

Lattice QCD at finite density: imaginary chemical potential

Maria Paola Lombardo*

Istituto Nazionale di Fisica Nucleare

Italy

E-mail: lombardo@lnf.infn.it

I describe the results for the critical line and the thermodynamics of different phases of QCD which have been obtained by lattice simulations with an imaginary chemical potential. I review motivations and merits of the different strategies – Taylor expansion, Fourier analysis and Pade' approximants – for analytic continuation from imaginary to real chemical potential. I consider phenomenological models which can be easily extended to the complex chemical potential plane, thus affording a direct comparison with lattice data at imaginary μ : the hadronic phase and the high temperature limit are amenable to a simple description, while a rather subtle interplay between thermodynamics and critical behaviour emerges in the hot phase close to T_c .

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*Speaker.

1. Introduction

In principle, the lattice formulation provides a rigorous framework for the study of the thermodynamics of QCD. In practise, however, the lattice regularisation is usually combined with importance sampling, which cannot be naively applied at nonzero baryon density, where the quark determinant becomes complex [1].

It has been realised that this problem can be circumvented in the high T , low μ part of the QCD phase diagram where one can take advantage of physical fluctuations. Interesting physical information can be obtained by computing the derivatives with respect to μ at zero chemical potential and high temperature [2, 3, 4, 5, 6, 7, 8]. Fodor and Katz proposed an improved reweighting affording a better overlap between simulation and target ensembles, and a first estimate of the critical endpoint [9, 10, 11, 12]. In refs. [13, 14, 15] the imaginary chemical potential approach was advocated and exploited in connection with the canonical formalism. In ref. [16] it was proposed that the analytic continuation from imaginary chemical potential could be practical at high temperature, and the idea was tested in the infinite coupling limit. In refs. [17] the method was applied successfully to QCD in 2+1 dimensions. In ref. [18] it was proposed that the critical line itself can be analytically continued and results for two flavor of staggered fermions were presented. The scaling of the critical line and thermodynamics of the four fermion models were studied in [19, 20, 21]

The purpose of this note is to review early and new results obtained by use of the imaginary chemical potential approach: the basic idea, the applications, and the potentiality.

The following Section is a short introduction into lattice QCD and the sign problem; Section 3 discusses the phase diagram in the T, μ^2 plane; Section 4 offers a short summary of the canonical approach, while Section 5 introduces the three series representations which have been used so far. Next, a short collection of results from models. The five central Sections are devoted to the presentations and discussion of results: Sections 7 focuses on the analytic continuation from lattice data to real chemical potential by use of a Taylor expansion, section 8 shows how to extend the results towards larger μ / smaller temperatures by using Pade' approximants. Sections 9 and 10 use phenomenological models extended to entire complex lane to parametrise thermodynamics in the different phases of QCD, section 11 shows how critical behaviour at imaginary chemical potential can influence the thermodynamics of the strong interactive Quark Gluon Plasma.

Introductory material and other surveys of results can be found e.g. in refs. [23, 24, 25, 26].

2. The sign problem, and an imaginary μ_B

Let us remind ourselves how to introduce a chemical potential μ for a conserved charge \hat{N} in the density matrix $\hat{\rho}$ in the Grand Canonical formalism, which is the one appropriate for a relativistic field theory:

$$\hat{\rho} = e^{-(H - \mu \hat{N})/T} \quad (2.1)$$

$$\mathcal{Z}(\mathcal{T}, \mu) = \text{Tr} \hat{\rho} = \int d\phi d\psi e^{-S(\phi, \psi)} \quad (2.2)$$

The temperature T on a lattice is the same as in the continuum: $T = 1/N_t a$, $N_t a$ being the lattice extent in the imaginary time direction (while, ideally, the lattice spatial size should be infinite).

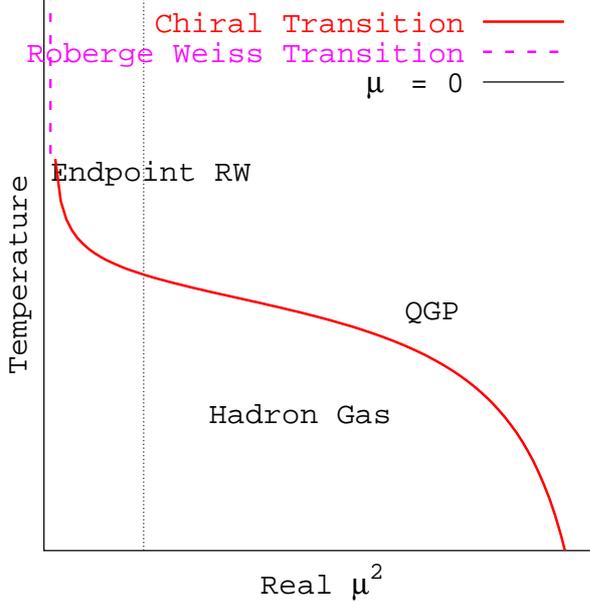


Figure 1: The phase diagram in the T, μ^2 plane. Simulations are possible in the $\mu^2 \leq 0$ halfplane, and results in the physical $\mu^2 \geq 0$ halfplane have to be inferred from the results.

A lattice realisation of a finite density of baryons, instead, poses specific problems: the naive discretization of the continuum expression $\mu \bar{\psi} \gamma_0 \psi$ would give an energy $\varepsilon \propto \frac{\mu^2}{a^2}$ diverging in the continuum ($a \rightarrow 0$) limit. The problem could be cured by introducing appropriate counterterms, however the analogy between μ and an external field in the 0_{th} (temporal) direction offers a nicer solution by considering the appropriate lattice conserved current [1]. This amounts to the following modification of the fermionic part of the Lagrangian for the 0_{th} direction L_F^0 :

$$L_F^0(\mu) = \bar{\psi}_x \gamma_0 e^{\mu a} \psi_{x+\hat{0}} - \bar{\psi}_{x+\hat{0}} \gamma_0 e^{-\mu a} \psi_x \quad (2.3)$$

We then integrate out fermions exactly, by taking advantage of the bilinearity of the fermionic part of the Lagrangian $L = L_{YM} + L_F = L_{YM} + \bar{\psi} M(U) \psi$:

$$\int dU d\psi d\bar{\psi} \mathcal{Z}(T, \mu, \bar{\psi}, \psi, U) = \int dU e^{-(S_{YM}(U) - \log(\det M))} \quad (2.4)$$

When $\det M > 0$ the functional integral can be evaluated with statistical methods, sampling the configurations according to their importance ($S_{YM}(U) - \log(\det M)$). For this to be possible the would-be-measure ($\det M$) has to be positive.

