general continuum effective action that is consistent with the symmetries of the bare lattice action is not the (2,2) 2d Wess-Zumino model. This raises the question of whether or not the good behavior of perturbation theory persists at a nonperturbative level. The results of [9, 10] give hope that the desired continuum limit is obtained beyond perturbation theory. If so, this would be one of the few examples of a supersymmetric field theory that can be latticized and studied nonperturbatively by Monte Carlo simulation without the need for fine-tuning of counterterms.

In our recent work [3], we have shown that features of the continuum theory associated with the R-symmetry are recovered in the continuum limit; this provides further evidence that the correct theory is obtained. The symmetry that we study persists in the infrared effective theory in a strongly coupled regime. Thus, we are testing aspects of the lattice theory that lie beyond the reach of perturbation theory. In the remainder of this note, we summarize the main results of [3].

The continuum Euclidean action is

$$S = \int d^2z \left[-4\bar{\phi}\partial_z\partial_{\bar{z}}\phi - 2i\bar{\psi}_-\partial_{\bar{z}}\psi_- + 2i\psi_+\partial_z\bar{\psi}_+ - \bar{F}F \right.$$
$$\left. + W'(\phi)F + \overline{W}'(\bar{\phi})\bar{F} - W''(\phi)\psi_+\psi_- - \overline{W}''(\bar{\phi})\bar{\psi}_-\bar{\psi}_+ \right] \tag{1}$$

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[1]This is directly related to the breaking of the so-called $U(1)_V$ symmetry, that was pointed out in [12].

We specialize to the superpotential $W(\phi) = \frac{m}{2}\phi^2 + \frac{g}{3!}\phi^3$. It is convenient to make the field re-definition $\phi = -\frac{m}{g} + \sigma$. In this case $W(\sigma) = -\lambda\sigma + \frac{g}{3!}\sigma^3$, $\lambda = \frac{m^2}{2g}$. The scalar potential is just $V = |W'(\sigma)|^2 = \left|\lambda - \frac{g}{2}\sigma^2\right|^2$. Degenerate minima occur: $\sigma_\pm = \pm\sqrt{2\lambda/g} = \pm m/g$. The action possesses a (2,2) supersymmetry, characterized by the algebra $\{Q_-, \bar{Q}_-\} = -2i\partial_z$, $\{Q_+, \bar{Q}_+\} = 2i\partial_{\bar{z}}$. In the case of $\lambda = 0$, there is a $U(1)_R$ symmetry: $\sigma \to e^{2i\alpha/3}\sigma$, $\psi_\pm \to e^{-i\alpha/3}\psi_\pm$, $F \to e^{-4i\alpha/3}F$. In addition there is a $U(1)_A$ symmetry: $\psi_\pm \to e^{\pm i\omega}\psi_\pm$, $\bar{\psi}_\pm \to e^{\mp i\omega}\psi_\pm$, with all other fields neutral. If $\lambda \neq 0$, the symmetry breaking $U(1)_R \times U(1)_A \to Z_2(R) \times U(1)_A$ occurs, with $Z_2(R)$ described by: $\sigma \to -\sigma$, $\bar{\sigma} \to -\bar{\sigma}$, $\psi_\pm \to \pm\psi_\pm$, $\bar{\psi}_\pm \to \pm\bar{\psi}_\pm$. An important property of the theory is that there is only wavefunction renormalization: $W(m, g|\phi) \equiv W(m_r, g_r|\phi_r)$, where $m_r = Zm$, $g_r = Z^{3/2}g$, $\lambda_r = \sqrt{Z}\lambda$. This is the so-called nonrenormalization theorem: mass and coupling counterterms vanish identically. It follows that $m = 0$ is a critical point for any g.

