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Color Superconductivity in High Density QCD

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We describe the effects of the strange quark mass and of color and electric neutrality on the superconducing phases of QCD. We discuss various phases pointing out the corresponding problems, typically arising from a chromomagnetic instability. We show that, at least for particular configurations, the LOFF phase dominates over the gCFL phase, when close to the transition from gCFL to the normal phase.

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Contents

1. Introduction

It is now a well established fact that at zero temperature and sufficiently high densities quark matter is a color superconductor [1, 2] (see also Alford and Rischke contributions at this workshop). The study starting from first principles was done in [3, 4, 5]. At chemical potentials much higher than the masses of the quarks *u*, *d* and *s*, the favored state is the so-called Color-Flavor-Locking (CFL) state, whereas at lower values the strange quark decouples and the relevant phase is called two-flavor color superconducting (2SC).

An interesting possibility is that in the interior of compact stellar objects (CSO) some color superconducting phase might exist. In fact we recall that the central densities for these stars could be up to 10^{15} g/cm^3, whereas the temperature is of the order of tens of keV. However the usual assumptions leading to prove that for three flavors the favored state is CFL should now be reviewed. Matter inside a CSO should be electrically neutral and should not carry any color. Also conditions for β-equilibrium should be fulfilled. As far as color is concerned, it is possible to impose a simpler condition, that is color neutrality, since in [6] it has been shown that there is no free energy cost in projecting color singlet states out of color neutral states. Furthermore one has to take into account that at the interesting densities the mass of the strange quark is a relevant parameter. All these three effects:

1. mass of the strange quark,
2. β-equilibrium,
3. color and electric neutrality

imply that the radii of the Fermi spheres of the quarks that would pair are not of the same size, thus creating a problem with the usual BCS pairing. Let us start from the first point. Suppose to have two fermions of masses $m_1 = M$ and $m_2 = 0$ at the same chemical potential μ. The corresponding Fermi momenta are

$$p_{F_1} = \sqrt{\mu^2 - M^2}, \quad p_{F_2} = \mu. \tag{1.1}$$

Therefore the radius of the Fermi sphere of the massive fermion is smaller than the one of the massless particle. If we assume $M \ll \mu$ the massive particle has an effective chemical potential

$$\mu_{\rm eff} = \sqrt{\mu^2 - M^2} \approx \mu - \frac{M^2}{2\mu}, \tag{1.2}$$

and the mismatch between the two Fermi spheres is

$$\delta\mu \approx \frac{M^2}{2\mu}. \tag{1.3}$$

This shows that the quantity $M^2/(2\mu)$ behaves as a chemical potential. Therefore for $M \ll \mu$ the mass effects can be taken into account through the introduction of the mismatch between the chemical potentials of the two fermions given by eq. (1.3). This is the way we will follow in our study.

Now let us discuss the β-equilibrium. If electrons are present (as generally required by electrical neutrality) chemical potentials of quarks of different electric charge are different. In fact, when at the equilibrium for the process $d \to ue\bar{\nu}$ we have

$$\mu_d - \mu_u = \mu_e. \tag{1.4}$$

From this condition we get that for a quark of charge Q_i the chemical potential μ_i is given by

$$\mu_i = \mu + Q_i \mu_Q, \tag{1.5}$$

where μ_Q is the chemical potential associated to the electric charge. Therefore

$$\mu_e = -\mu_Q. \tag{1.6}$$

Notice also that μ_e is not a free parameter since it is determined by the neutrality condition

$$Q = -\frac{\partial\Omega}{\partial\mu_e} = 0. \tag{1.7}$$

At the same time the chemical potentials associated to the color generators T_3 and T_8 are determined by the color neutrality conditions

$$\frac{\partial\Omega}{\partial\mu_3} = \frac{\partial\Omega}{\partial\mu_8} = 0. \tag{1.8}$$

We see that the general there is a mismatch between the quarks that should pair according to the BCS mechanism for $\delta\mu = 0$. Therefore, in general, the system will go to a normal phase, since the mismatch, as we shall see, tends to destroy the BCS pairing, or a different phase will be formed. In the next Sections we will explore some of these possible phases.

2. Pairing Fermions with Different Fermi Momenta

In order to discuss the pairing of fermions with different Fermi momenta let us review the gap equation for the BCS condensate. The condensation phenomenon is the key feature of a degenerate Fermi gas with attractive interactions. Once one takes into account the condensation the physics can be described using the Landau's idea of quasi-particles. In this context quasi-particles are nothing but fermionic excitations around the Fermi surface described by the following dispersion relation

$$\varepsilon(\vec{p},\Delta_0) = \sqrt{\xi^2+\Delta_0^2}, \tag{2.1}$$

with

$$\xi = E(\vec{p}) - \mu \approx \frac{\partial E(\vec{p})}{\partial \vec{p}}\Big|_{\vec{p}=\vec{p}_F} \cdot (\vec{p}-\vec{p}_F) = \vec{v}_F \cdot (\vec{p}-\vec{p}_F), \tag{2.2}$$

and Δ_0 the BCS condensate. The quantities $\vec{v}_F$ and $(\vec{p}-\vec{p}_F)$ are called the Fermi velocity and the residual momentum respectively. A easy way to understand how the concept of quasi-particles comes about in this context is to study the gap equation at finite temperature. For simplicity let us consider the case of a four-fermi interaction. The euclidean gap equation is given by

$$1 = g\int \frac{d^4p}{(2\pi)^4}\frac{1}{p_4^2+|\vec{p}|^2+\Delta_0^2}. \tag{2.3}$$

From this expression it is easy to get the gap equation at finite temperature. We need only to convert the integral over p_4 into a sum over the Matsubara frequencies

$$1 = gT\int \frac{d^3p}{(2\pi)^3}\sum_{n=-\infty}^{+\infty}\frac{1}{((2n+1)\pi T)^2+\varepsilon^2(\vec{p},\Delta_0)}. \tag{2.4}$$