Consider now the relationship $M^\dagger(\mu_B) = -M(-\mu_B)$: this implies that reality is lost when $\text{Re} \mu \neq 0$. Clearly, for real μ , the imaginary part of the determinant cancels out in the statistical ensemble, and it is even possible to cancel it exactly on a finite number of configurations by considering the appropriate symmetry transformation [29, 30]; but still one has to face a sign problem, and it is not clear in which dynamical region this problem becomes significant [31].

Consider instead $\text{Re} \mu = 0$ and $\text{Im} \mu \neq 0$: in this case $M^\dagger(\mu_B) = -M(-\mu_B)$ implies $M^\dagger = -M^*$ and standard lattice simulations using $M^\dagger M$ as a weight are possible.

It has been shown by Roberge and Weiss [27] that $Z(i\mu/T)$ is always periodic $2\pi/3$, for any physical temperature. At low T strong coupling calculations predict a smooth behaviour, whereas at high T weak coupling calculations predict discontinuities in the thermodynamics observables at $T = 2\pi/3(k + 1/2)$.

This scenario for the phase diagram of QCD in the $T - i\mu_t$ plane has been indeed confirmed by lattice studies [18, 19] and model calculations [28]. It was also found that the Roberge-Weiss line of discontinuities ends around a critical temperature $T_{RW} > T_c$.

3. Mapping the phase diagram in the T, μ complex μ plane to the T, μ^2 plane

The Gran Canonical Partition function $\mathcal{Z}(\mu)$ is an even function of μ , which is real valued for either real and purely imaginary μ , and complex otherwise.

We can map the complex μ plane onto the complex μ^2 plane, and consider $\mathcal{Z}(\mu^2)$. Note that because of the symmetries of the partition function this can be done without any loss of generality.

Then, \mathcal{Z} is real valued on the real μ^2 axis, complex elsewhere : the situation is analogous to e.g. the partition function as a function of a magnetic field, which becomes complex as soon as the external field becomes complex, and the physical domain (real partition function) is associated with real values of the couplings. The critical behaviour of the system is then dictated by the zeroes of the partition function (Lee-Yang zeros) in the complex μ^2 plane. The locus of the Lee Yang zeros is thought to be associated to a general surface of phase separation [32], and phase transition points, for each value of the temperature, are associated with the Lee-Yang edge building up in the infinite volume limit, thus defining a curve in the T, μ^2 plane.

This simple reasoning shows that it is very natural to re-think the phase diagram in the T, μ^2 plane, allowing μ^2 to take both negative and positive values (Fig. 1). The critical line itself should be a smooth function $T(\mu^2)$, making it natural the analytic continuation from positive to negative μ^2 values. The vertical dash line is the Roberge-Weiss discontinuity, ending at T_{RW} , and it is still not completely clear how it morphs with the chiral transition [33].

Experience with statistical models shows that not only the critical line, but also the critical exponents are smooth functions of the couplings [34] (aside of course from endpoints, bifurcation points, etc.). Hence, they can be safely expanded, either via Taylor expansion or a suitable ansatz. In particular, $\mu_c^2 = 0$ has no special character: it is just the point where the Lee Yang edge hits the real axis where $T = T_c$.

All in all, our task is to simulate the theory in the $\mu^2 \leq 0$ strip, and to do our best to 'extrapolate' the results to real values of μ . There are two main possibilities : one is via a canonical approach, and the other via an analytic continuation.

4. The Canonical Approach

An imaginary chemical potential μ in a sense bridges Canonical and Grand Canonical ensemble:

$$\mathcal{Z}_{\mathcal{G}}(\mathcal{N}) = \frac{\beta}{2\pi} \int_0^{2\pi/\beta} d\mu \mathcal{Z}_{\mathcal{G}}(i\mu) e^{-i\beta\mu N} \quad (4.1)$$

hence, $Z(V, T, i\mu_I)$ can also be used to reconstruct the canonical partition function $Z(V, T, n)$ at fixed quark number n [27], *i.e.* at fixed density:

$$\begin{aligned} Z(V, T, n) &= \text{Tr} \left(e^{-\frac{H_{\text{QCD}}}{T}} \delta(N - n) \right) = \frac{1}{2\pi} \text{Tr} \left(e^{-\frac{H_{\text{QCD}}}{T}} \int_0^{2\pi} d\theta e^{i\theta(N-n)} \right) \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i\theta n} Z(V, T, i\theta T). \end{aligned} \quad (4.2)$$

As n grows, the factor $e^{-i\theta n}$ oscillates more and more rapidly and the error in the numerical integration grows exponentially with n : this makes the application of the method difficult especially at low temperatures where $Z(V, T, i\mu_I)$ depends very weakly on μ_I . The method has been applied in QCD [14] and in the 2-d Hubbard model [13] [15], where $Z(V, T, n)$ has been reconstructed up to $n = 6$ [15], and more recently again to QCD (see later for some comparison with other approaches) [35, 37, 36]

5. Analytic continuation: Taylor, Fourier, Pade'

In principle, if one were able to determine thermodynamic observables as a function $F(\mu)$ of the imaginary part of μ with infinite accuracy, standard complex analysis arguments would guarantee that the result will be valid within the entire analytic domain. Since $F(\mu)$ is not known a priori, analytic continuation must rely on some series representation, or phenomenological insight. In the following of this section, I will discuss Taylor series, Fourier series and Pade' approximants, which are ratios of polynomials defined as

$$P[N, M](\mu_i) = \frac{a_0 + a_1\mu_i + \dots + a_M\mu_i^M}{b_0 + b_1\mu_i + \dots + b_N\mu_i^N} \quad (5.1)$$

Note that $P[0, N]$ is the Taylor N-th partial sum.)