The lattice action preserves the nilpotent subalgebra $Q^2 = 0$ where $Q = Q_- + \bar{Q}_+$. The action of Q on lattice fields is defined by a discretized version of the continuum supersymmetry:[2] $Q\phi = \psi_-$, $Q\psi_+ = F + 2i\Delta_{\bar{z}}\phi$, $Q\psi_- = 0$, $QF = -2i\Delta_{\bar{z}}\psi_-$, $Q\bar{\phi} = \bar{\psi}_+$, $Q\bar{\psi}_+ = 0$, $Q\bar{\psi}_- = \bar{F} - 2i\Delta_z\bar{\phi}$, $Q\bar{F} = 2i\Delta_z\bar{\psi}_+$. The action is Q-exact:

$$S = Q\left(-F\bar{\psi}_- - 2i\psi_+\Delta_z\bar{\phi} + W'(\phi)\psi_+ + \overline{W}'(\bar{\phi})\bar{\psi}_-\right) \tag{2}$$

The auxiliary fields $F, \bar{F}$ can be eliminated by their equation of motion. To lift spectrum doublers, a Wilson mass term is introduced into the superpotential:

$$W(\phi) = \sum_m \left(-\frac{r}{4}\phi_m\Delta^2\phi_m + \frac{m}{2}\phi_m^2 + \frac{g}{3!}\phi_m^3\right) \tag{3}$$

Note that $W'_m = \partial W/\partial\phi_m$, etc. We can also make use of the $\phi \to \sigma$ field redefinition to obtain

$$W(\sigma) = \sum_m \left(-\frac{r}{4}\sigma_m\Delta^2\sigma_m - \lambda\sigma_m + \frac{g}{3!}\sigma_m^3\right) \tag{4}$$

Note that the Wilson mass term (an irrelevant operator) violates the R-symmetries of the continuum theory. In [3] we have shown that the effective potential nevertheless has the continuum symmetries: the effect of the irrelevant symmetry breaking operator is negligible for small lattice spacing.

We probe the symmetry of the effective potential by introducing a background field in the scalar potential: $\Delta V(h) = -\sum_m \left(\bar{h}\sigma_m - h\bar{\sigma}_m\right)$. This allows us to explore the extent to which the lattice theory is symmetric w.r.t. $\sigma \to -\sigma$, or the phase rotation $\sigma \to e^{i\theta}\sigma$. Define the generating function $w(h) = \ln Z(h)$, where $Z(h)$ is the partition function that is obtained when $\Delta V(h)$ is added to the lattice action. $Z_2(R)$ symmetry of the effective potential is equivalent to $w(-h) = w(h)$. Similarly, $U(1)_R$ symmetry of the effective potential is equivalent to $w(e^{i\theta}h) = w(h)$. Note also that $\langle\sigma\rangle_h = \partial w(h)/\partial\bar{h}$, where $\langle\sigma\rangle_h$ is the expectation value of σ in the background h. It follows that in the case of $Z_2(R)$ symmetry we have the prediction $\langle\sigma\rangle_{-h} = -\langle\sigma\rangle_h$. In the case of $U(1)_R$ symmetry we have the much stronger prediction $\langle\sigma\rangle_{e^{i\theta}h} = e^{i\theta}\langle\sigma\rangle_h$. Equivalently, since we take $m > 0, g > 0$ and will find below that $\langle\sigma\rangle_h > 0$ if h is real and positive,

$$\arg\langle\sigma\rangle_h = \arg h, \quad \text{and} \quad |\langle\sigma\rangle_h| = \text{const.}, \quad \text{fixed } |h| \tag{5}$$

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[2]Here and below the difference operators are defined as $\Delta_\mu^S = \frac{1}{2}\left(\Delta_\mu^+ + \Delta_\mu^-\right)$, $\Delta^2 = \sum_{\mu=1,2}\Delta_\mu^+\Delta_\mu^-$, $\Delta_z = \frac{1}{2}\left(\Delta_1^S - i\Delta_2^S\right)$, $\Delta_{\bar{z}} = \frac{1}{2}\left(\Delta_1^S + i\Delta_2^S\right)$, where Δ_μ^+ and Δ_μ^- are forward and backward difference operators in the μ direction.