Performing the sum we get

$$1 = \frac{g}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1-n_u-n_d}{\varepsilon(\vec{p},\Delta_0)}. \tag{2.5}$$

Here n_u and n_d are the finite-temperature distribution functions for the excitations (quasi-particles) corresponding to the original pairing fermions

$$n_u = n_d = \frac{1}{e^{\varepsilon(\vec{p},\Delta_0)/T}+1}. \tag{2.6}$$

At zero temperature ($n_u = n_d \to 0$) we find (restricting the integration to a shell around the Fermi surface)

$$1 = \frac{g}{2}\int \frac{d\Omega_p\, p_F^2\, d\xi}{(2\pi)^3}\frac{1}{\sqrt{\xi^2+\Delta_0^2}}. \tag{2.7}$$

In the limit of weak coupling we get

$$\Delta_0 \approx 2\,\bar{\xi}\, e^{-2/(g\rho)}, \tag{2.8}$$

where $\bar{\xi}$ is a cutoff and

$$\rho = \frac{p_F^2}{\pi^2 v_F} \tag{2.9}$$

is the density of states at the Fermi surface. This shows that decreasing the density of the states the condensate decreases exponentially. From a phenomenological point of view, one determines the coupling g requiring that the same four-fermi interaction, at zero temperature and density, gives rise to a constituent mass of the order of 400 MeV. From this requirement, using values for $\mu \approx 400 \div 500$ MeV (interesting for the physics of compact stellar objects), one obtains values of Δ_0 in the range $20 \div 100$ MeV. However, since at very high density it is possible to use perturbative QCD, one can evaluate the gap from first principles [3]. The result is

$$\Delta_0 \approx 2b\mu e^{-3\pi^2/\sqrt{2}g_s}, \tag{2.10}$$

with

$$b \approx 256\pi^4 (2/N_f)^{5/2} g_s^{-5}. \tag{2.11}$$

It is interesting to notice that from Nambu-Jona Lasinio type of models one would expect a behavior of the type $\exp(-c/g_s^2)$ rather than $\exp(-c/g_s)$. This is due to an extra infrared singularity from the gluon propagator. Although this result is strictly valid only at extremely high densities, if extrapolated down to densities corresponding to $\mu \approx 400 \div 500$ MeV, one finds again $\Delta_0 \approx 20 \div 100$ MeV.

We start now our discussion considering a simple model with two pairing quarks, u and d, with chemical potentials

$$\mu_u = \mu + \delta\mu, \quad \mu_d = \mu - \delta\mu, \tag{2.12}$$

and no further constraints. The gap equation has the same formal expression as given in eq. (2.5) for the BCS case, but now $n_u \neq n_d$

$$n_{u,d} = \frac{1}{e^{(\varepsilon(\vec{p},\Delta)\pm\delta\mu)/T}+1}. \tag{2.13}$$

In the limit of zero temperature we obtain

$$1 = \frac{g}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\varepsilon(\vec{p},\Delta)}\left(1-\theta(-\varepsilon-\delta\mu)-\theta(-\varepsilon+\delta\mu)\right). \tag{2.14}$$

The meaning of the two step functions is that at zero temperature there is no pairing when $\varepsilon(\vec{p},\Delta) < |\delta\mu|$. In other words the pairing may happen only for excitations with positive energy. However, the presence of negative energy states, as in this case, implies that there must be gapless modes. When this happens there are blocking regions in the phase space, that is regions where the pairing cannot occur. The effect is to inhibit part of the Fermi surface to the pairing giving rise a to a smaller condensate with respect to the BCS case where all the surface is used. In the actual case the gap equation at $T = 0$ has two different solutions (see for instance ref. [7])

$$a) \quad \Delta = \Delta_0, \qquad b) \quad \Delta^2 = 2\delta\mu\Delta_0 - \Delta_0^2, \tag{2.15}$$

where Δ_0 is the BCS solution of the gap equation for $\delta\mu = 0$ The free energy of the two solutions are given by

$$\begin{aligned} a) \quad & \Omega(\delta\mu) = \Omega_0(\delta\mu) - \frac{\rho}{4}(-2\delta\mu^2 + \Delta_0^2), \\ b) \quad & \Omega(\delta\mu) = \Omega_0(\delta\mu) - \frac{\rho}{4}(-4\delta\mu^2 + 4\delta\mu\Delta_0 - \Delta_0^2), \end{aligned} \tag{2.16}$$

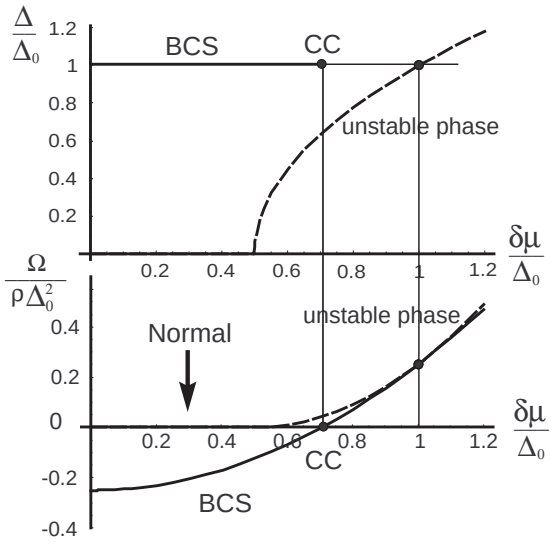

Figure 1: The two solutions of the gap equation with a mismatch $\delta\mu$. The continuous line is the BCS solution, the dashed one is called the Sarma solution.