Taylor series was the first idea which came to mind: one fits the numerical results to a polynomial, whose coefficients can be interpreted in terms of a Taylor series centred at $\mu = 0$. This allows an easy contact with the calculations of [2, 3, 4, 5, 6, 7, 8] where the various coefficients are computed as derivatives of the various observables at $\mu = 0$. A truncated polynomial is of course an analytic function, which can be evaluated everywhere on the complex plane as a function of the complex variable say z :

$$O(\mu) = \sum_k a_k \mu^k \quad (5.2)$$

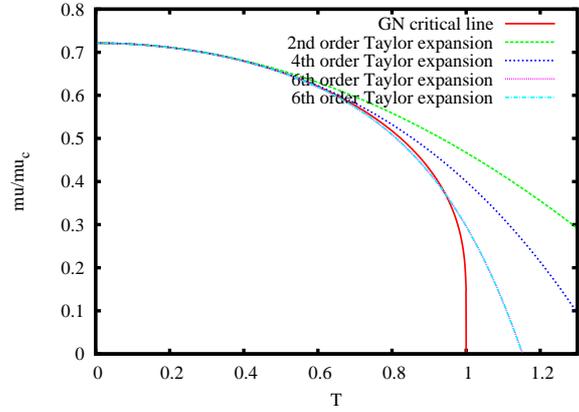


Figure 2: The Gross Neveu critical line and its Taylor approximants: clearly one would need an infinite number of terms to reproduce the infinite slope of the critical line at $T = 0$. Pade' guarantees a faster converges.

However, convergence will only be achieved within a circle (the circle of convergence of the Taylor series). One can make a virtue of this limitation by estimating the position of singularities in the complex plane from the value of the radius of convergence estimated from the behaviour of the series itself [38].

We can now ask ourselves, how can we analytically continue beyond the radius of convergence of the Taylor series. This of course must be possible, because of the general argument recalled above.

let us remind ourselves that an analytic function is locally representable as a Taylor series. The convergence disks can be chosen in such a way that they overlap two by two, and cover the analytic domain. Thus, one way to build the analytic continuation is by connecting all of these convergence disks. The arcs of the convergence circles which are within the region where f is analytic have a pure geometric meaning, and by no means are an obstacle to the analytic continuation. Assume now that the circle of convergence about $z = (0,0)$ has radius unit, i.e. is tangent to the lines which limit the analytic domain; take now a z value, say $z_1 = (0,a)$, $1/2 < a < 1$ inside the convergence disk as the origin of a new series expansion, which is explicitly defined by the rearrangement $(z - z_0)^n = (z - z_1 + z_1 - z_0)^n$. As the radius of convergence of the new series will be again one, this procedure will extend the domain of definition of our original function (the two series define restrictions of the same function to the intersection between the two disks), and by 'sliding' the convergence disk we can cover all the analytic strip.

An alternative parametrisation, advocated in [19, 40, 41] for dealing with the data in hadronic phase, uses the Fourier Analysis:

$$O(\mu_I) = \sum_k a_k \exp(ki\mu_I) O(\mu) = \sum_k a_k \exp(k\mu) \quad (5.3)$$

The latest proposal is to use Pade' Approximants [42, 43]. Pade' approximants have a time honored history in statistical mechanics, and we have borrowed these idea to try and improve the analytic continuation in our context. We have sketched above the standard theoretical argument to demonstrate the feasibility of analytic continuation beyond the radius of convergence of the Taylor series, and we will show that the Pade' series is one practical way to accomplish it.

6. Interlude : Insight from models

We collect here a miscellaneous (and by no means systematic or complete) set of results for models, which might be useful to keep in mind for further experimentation.

6.1 The Gross-Neveu model

The Gross-Neveu model in three dimensions is interacting, renormalizable and can be chosen with the same global symmetries as those of QCD which, when spontaneously broken at strong coupling, produce Goldstone particles and dynamical mass generation. As such, the Gross-Neveu model (as well as other four fermion models) can provide some guidance to the understanding of the QCD critical behaviour (see e.g. refs. [49, 50, 51]).

The critical line for the three dimensional GN model was calculated in ref. [52] and reads

$$1 - \mu/\Sigma_0 = 2T/\Sigma_0 \ln(1 + e^{-\mu/T}) \quad (6.1)$$

where Σ_0 is the order parameter in the normal phase. Setting $\mu = 0$ in the above equation gives the critical temperature at zero chemical potential, $T_c(\mu = 0) = \Sigma_0/2 \ln 2 \simeq .72\Sigma_0$

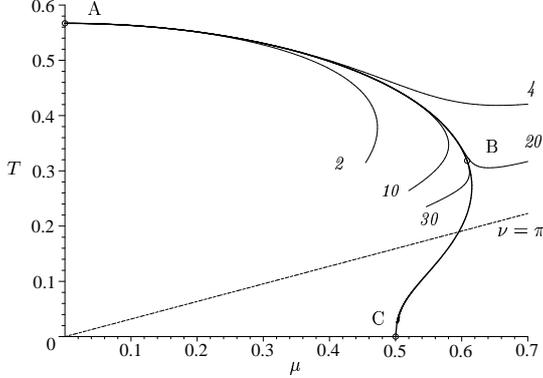


Figure 3: 2nd order critical line in comparison to analytic continuation from imaginary to real μ via power series expansion around $\theta = 0$. The curves correspond to 2nd, 4th, 10th, 20th, 30th order in θ and reach beyond the tricritical point B. The series converges above the line $\nu = \pi$. From ref. [39]

of the analytic continuation past a tricritical point, see Fig. 3: they find that the tricritical point does not limit the radius of convergence of the critical line, and propose a strategy based on the analysis of the effective potential for identifying it.

6.2 Random Matrix Theories

As it is well known (see e.g. [66]), there is a remarkable relation between the symmetry breaking classes of QCD and the classification of chiral Random matrix Ensembles.

For QCD with fermions in the complex representation (i.e. $N_c > 2$, fundamental fermions) with pattern of SSB $SU(N_f)_R \times SU(N_f)_L \rightarrow SU(N_f)$, the corresponding RMT is chiral unitary, $\beta = 2$ in the Dyson representation. On the lattice, staggered fermions have unusual patterns of χ SB: all real and pseudoreal representations are swapped. However, for complex representations, the corresponding RMT ensemble remains chiral unitary [66]. The critical line in the T, μ plane for this ensemble derived in ref. [67]:

$$(\mu^2 + T^2)^2 + \mu^2 - T^2 = 0 \quad (6.3)$$

is thus valid both on the lattice and in the continuum. Expanding it to $O(\mu^2)$ we obtain

$$T^2 = T_c^2 - 3\mu^2 \quad (6.4)$$

and by comparison with the exact result 6.3, we note that this simple expression describes well the critical line basically till its endpoint. Again, note that the parametrisation is a Jordan curve with infinite slope at $T = 0$.