× m=0.00,   ◇ m=0.03,   □ m=0.10
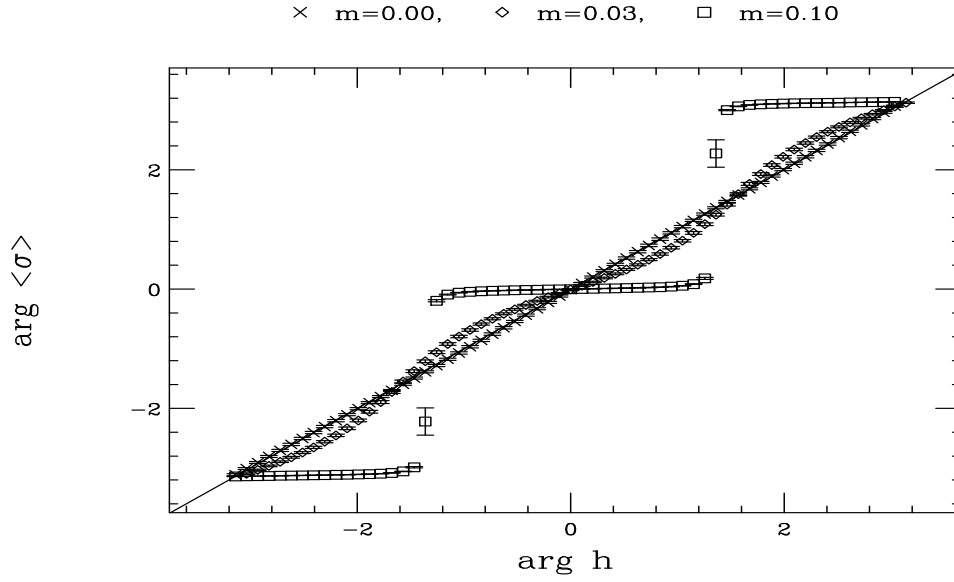
Figure 1: A test of $U(1)_R$ symmetry, by comparison to the prediction $\arg\langle\sigma\rangle_h = \arg h$.

In [3] we have shown by various means that, up to statistical errors, simulation results are supportive of the $Z_2(R)$ symmetry prediction $\langle\sigma\rangle_{-h} = -\langle\sigma\rangle_h$. We have also shown that $m = 0$ is a critical point. For brevity we do not discuss the details here. Rather, we will concentrate on the $U(1)_R$ symmetry at $m \to 0$; however, some of the results we review here also indicate the $Z_2(R)$ symmetry at $m \neq 0$.

In Fig. 1 we display $\arg\langle\sigma\rangle_h$ versus $\arg h$ at $(g, |h|, N) = (0.03, 0.001, 16)$ for three different mass values, $m = 0, 0.03, 0.10$. For $m = 0$, the data passes through the (diagonal) straight line $\arg\langle\sigma\rangle = \arg h$, showing that $U(1)_R$ is a symmetry of the effective potential. For $m = 0.03$, the data deviates slightly from the straight line, indicating that the symmetry is only slightly violated, breaking to $Z_2(R)$. Finally, at $m = 0.10$, the $U(1)_R$ symmetry is completely broken. The fact that $\arg\langle\sigma\rangle_h \approx \pm\pi$ in this case can be understood as follows. For larger values of m and the very small h that we choose, the potential $V = |W'(\sigma)|^2$ dominates over the source potential $\Delta V(h)$. In that case, $\langle\sigma\rangle_h \approx \pm\langle\sigma\rangle_{0^+} \equiv \pm v$. For m, g positive, $v > 0$. The role of h then is just as a perturbation to pick the sign of $\pm v$. It follows that $\arg\langle\sigma\rangle_h \approx 0, \pm\pi$.