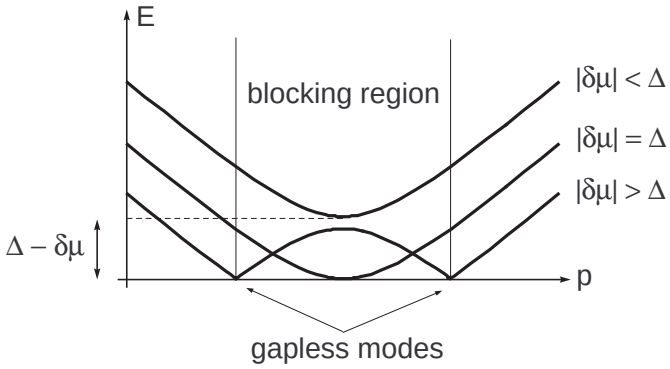

Figure 2: The spectrum of quasi-particles for different values of the mismatch $\delta\mu$.

with $\Omega_0(\delta\mu)$ the free energy for unpaired fermions. For two massless fermions $p_F = \mu$ and $v_F = 1$ and $\rho = \mu^2/\pi$. The two solutions are illustrated in Fig. 1. We see that the solution a) is always favored with respect to the solution b) (called the Sarma phase [8]). Furthermore the BCS phase goes to the normal phase at

$$\delta\mu_1 = \frac{\Delta_0}{\sqrt{2}}. \tag{2.17}$$

This point is called the Chandrasekhar-Clogston (CC) point [9] (denoted by CC in Fig. 1). Ignoring for the moment that in this case, after the CC point the system goes to the normal phase, we notice that the gaps of the two solutions coincide at $\delta\mu = \Delta_0$. This is a special point, since in presence of

a mismatch the spectrum of the quasi-particles is modified as follows

$$E_{\delta\mu=0} = \sqrt{(p-\mu)^2+\Delta^2} \to E_{\delta\mu} = \left|\delta\mu \pm \sqrt{(p-\mu)^2+\Delta^2}\right|. \tag{2.18}$$

Therefore for $|\delta\mu| < \Delta$ we have gapped quasi-particles with gaps $\Delta \pm \delta\mu$ (see Fig. 2). However, for $|\delta\mu| = \Delta$ a gapless mode appears and from this point on there are regions of the phase space which do not contribute to the gap equation (blocking regions). The gapless modes are characterized by

$$E(p) = 0 \Rightarrow p = \mu \pm \sqrt{\delta\mu^2 - \Delta^2}. \tag{2.19}$$

Since the energy cost for pairing two fermions belonging to Fermi spheres with mismatch $\delta\mu$ is $2\delta\mu$ and the energy gained in pairing is 2Δ, we see that the fermions begin to unpair for

$$2\delta\mu \geq 2\Delta. \tag{2.20}$$

These considerations will be relevant for the study of the gapless phases when neutrality is required.

3. The g2SC Phase

The g2SC phase [10] has the same condensate as the 2SC

$$\langle 0|\psi^{\alpha}_{aL}\psi^{\beta}_{bL}|0\rangle = \Delta\varepsilon^{\alpha\beta 3}_{ab3}, \quad \alpha,\beta \in SU_c(3), \quad a,b \in SU(2)_L, \tag{3.1}$$

and technically, it is distinguished by 2SC due to the presence of gapless modes starting at $\delta\mu = \Delta$. In this case only two massless flavors are present (quarks u and d) and there are 2 quarks ungapped q_{ub}, q_{db} and 4 gapped q_{ur}, q_{ug}, q_{dr}, q_{dg}, where the color indices $1,2,3$ have been identified with r,g,b (red, green and blue). The difference with the usual 2SC phase is that color and electrical neutrality are required:

$$\frac{\partial\Omega}{\partial\mu_e} = \frac{\partial\Omega}{\partial\mu_3} = \frac{\partial\Omega}{\partial\mu_8} = 0. \tag{3.2}$$

This creates a mismatch between the two Fermi spheres given by

$$\delta\mu = \frac{p_F^d - p_F^u}{2} = \frac{\mu_d - \mu_u}{2} = \frac{\mu_e}{2}. \tag{3.3}$$

Furthermore the gap equation must be satisfied

$$\frac{\partial\Omega}{\partial\Delta} = 0. \tag{3.4}$$

The solutions to these equations are plotted in the plane (μ_e, Δ) in Fig. 3. In this figure we see the two branches of solutions of the gap equation corresponding to the BCS phase and to the Sarma phase (compare with Fig. 1). Therefore the solution to the present problem belongs to the Sarma branch. In [10] it is also shown that the solution is a minimum of the free energy following the neutrality line. On the other hand this point is a maximum following the appropriate line $\mu_e = \text{const.}$. We see that the neutrality conditions promote the unstable phase (Sarma) to a stable one. However this phase has an instability connected to the Meissner mass of the gluons

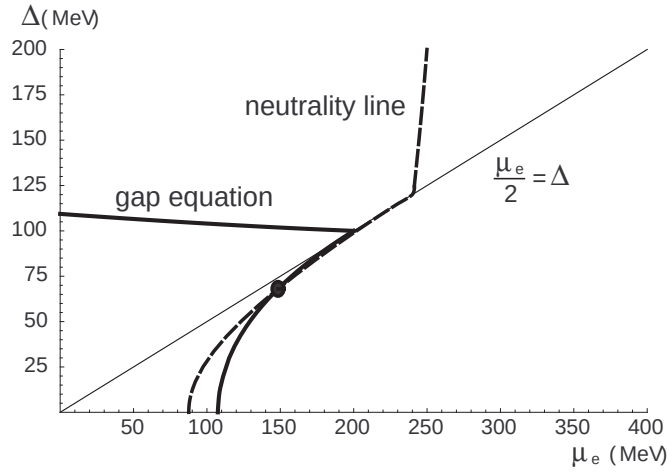

Figure 3: The plane (μ_e, Δ) showing the lines of the solutions of the gap equation (continuous) and to the neutrality condition (dashed). The common solution is marked by a black dot.

[11]. In this phase the color group $SU_c(3)$ is spontaneously broken to $SU_c(2)$ with 5 of the 8 gluons acquiring a mass; precisely the gluons 4,5,6,7,8. At the point $\delta\mu = \Delta$ where the 2SC phase goes into the g2SC one, all the massive gluons have imaginary mass. Furthermore the gluons 4,5,6,7 have imaginary mass already starting at $\delta\mu = \Delta/\sqrt{2}$, that is at the Chandrasekhar-Clogston point, see Fig. 4. This shows that both the g2SC and the 2SC phases are unstable. The instability of the g2SC phase seems to be a general feature of the phases with gapless modes [12].