Expanding now $\ln(1 + e^{(-x)}) \simeq \ln 2 - 1/2x + 1/8x^2$, and eliminating Σ_0 in favour of T_c , we get

$$(T - 1/2T_c)^2 + \mu^2/(8 \ln 2) = T_c^2/4 \quad (6.2)$$

This expression can be cross checked with different orders of the Taylor expansion: it is interesting to mention that, even barring problems connected with the radius of convergence we cannot expect that the Taylor expansion describes well the critical line: note the infinite slope at $T=0$, and we believe the Fig. 2 is self explanatory.

A more sophisticated study has been carried out recently, [39], addressing the very interesting question of the behaviour

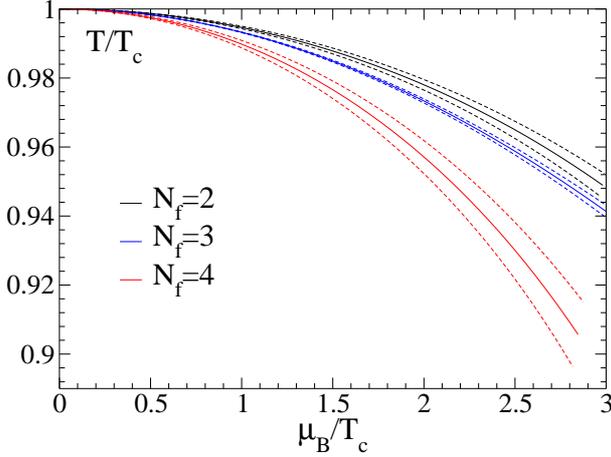


Figure 4: Summary plot[20] for the critical line for $N_f = 2$ [18], $N_f = 3$ [20], $N_f = 4$ [19] from imaginary chemical potential calculations.

such zeros does indeed confirm the general discussion of [55].

7. The critical line at small μ from the Taylor expansion

The first studies of the critical line have indeed found that a simple polynomial approximation suffices to describe the data, within the current precision. The precision is demonstrated in Fig. 4 [20] where the results on the three flavor model [20] are superimposed to the ones with two[18] and four flavors[19].

The overall trend is consistent with the transition becoming weaker with increasing the number of flavor.

Another nice consistence check can be done by exploiting the four flavor model, which was studied in a greater detail, and it is shown in Figs. 5,6.

In Fig. 5 we see a compilation of results from different methods. In Fig. 6, left the results are contrasted with a generalised imaginary chemical potential approach [45]. They define

$$S = S_{PG} + ma \sum_n \bar{\psi}_n \psi_n + \frac{1}{2} \sum_n \sum_{i=1}^3 \bar{\psi}_n \eta_i(n) \left(U_{n,i} \psi_{n+i} - U_{n-i,i}^\dagger \psi_{n-i} \right) + S_\tau(x,y), \quad (7.1)$$

with

$$S_\tau(x,y) = x \frac{1}{2} \sum_n \bar{\psi}_n \eta_0(n) \left(U_{n,0} \psi_{n+0} - U_{n-0,0}^\dagger \psi_{n-0} \right) + y \frac{1}{2} \sum_n \bar{\psi}_n \eta_0(n) \left(U_{n,0} \psi_{n+0} + U_{n-0,0}^\dagger \psi_{n-0} \right), \quad (7.2)$$

where x and y are two independent parameters. The QCD action is recovered by setting $x = \cosh(\mu a)$ and $y = \sinh(\mu a)$. By moving in the x, y space it is possible to include low temperature regions within the radius of convergence of the Taylor series. In Fig. 6, right we see the

6.3 One Dimensional QCD

Another amusing example is one dimensional QCD, whose free energy as a function of fugacity can be analytically computed [68]. In terms of the fugacity $f = e^{3\mu/T}$ \mathcal{Z} reads:

$$\mathcal{Z}(f) = f^2 + 1 + f \cosh(3m/T) = P[2,2](f) \quad (6.5)$$

We note that the Padé expansion is exact, while a Taylor series, again, will not converge outside its radius of convergence. The model is also an excellent testbed for the general pattern of complex zeros of the partition function: the analysis [57] of

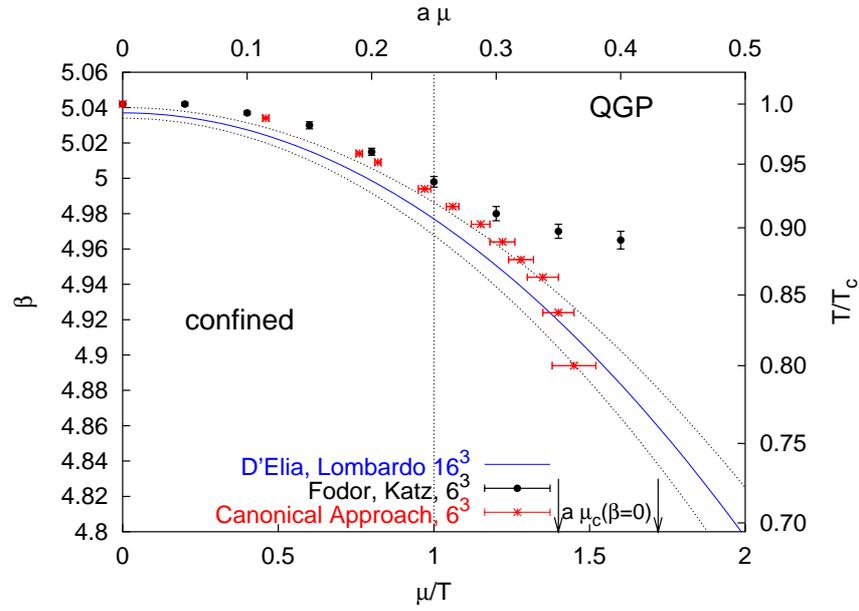


Figure 5: Comparison among different techniques for the critical line of four flavor QCD I : The analytic continuation relying on the Taylor expansion cannot be automatically trusted beyond the vertical line. From Ref. [37]