The second part of the conjecture (5) was studied through the quantity

$$R(|\langle\sigma\rangle_h|) = \frac{|\langle\sigma\rangle_h| - \overline{|\langle\sigma\rangle|}}{\overline{|\langle\sigma\rangle|}}, \quad \overline{|\langle\sigma\rangle|} = \frac{1}{n}\sum_{j=1}^{n}|\langle\sigma\rangle_{h_j}| \tag{6}$$

where $h_j = |h|\exp(2\pi i j/n)$ corresponds to the values of h that were used in the data set. R measures the relative shift of $|\langle\sigma\rangle_h|$ away from the mean w.r.t. $\arg h$. In Fig. 2, one sees that $U(1)_R$ is restored at $m = 0$.

In research in progress, we are performing another nonperturbative check of the lattice. As mentioned at the outset, the continuum theory in the critical domain is believed to afford a Landau-Ginzburg description of the minimal discrete series of $\mathcal{N} = 2$ superconformal field theories; the
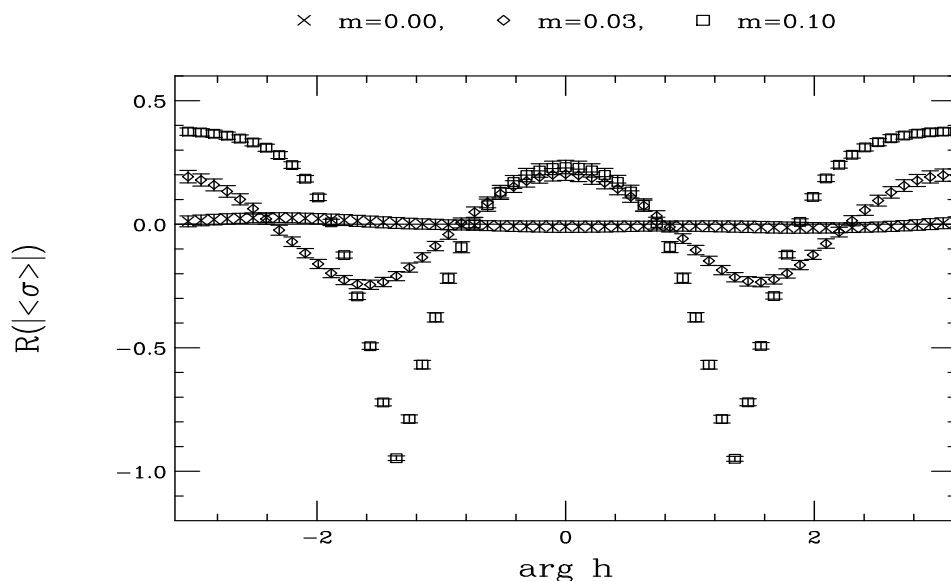
Figure 2: A test of $U(1)_R$ symmetry, by comparison to the prediction that $|\langle\sigma\rangle_h|$ should be independent of $\arg h$. Relative deviation from the average w.r.t. $\arg h$ is measured by R.

critical exponents of relevant operators are known exactly. The lattice theory should reproduce these exponents. We are studying this through an examination of hyperscaling (dependence on correlation length) and finite-size scaling (dependence on system size). We hope to report the results of that study in the near future. Unfortunately, on the large lattice sizes required for such an analysis, a variety numerical obstacles have been found to arise in our simulations. Due to space limitations, we do not detail them here.

The simulation results related to R-symmetries are quite encouraging. The explicit breaking due to the Wilson mass term in the superpotential is harmless in the continuum limit; the continuum R-symmetry is recovered without the need for counterterms.

Undoubtedly these positive results are related to the following features: (i) the symmetry breaking is due to irrelevant operators; (ii) 1PI diagrams of UV degree $D \geq 0$ do not occur in the lattice perturbation series. The cancellations of $D = 0$ contributions of subdiagrams in lattice perturbation theory is intimately related to the exact lattice supersymmetry [12]. It would be very interesting to know whether or not other lattice actions with an exact supersymmetry, such as the super-Yang-Mills examples that have been recently proposed [14], have a finite lattice perturbation series, in the sense that they have no $D \geq 0$ 1PI diagrams. However, a careful power-counting analysis, comparable to that done by Reisz for 4d Yang-Mills [15], has yet to be performed.

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