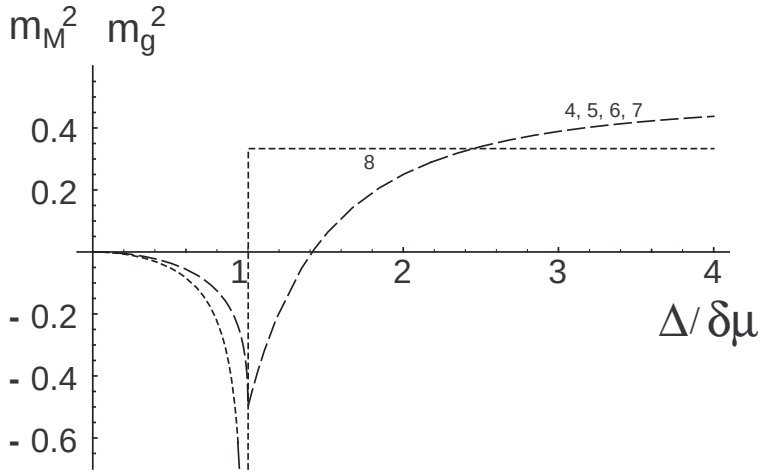

Figure 4: Plot of m_M^2/m_g^2 vs. $\Delta/\delta\mu$. Here $m_g^2 = \mu^2 g^2/(3\pi^2)$. The long-dashed line corresponds to the gluons 4,5,6,7, whereas the short-dashed one to the gluon 8.

4. The gCFL phase

The gCFL phase is a generalization of the CFL phase which has been studied both at $T = 0$ [13, 14] and $T \neq 0$ [15]. The condensate has now the following form

$$\langle 0|\psi^{\alpha}_{aL}\psi^{\beta}_{bL}|0\rangle = \Delta_1\varepsilon^{\alpha\beta 1}\varepsilon_{ab1} + \Delta_2\varepsilon^{\alpha\beta 2}\varepsilon_{ab2} + \Delta_3\varepsilon^{\alpha\beta 3}\varepsilon_{ab3}. \tag{4.1}$$

The CFL phase corresponds to all the three gaps Δ_i being equal. Varying the gaps one gets many different phases. In particular we will be interested to CFL, to g2SC characterized by $\Delta_3 \neq 0$ and $\Delta_1 = \Delta_2 = 0$ and to the gCFL phase with $\Delta_3 > \Delta_2 > \Delta_1$. Notice that, in the actual context, the strange quark is present also in the g2SC phase but unpaired. The matrix of the condensates in the color (r,g,b) and flavor (u,d,s) space is given below:

	ru	*gd*	*bs*	*rd*	*gu*	*rs*	*bu*	*gs*	*bd*
ru		Δ_3	Δ_2						
gd	Δ_3		Δ_1						
bs	Δ_2	Δ_1							
rd					$-\Delta_3$				
gu				$-\Delta_3$					
rs							$-\Delta_2$		
bu						$-\Delta_2$			
gs									$-\Delta_1$
bd								$-\Delta_1$	

(4.2)

In flavor space the gaps Δ_i correspond to the following pairings

$$\Delta_1 \Rightarrow ds, \quad \Delta_2 \Rightarrow us, \quad \Delta_3 \Rightarrow ud. \tag{4.3}$$

The mass of the strange quark is taken into account by shifting all the chemical potentials involving the strange quark as follows:

$$\mu_{\alpha s} \rightarrow \mu_{\alpha s} - \frac{M_s^2}{2\mu}. \tag{4.4}$$

It has also been shown in ref. [16] that color and electric neutrality in CFL require

$$\mu_8 = -\frac{M_s^2}{2\mu}, \quad \mu_e = \mu_3 = 0. \tag{4.5}$$

At the same time the various mismatches are given by

$$\delta\mu_{bd-gs} = \frac{M_s^2}{2\mu}, \quad \delta\mu_{rd-gu} = \mu_e = 0, \quad \delta\mu_{rs-bu} = \mu_e - \frac{M_s^2}{2\mu}. \tag{4.6}$$

It turns out that in the gCFL the electron density is different from zero and, as a consequence, the mismatch between the quarks d and s is the first one to give rise to the unpairing of the corresponing quarks. This unpairing is expected to occur for

$$2\frac{M_s^2}{2\mu} > 2\Delta \;\Rightarrow\; \frac{M_s^2}{\mu} > 2\Delta. \tag{4.7}$$

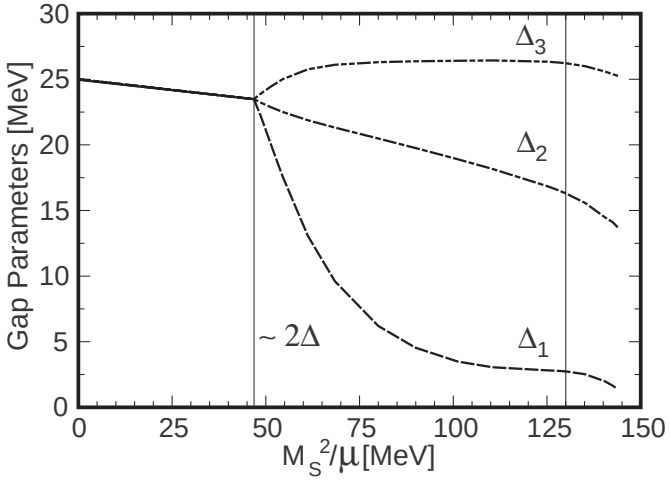

Figure 5: The behavior of the gap parameters in gCFL. The parameters has been chosen in such a way that $\Delta_0 = 25\ MeV$ and $\mu = 500\ MeV$ [14]. The vertical line at $M_s^2/\mu \approx 130\ MeV$ marks the transition from the gCFL phase to the normal one.