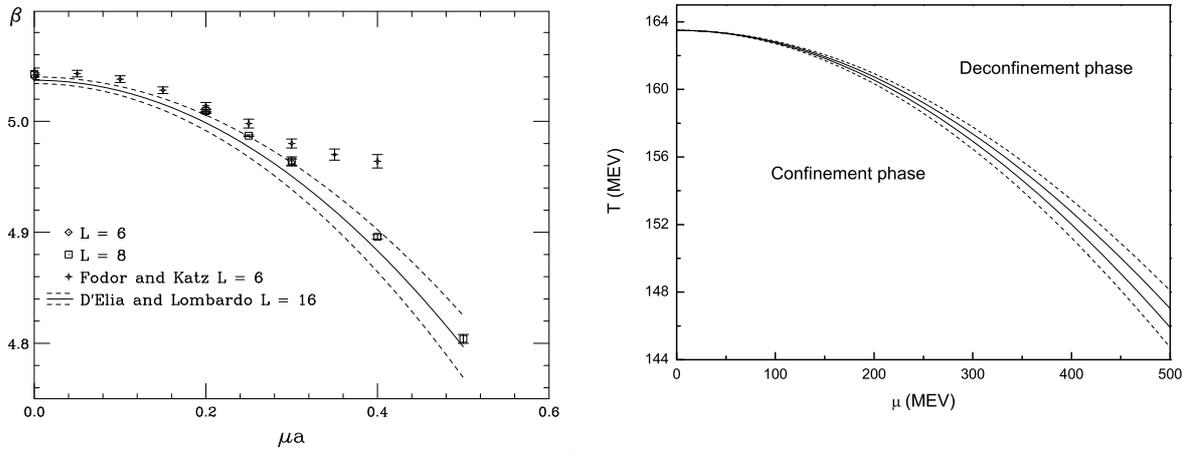


Figure 6: Results for the critical line from the generalised imaginary chemical potential approach of ref. [45] compared with those of ref. [19] (left); Comparison between Wilson (solid) and staggered (dash) fermions, from ref. [46]

comparison between the results for Wilson fermions and those for staggered fermions [46]. Recently, an interesting study with two flavor of Wilson fermions has appeared as well [47]. Results from different fermion discretions are indeed very important to control the continuum limit, and these cross checks nicely demonstrate the robustness of the results

However, the critical line estimated by use of a second order Taylor series, in principle cannot be trusted beyond the radius of convergence of the series itself. We will discuss in the next Section how to improve the situation by use of Pade' approximants.

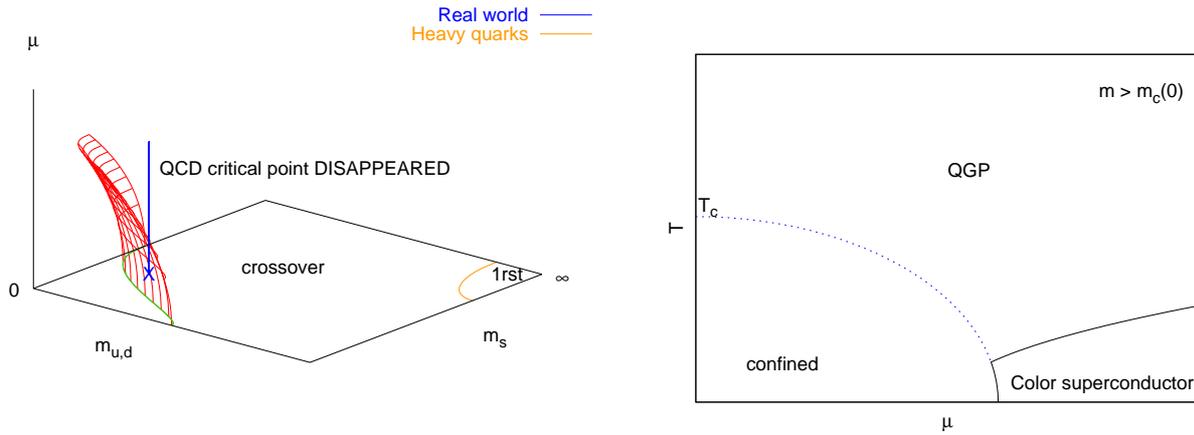


Figure 7: For $dm_c(\mu)/d\mu^2 < 0$, there is no critical point at all, the dotted line on the right is merely a crossover., from Ref. [56]

7.1 The Endpoint

A crucial issue remains the determination of the endpoint, expected of a theory with two plus one flavor. The first estimate was given within the reweighting method $T_E = 160 \pm 3.5 MeV$, $\mu_E = 725 \pm 35 MeV$ [10]. Results obtained at imaginary chemical potential and with improved precisions did show a subtle dependence on the mass values, and on the parameters of the algorithm [61, 60].

Such technical limitations are not specific of imaginary chemical potential. This makes mandatory an extrapolation to physical values of the quark masses, which, in turn, implies a good control on the continuum limit.

Recently, Forcrand and Philipsen took an alternate approach to the search of the endpoint : they observed [56] that the existence of the endpoint depends on general features of the critical surface in the temperature chemical potential and mass plane (Fig.7) . They arrived at the conclusion that, for the lattice spacing considered here, the curvature of the critical surface is negative, which seems hardly consistent with the presence of an endpoint.

This behaviour, if confirmed, (and, indeed, the results of ref. [58] support these findings) would be a strong indication that the behaviour of QCD is indeed much more subtle than that of the four fermion models with the same global symmetries, which were first used to suggest the existence of such endpoint in the T, μ plane [59]. Perhaps not surprisingly, given that the mechanism of chiral symmetry breaking in QCD is associated with long distance forces, as opposed to the strong local dynamics which is responsible for chiral breaking in four fermions models.

7.2 Chiral Symmetry, confinement, topology

Theoretical issues related with the nature of the critical line can be studied directly at imaginary chemical potential, without any continuation.

In Ref. [19, 21] we have demonstrated the correlation between chiral condensate and of the Polyakov loop, and argued that this correlation should be continued at real baryon density: to this end, we note that if $\beta_c(i\mu_I) = \beta_d(i\mu_I)$ over a finite imaginary chemical potential interval, then the

function $\Delta\beta(i\mu_I) = \beta_c(i\mu_I) - \beta_d(i\mu_I)$ is simply continued to be zero over the entire analyticity domain, thus demonstrating the correlation between the chiral and the deconfinement transitions ($\beta_c = \beta_d$) also for real values of μ . We refer to e.g refs. [62, 63] for an effective Lagrangian discussion of this issue.

It is interesting to briefly mention results in the two colour model: in ref [64] it is shown that Polyakov Loop, chiral condensate are correlated in the high temperature region of the phase diagram in two colour QCD, as they are in three colour QCD. In addition to this, also the topological charge (Fig.8) is correlated with the other observables, as it was at $\mu = 0$: all in all, a finite density of baryons does not change the nature of the transition from the hadron to the quark gluon plasma phase. The simulations were repeated for the Pisa order parameter in ref. [65].

8. Beyond small μ via Pade' approximants

In Fig. 9 we present the Pade' analysis [42] of data for the critical line of four flavor QCD (numerical results are from [19]) Results seem stable beyond $\mu_B = 500 MeV (\mu_B/T \simeq 1)$, with the Pade' analysis in good agreement with Taylor expansion for smaller μ values. At larger μ the Taylor expansion seems less stable, while the Pade' still converges, giving a slope of the critical line larger than the naive continuation of the second order Taylor approximations: we underscore that the possibility of analytically continue the results beyond the radius of convergence of the Taylor series by no means imply that one can blindly extrapolate a lower Taylor order approximation!

The same bending towards lower μ values is suggested by recent results within the canonical approach [37] and the DOS method [53], and it agrees with the qualitative features of the simple models discussed above.

Finally, we summarise in Fig. 10 the overall results for the critical line of four flavor QCD (note that Fig. 9 uses μ_B while Fig. 10 uses μ : the red lines is the same in the two plots).

8.1 Thermodynamics beyond $\mu/T \simeq 1$

Similarly, it is possible to apply the Pade' analysis to thermodynamics observables. The Pade' approximants to the results for the chiral condensate in the hot phase are shown in Fig. 11, upper diagram, from [42]. We see that the Pade' approximants converge well beyond the would-be radius of convergence of the Taylor expansion. Similar results have been obtained in two colour QCD [43] where it is also possible a direct comparison with exact results, in the spirit of ref. [70].

We underscore that, as noted in [21] the radius of convergence should tend to infinite in the infinite temperature limit, and indeed it has been estimated to be large by the Bielefeld–Swansea collaboration [71]: by increasing T Pade' and Taylor should become equivalent.

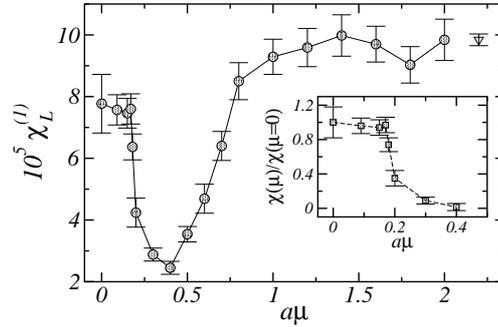


Figure 8: Topological susceptibility as a function of the chemical potential μ [64].

9. The Hadronic Phase and the Fourier Analysis

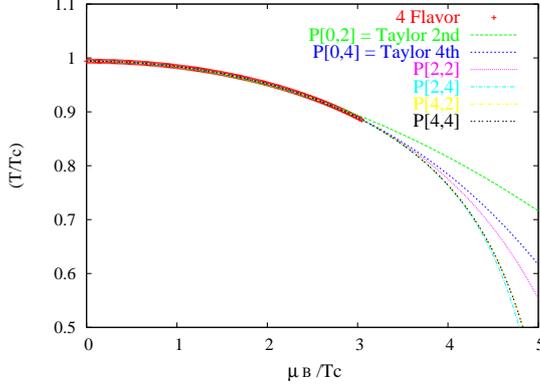


Figure 9: Pade' approximants for the critical line of four flavor QCD [42].

of the number density[21] - even and odd observables, respectively - we limited ourselves to $n = 0, 1, 2$ and we assessed the validity of the fits via both the value of the $\chi^2/\text{d.o.f.}$ and the stability of $a_F^{(0)}$ and $a_F^{(1)}$ given by one and two cosine [sine] fits: we found that one cosine [sine] fit describes reasonably well the data up to $T \simeq 0.985T_c$ (Fig. 12a); further terms in the expansion did not modify much the value of the first coefficients and does not particularly improve the $\chi^2/\text{d.o.f.}$: the data are well approximated by the hadron resonance gas prediction $\Delta P \propto (\cosh(\mu_B/T) - 1)$ in the broken phase up to $T \simeq 0.985T_c$. This behaviour has been confirmed by an improved analysis in [40], Fig. 13. The analysis of the corrections requires better precision [74].

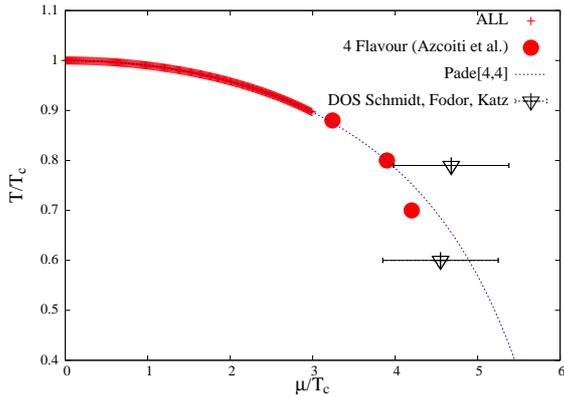


Figure 10: The critical line of four flavor QCD : the red line is the small chemical potential region amenable to a Taylor expansion. Using it as an input to the Pade' analysis, the results have been extended deep into the low temperature phase, in good agreement with other approaches

The grand canonical partition function of the Hadron Resonance Gas model[73, 72] has a simple hyperbolic cosine behaviour. This can be framed in our discussion of the phase diagram in the temperature-imaginary chemical potential plane which suggests to use Fourier analysis in this region, as observables are periodic and continuous there[19]. For observables which are even (O_e) or odd (O_o) under $\mu \rightarrow -\mu$ the analytic continuation to real chemical potential of the Fourier series read $O_e[o](\mu_I, N_I) = \sum_n a_F^{(n)} \cosh[\sinh](nN_I N_c \mu_I)$. In our Fourier analysis of the chiral condensate [19] and

The analytic continuation (Fig. 12b) of any observable O is valid within the analyticity domain, i.e. till $\mu < \mu_c(T)$, where $\mu_c(T)$ has to be measured independently. The value of the analytic continuation of O at μ_c , $O(\mu_c)$, defines its critical value. When O is an order parameter which is zero in the quark gluon plasma phase, the calculation of $O(\mu_c)$ allows the identification of the order of the phase transition: first, when $O(\mu_c) \neq 0$, second, when $O(\mu_c) = 0$ [19, 21].