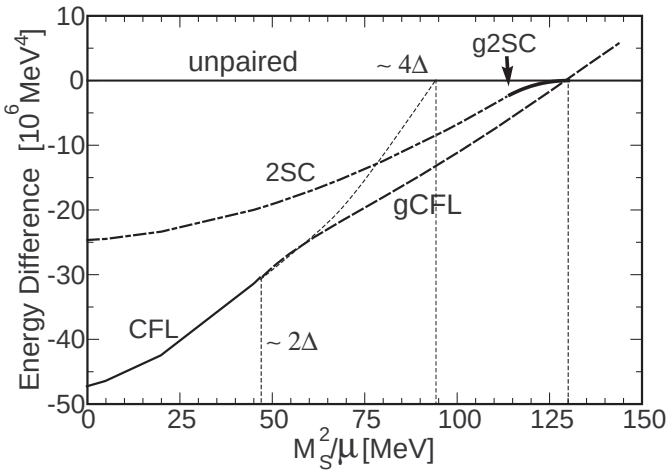

Figure 6: We give here the free energy of the various phases with reference to the normal phase [14], named unpaired in the figure.

This has been substantiated by the calculations in a NJL model modeled on one gluon-exchange in [14]. The results for the gaps are given in Fig. 5. We see that the transition from the CFL phase, where all gaps are equal, to the gapless phase occurs roughly at $M_s^2/\mu = 2\Delta$. In Fig. 6 we show the free energy of the various phases with reference to the normal phase. The CFL phase is the stable one up to $M_s^2/\mu \approx 2\Delta$. Then the gCFL phase takes over up to about 130 MeV where the system goes to the normal phase. Notice that except in a very tiny region around this point, the CFL and gCFL phases win over the corresponding 2SC and g2SC ones. The thin short-dashed line represents the free energy of the CFL phase up to the point where it becomes equal to the

free-energy of the normal phase. This happens for $M_s^2/\mu \approx 4\Delta$. This point is the analogue of the Chandrasekhar-Clogston point of the two-flavor case.

The gCFL phase has gapless excitations and, as a consequence, the chromomagnetic instability discussed in the case of the g2SC phase shows up here too. This has been shown in [17, 18]. The results of ref. [17] are given in Fig. 7 for the various gluon masses.

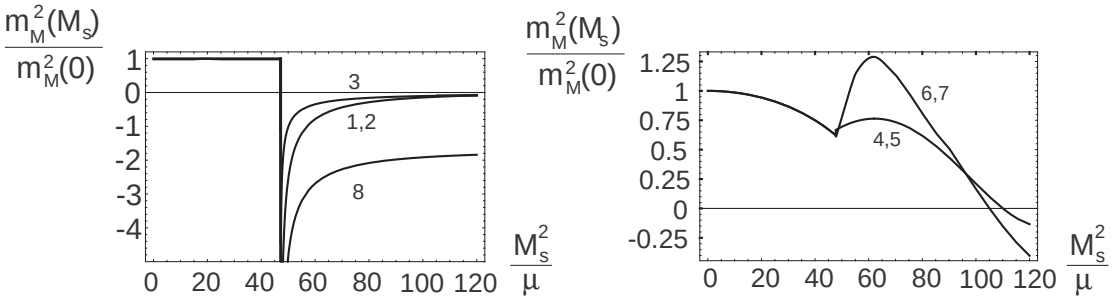

Figure 7: The figure shows, for the gCFL case, the masses of the gluons 1,2,3, 8 (left panel) and 4,5,6,7 (right panel) vs. M_s^2/μ.

The existence of the chromomagnetic instability is a serious problem for the gapless phases (g2SC and gCFL) but also for the 2SC phase, as we have discussed previously. A way out of this problem would be to have gluon condensation. For instance, if one assumes artificially $\langle A_\mu^3 \rangle$ and $\langle A_\mu^8 \rangle$ not zero and with a value of about 10 *MeV* it can be shown that the instability disappears [17]. Also, very recently in [19], it has been shown the possibility of eliminating the chromomagnetic instability in the 2SC phase through a gluonic phase. However it is not clear if the same method can be extended to the gapless phases.

Another interesting possibility has been considered in three papers by Giannakis and Ren, who have considered the LOFF phase, that is a nonhomogeneous phase first studied in a condensed matter context [20, 21] and then in QCD in [22, 23] (for recent reviews of the LOFF phase, see [7, 24]). The results obtained by Giannakis and Ren in the two-flavor case are the following:

- The presence of the chromomagnetic instability in g2SC is exactly what one needs in order that the LOFF phase is energetically favored [25].
- The LOFF phase in the two-flavor case has no chromomagnetic instabilities (though it has gapless modes) at least in the weak coupling limit [26, 27].

Of course these results make the LOFF phase a natural candidate for the stable phase of QCD at moderate densities. In the next Sections we will describe the LOFF phase in its simplest version and a very simple approach to the problem with three flavors.

5. The LOFF Phase

According to the authors of refs. [20, 21] when fermions belong to different Fermi spheres, they might prefer to pair staying as much as possible close to their own Fermi surface. When they

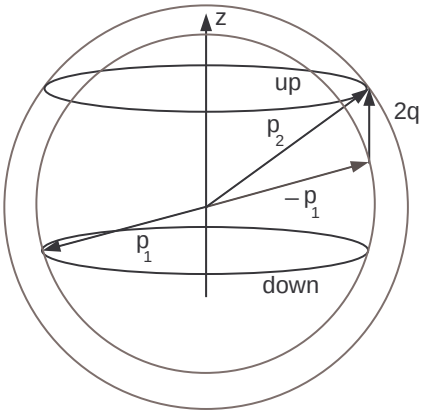

Figure 8: Pairing of fermions belonging to two Fermi spheres of different radii according to LOFF.

are sitting exactly at the surface, the pairing is as shown in Fig. 8. We see that the total momentum of the pair is $\vec{p}_1+\vec{p}_2=2\vec{q}$ and, as we shall show, $|\vec{q}|$ is fixed variationally whereas the direction of $\vec{q}$ is chosen spontaneously. Since the total momentum of the pair is not zero the condensate breaks rotational and translational invariance. The simplest form of the condensate compatible with this breaking is just a simple plane wave (more complicated possibilities will be discussed later)

$$\langle\psi(x)\psi(x)\rangle \approx \Delta e^{2i\vec{q}\cdot\vec{x}}. \tag{5.1}$$

It should also be noticed that the pairs use much less of the Fermi surface than they do in the BCS case. In fact, in the case considered in Fig. 8 the fermions can pair only if they belong to the circles in figure. More generally there is a quite large region in momentum space (the so called blocking region) which is excluded from pairing. This leads to a condensate generally smaller than the BCS one.