10. The Hot Phase and the QGP

The behaviour of the number density (Fig. 14, from ref.[83]) approaches the lattice Steffan-Boltzmann prediction,

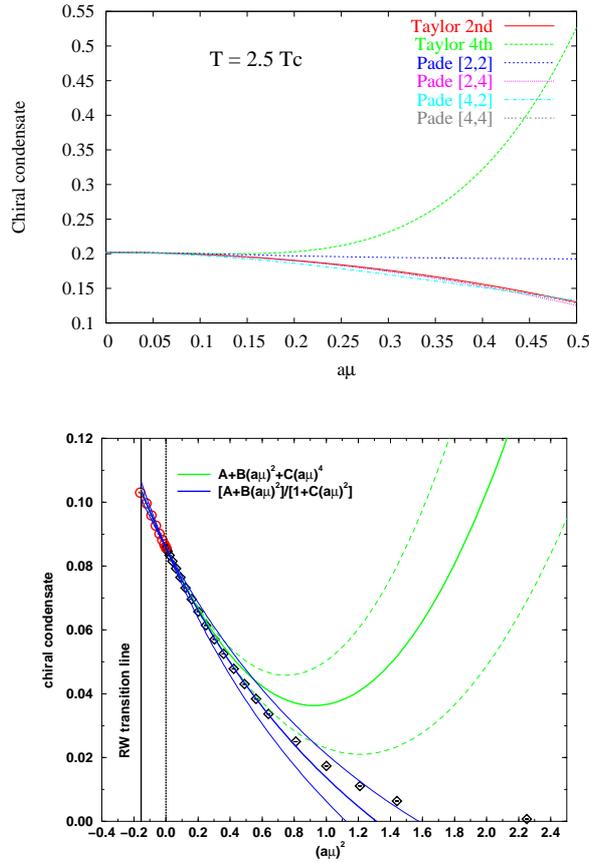


Figure 11: Analytic continuation via Pade' approximants for the chiral condensate: three colour QCD (top diagram) [42] and two colour QCD (bottom diagram) [43]. For real QCD convergence seems to be achieved for Pade'[N,M], $N + M \geq 6$. For two colour QCD Pade'[2,2] suffices.

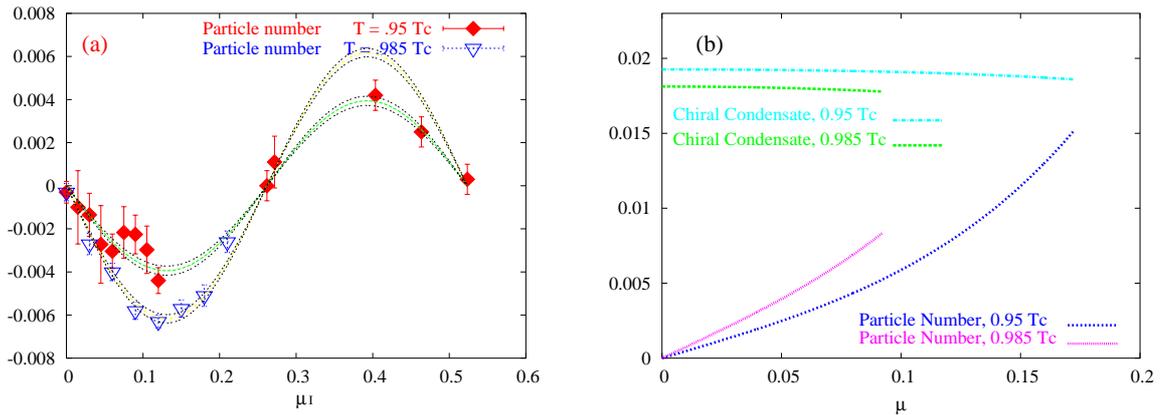


Figure 12: Hadronic Phase: (a) One Fourier coefficient fit to the particle number, showing that the Hadron Resonance Model is adequate to describe this data. (b) Compilation of the results for the chiral condensate and the particle number as a function of real chemical potential: the lines are cut in correspondence with μ_c , showing the first order character of the phase transition (inferred from the chiral condensate) and the critical density [83].

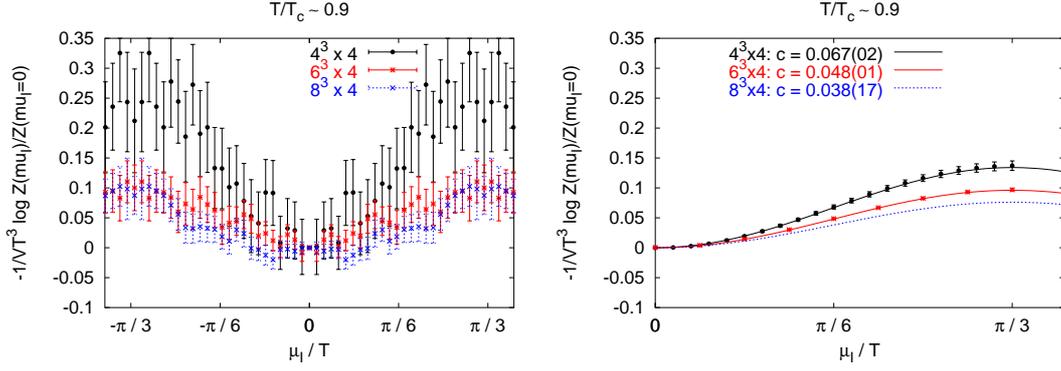


Figure 13: $\frac{\Delta F(T, \mu_I)}{VT^4}$ as a function of $\frac{\mu_I}{T}$ for $\frac{T}{T_c} \sim 0.9$. The results from an improved analysis at imaginary chemical potential confirm that the data are well accounted for by one component Fourier fit [40].

with some residual deviation. The deviation from a free field behaviour can be parametrised as [80, 82]

$$\Delta P(T, \mu) = f(T, \mu) P_{free}^L(T, \mu) \quad (10.1)$$

where $P_{free}^L(T, \mu)$ is the lattice free result for the pressure. For instance, in the discussion of Ref. [82]

$$f(T, \mu) = 2(1 - 2\alpha_s/\pi) \quad (10.2)$$

and the crucial point was that α_s is μ dependent.