Let us now consider in more detail the LOFF phase. For two fermions at different densities we have an extra term in the hamiltonian which can be written as

$$H_I = -\delta\mu\sigma_3, \tag{5.2}$$

where, in the original LOFF papers [20, 21] $\delta\mu$ is proportional to the magnetic field due to the impurities, whereas in the actual case $\delta\mu=(\mu_1-\mu_2)/2$ and σ_3 is a Pauli matrix acting on the two fermion space. According to refs. [20, 21] this favors the formation of pairs with momenta $\vec{p}_1=\vec{k}+\vec{q}$, $\vec{p}_2=-\vec{k}+\vec{q}$. We will discuss in detail the case of a single plane wave (see eq. (5.1)). The interaction term of eq. (5.2) gives rise to a shift in ξ (see eq. (2.2)) due both to the non-zero momentum of the pair and to the different chemical potentials

$$\xi = E(\vec{p})-\mu \to E(\pm\vec{k}+\vec{q})-\mu\mp\delta\mu \approx \xi\mp\bar{\mu}, \tag{5.3}$$

with

$$\bar{\mu}=\delta\mu-\vec{v}_F\cdot\vec{q}. \tag{5.4}$$

Notice that the previous dispersion relations show the presence of gapless modes at momenta depending on the angle with $\vec{q}$. Here we have assumed $\delta\mu \ll \mu$ (with $\mu = (\mu_1 + \mu_2)/2$) allowing us to expand E at the first order in $\vec{q}/\mu$ (see Fig. 8).

The gap equation for the present case is obtained simply from eq. (2.14) via the substitution

$$\delta\mu \to \bar{\mu}. \tag{5.5}$$

By studying eq. (2.14) one can show that increasing $\delta\mu$ starting from zero, we have first the BCS phase. Then at $\delta\mu = \delta\mu_1$ there is a first order transition to the LOFF phase [20, 22], and at $\delta\mu = \delta\mu_2 > \delta\mu_1$ there is a second order phase transition to the normal phase [20, 22]. We start comparing the grand potential in the BCS phase to the one in the normal phase. Their difference is given by

$$\Omega_{\rm BCS} - \Omega_{\rm normal} = -\frac{p_F^2}{4\pi^2 v_F}\left(\Delta_0^2 - 2\delta\mu^2\right), \tag{5.6}$$

where the first term comes from the energy necessary to the BCS condensation, whereas the last term arises from the grand potential of two free fermions with different chemical potential. We recall also that for massless fermions $p_F = \mu$ and $v_F = 1$. We have again assumed $\delta\mu \ll \mu$. This implies that there should be a first order phase transition from the BCS to the normal phase at $\delta\mu = \Delta_0/\sqrt{2}$ [9], since the BCS gap does not depend on $\delta\mu$. The situation is represented in Fig. 9. In order to compare with the LOFF phase one can expand the gap equation around the point

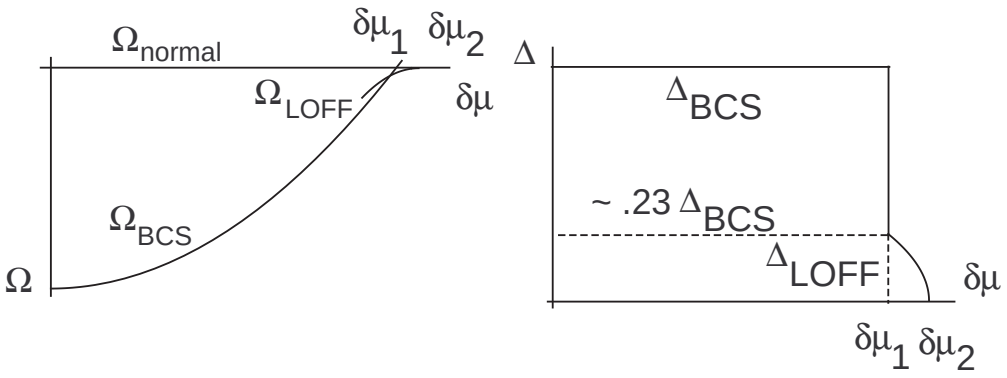

Figure 9: The grand potential (left panel) and the condensates of the BCS and LOFF phases vs. $\delta\mu$ (right panel).

$\Delta = 0$ (Ginzburg-Landau expansion) to explore the possibility of a second order phase transition [20]. The result for the free energy is

$$\Omega_{\rm LOFF} - \Omega_{\rm normal} \approx -0.44\,\rho(\delta\mu - \delta\mu_2)^2. \tag{5.7}$$

At the same time, looking at the minimum in q of the free energy one finds

$$q v_F \approx 1.2\,\delta\mu. \tag{5.8}$$

We see that in the window between the intersection of the BCS curve and the LOFF curve in Fig. 9 and $\delta\mu_2$, the LOFF phase is favored. Also at the intersection there is a first order

transition between the LOFF and the BCS phase. Furthermore, since $\delta\mu_2$ is very close to $\delta\mu_1$ the intersection point is practically given by $\delta\mu_1$. In Fig. 9 we show, in the right panel, the behaviour of the condensates. Although the window $(\delta\mu_1, \delta\mu_2) \simeq (0.707, 0.754)\Delta_0$ is rather narrow, there are indications that, considering the realistic case of QCD [28], the window opens up. Such opening occurs also for different crystalline structures than the single plane wave considered here [23, 30].

6. The LOFF phase with three flavors

In the last Section we would like to illustrate some preliminary result about the LOFF phase with three flavors. This problem has been considered in [29] under various simplifying hypothesis:

- The study has been made in the Ginzburg-Landau approximation.
- Only electrical neutrality has been required and the chemical potentials for the color charges T_3 and T_8 have been put equal to zero (see later).
- The mass of the strange quark has been introduced as it was done previously previously for the gCFL phase.
- The study has been restricted to plane waves, assuming the following generalization of the gCFL case:

$$\langle \psi^{\alpha}_{aL}\, \psi^{\beta}_{bL} \rangle = \sum_{I=1}^{3} \Delta_I(\vec{x}) \varepsilon^{\alpha\beta I} \varepsilon_{abI}, \quad \Delta_I(\vec{x}) = \Delta_I e^{2i\vec{q}_I\cdot\vec{x}} \tag{6.1}$$

- The condensate depends on three momenta, meaning three lengths of the momenta q_i and three angles. In [29] only four particular geometries have been considered: 1) all the momenta parallel pointing upward the z-axis, then 2), 3) and 4) are obtained by inverting respectively the momentum $\vec{q}_1$, $\vec{q}_2$ and $\vec{q}_3$.

Under the previous hypothesis the free energy (with reference to the normal state) has the expansion

$$\Omega - \Omega_{normal} = \sum_{I=1}^{3} \left(\frac{\alpha_I}{2}\Delta_I^2 \,+\, \frac{\beta_I}{4}\Delta_I^4 \,+\, \sum_{I\neq J} \frac{\beta_{IJ}}{4}\Delta_I^2\Delta_J^2 \right) \,+\, O(\Delta^6) \tag{6.2}$$

with

$$\alpha_I(q_I, \delta\mu_I) = -\frac{4\mu^2}{\pi^2}\left(1 - \frac{\delta\mu_I}{2q_I}\log\left|\frac{q_I+\delta\mu_I}{q_I-\delta\mu_I}\right| - \frac{1}{2}\log\left|\frac{4(q_I^2-\delta\mu_I^2)}{\Delta_0^2}\right|\right) \tag{6.3}$$

$$\beta_I(q_I, \delta\mu_I) = \frac{\mu^2}{\pi^2}\frac{1}{q_I^2-\delta\mu_I^2} \tag{6.4}$$

$$\beta_{12} = -2\frac{\mu^2}{\pi^2}\int \frac{d\mathbf{n}}{4\pi}\, \frac{1}{(2\mathbf{q_1}\cdot\mathbf{n} + \mu_s - \mu_d + i0^+)\,(2\mathbf{q_2}\cdot\mathbf{n} + \mu_s - \mu_u + i0^+)} \tag{6.5}$$

and the other β_{IJ}, $I \neq J$ obtained by the exchange

$$12 \to 23,\ \mu_s \leftrightarrow \mu_d, \quad 12 \to 13,\ \mu_s \leftrightarrow \mu_u \tag{6.6}$$

The $\delta\mu_I$ are obtained from

$$\mu_u = \mu - \frac{2}{3}\mu_e, \;\; \mu_d = \mu + \frac{1}{3}\mu_e, \;\; \mu_s = \mu + \frac{1}{3}\mu_e - \frac{M_s^2}{2\mu} \tag{6.7}$$

In particular the coefficients of Δ_I^2 are the same as for LOFF with two flavors. Therefore the minimization with respect to the $|\vec{q}_I|$'s leads to the same result as in eq. (5.8)

$$|\vec{q}_I| = 1.2\delta\mu_I \tag{6.8}$$

Then, one has to minimize with respect to the gaps and to μ_e in order to require electrical neutrality. It turns out that the configurations 3 and 4 have an extremely small gap. Furthermore for M_s^2/μ greater than about 80 MeV the solution gives $\Delta_1 = 0$ and $\Delta_2 = \Delta_3$. In this case the configurations 1‘ and 2 have the same free energy. The results for the free energy and for the gap of this solution are given in Fig. 10 and 11. In this study, the following choice of the parameters has been made: the BCS gap, $\Delta_0 = 25$ MeV, and the chemical potential $\mu = 500$ MeV. The values are the same discussed previously for gCFL in order to allow for a comparison of the results.

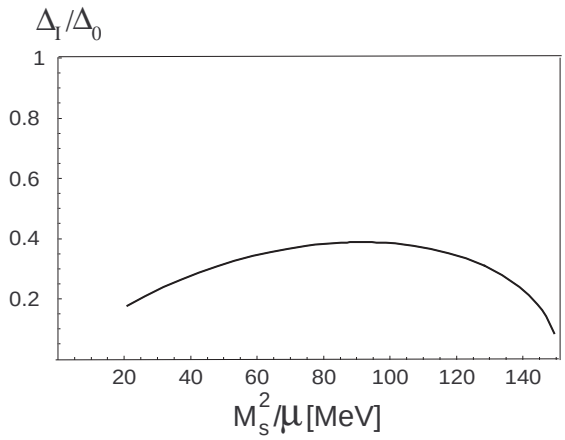

Figure 10: The gap for LOFF with three flavors vs. M_s^2/μ. The line corresponds to the most favored solution, that is to the configurations 1 and 2.

We are now in the position to compare these results with the ones obtained in [14] for the gCFL phase. The comparison is made in Fig. 12. Ignoring the chromomagnetic instabilities of the gapless phases and of 2SC we see that LOFF takes over with respect to gCFL at about $M_s^2/\mu = 128$ MeV and goes over to the normal phase for $M_s^2/\mu \approx 150$ MeV.

However, since the instability exists it should be cured in some way. The results for the LOFF phase, assuming that also for three flavors the chromomagnetic instability does not show up, say that it could be the LOFF phase to takes over the CFL phase before the transition to gCFL. For this it is necessary that the window for the LOFF phase gets enlarged. However, in [30] it has been show that for structures more general than the plane wave the windows may indeed becomes larger. If define the window for the single plane wave as $(\delta\mu_2 - \delta\mu_1)/\delta\mu_2$ (see the previous Section) we would get 0.06. The analogous ratio in going from one to three plane waves goes to

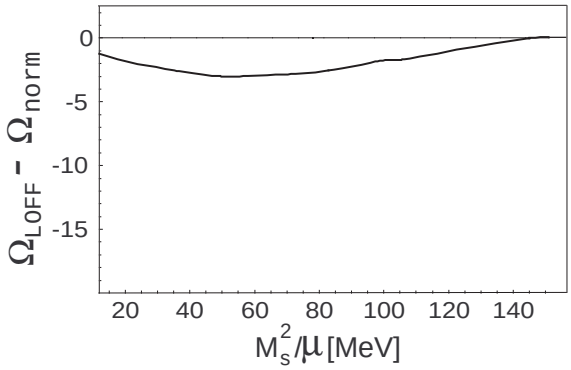

Figure 11: The free energy of the most favored configurations (1 and 2) considered for LOFF with three flavors vs. M_s^2/μ.

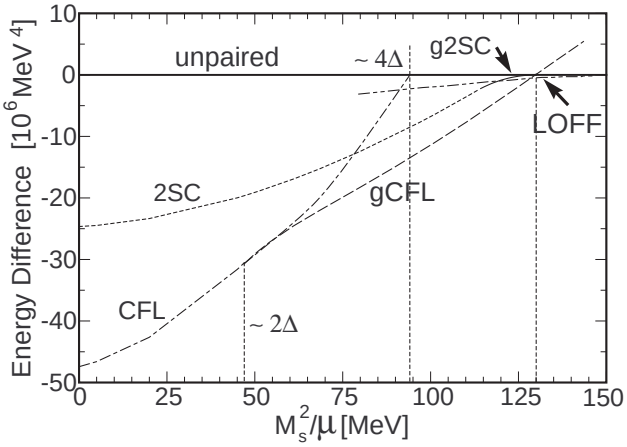

Figure 12: Comparison of the free energy of the various phases with the LOFF phase with three flavors.

about $(150-115)/150 = .23$, with a gain of almost a factor 4. On the other hand, in [30] it has been shown that considering some of the crystalline structures already taken in exam in [22], as the face centered cube or the cube the windows becomes about $(1.32-0.707)/1.32 = 0.46$ with a gain of about 7.7 with respect to the single plane wave. If these gains would be maintained in going from two to three flavors with the face centered cube structure, one could expect a gain from 4 to 7.7 with an enlargement of the window between 88 and 170 *MeV*, which would be enough to cover the region of gCFL (which is about 70 *MeV*).

At last we want to comment about the approximation in neglecting the color neutrality condition and assuming $\mu_3 = \mu_8 = 0$. In Fig. 13 we show the chemical potentials μ_e, μ_3, $\mu_8$for the gCFL phase in the left panel, and μ_e for the LOFF phase in the right panel. We can make two observations: first of all, in the region of interest where LOFF dominates over gCFL the behaviour of μ_e in the two phases is pretty much similar, and μ_3, $\mu_8 \ll \mu_e$ for gCFL. This suggests that also in the LOFF case μ_3 and μ_8 are small. Second and more important the result of [29] shows that $\mu_e \approx M_s^2/(4\mu)$ as for the case of 3 color and 3 flavor unpaired quarks [16]. As can be seen from eq.

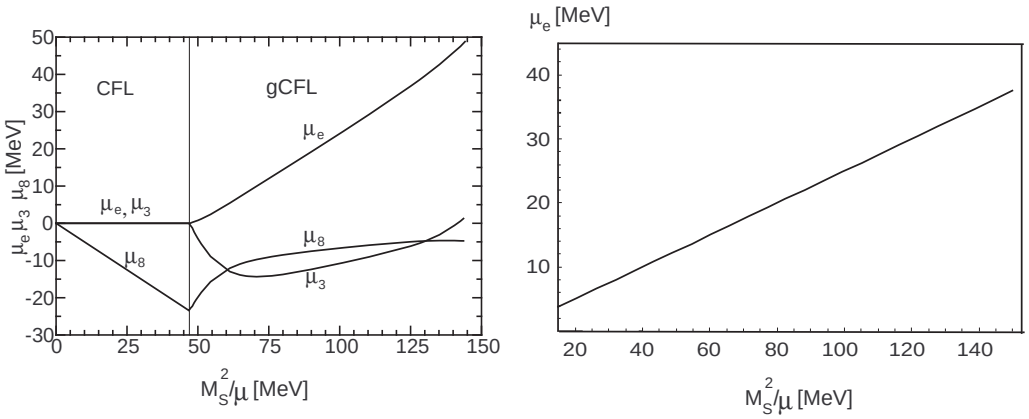

Figure 13: The chemical potential for gCFL (left panel) and LOFF (right panel) vs. M_s^2/μ.

(6.7) this coreesponds to a symmetrical spli of the *s* and *d* Fermi surfaces around the *u* Fermi surface. Therefore $\mathbf{q}_2 = \mathbf{q}_3$ and the gaps Δ_2 and Δ_3 must coincide. At the same time the separation of the *d* and *s* surfaces is the double and therefore $\Delta_1 = 0$. The unpaired quarks have also $\mu_3 = \mu_8 = 0$. Also, from Fig. 10 we see that in our approximations the transition from the LOFF to the normal phase is very close to be continuous. Since we expect also the chemical potentials to be continuous at the transition point very close to the critical point we should have $\mu_3 = \mu_8 = 0$ also on the LOFF side. This means the color neutrality condition should be $\mu_3 = \mu_8 = 0$ in the neighborhood of the transition. Therefore we expect the determination of the point $M_s^2/\mu = 150\ MeV$ to be safe. On the other hand, the requirement of color neutrality could change the intersection point with gCFL. Nevertheless, since the critical point for LOFF is higher than the one of gCFL, for increasing M_s the system must to go into the LOFF phase.

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