We can search for such a non trivial prefactor $f(T, \mu)$ by taking the ratio between the numerical data and the lattice free field result $n_{free}^L(\mu_I)$ at imaginary chemical potential:

$$R(T, \mu_I) = \frac{n(T, \mu_I)}{n_{free}^L(\mu_I)} \quad (10.3)$$

A non-trivial (i.e. not a constant) $R(T, \mu_I)$ would indicate a non-trivial $f(T, \mu)$. In Fig. 14 we plot $R(T, \mu_I)$ versus μ_I/T : the results for $T \geq 1.5T_c$ seem consistent with a free lattice gas, with an fixed effective number of flavors $N^{eff} = 0.92(0.89) \times 4$ for $T = 3.5(1.5)T_c$: in short, the deviation from free field are probably trivial, and certainly μ independent.

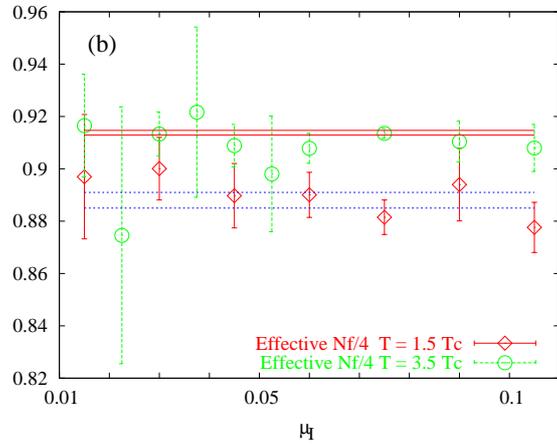


Figure 14: $T \geq 1.5T_c$: Ratio of the lattice results to the lattice free field: the deviation from free field can be simply described by an effective reduced number of flavor, from ref. [83]

11. The non-perturbative Quark Gluon Plasma, and the critical line of QCD in the T, μ_I plane.

When temperature is not much larger than the critical temperature – say, $T_c < T < \simeq 2T_c$ – strong interactions among the constituents give rise to non-perturbative effects: in short, at large T the QGP is a gas of nearly free quarks, which becomes strongly interacting at lower temperatures $T = (1 - 3)T_c$, see e.g. [75, 76] for recent reviews and a complete set of references.

Several proposals have been made to characterise the properties of the system in such non-perturbative phase. For instance the above mentioned strong interactions might be enough to preserve bound states above T_c , while coloured states might appear, deeply affecting the thermodynamics of the system [77]. High temperature expansions are being refined more and more, so to be able to capture the features of dense systems down to T_c (see e.g [84, 85, 86]. Model theories of quasiparticle physics have been considered as well [78].

Here we would like to frame the discussion of the strongly interacting QGP in the context of the critical behaviour at imaginary μ [87]. Consider again the phase diagram in the T, μ^2 plane (Figure 1): we note that the candidate sQGP region right above T_c is limited by a chiral transition at negative μ^2 : it is then all a critical region, whose features might well carry over to real baryochemical potential.

11.1 Evidence for strong interactions in the plasma

In this final subsection I report on our attempt to confront these speculative idea with lattice data [87].

First, consider the data against a simple free field behaviour: as done in Fig. 15, we take the ratio between the numerical results and the free field results $n(\mu_I)/n(\mu_I)_{free}$ (Fig. 16, left) [41, 87]: $n(\mu_I)/n(\mu_I)_{free}$ is far from being constant (as opposed to the findings at higher temperature, Fig. 14), which cannot be accounted for by any simple renormalisation of the degrees of freedom.

In ref. [40], Fig. 15, it was confirmed that the coefficients of a polynomial fit are not in a simple relations with those of a lattice free field.

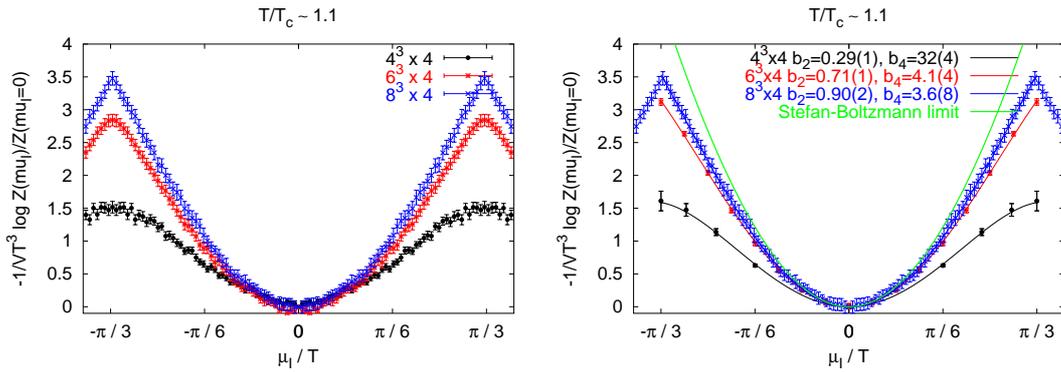


Figure 15: $\frac{\Delta F(T, \mu_I)}{VT^4}$ as a function of $\frac{\mu_I}{T}$ for $\frac{T}{T_c} \sim 1.1$. (left) The histogram method; (right) the reweighting method [35], supplemented by the histogram results for $8^3 \times 4$. A simple modification of the free gas expression describes all the data. As the volume increases, the data come close to the Stefan-Boltzmann limit ($T \rightarrow \infty$) even though $\frac{T}{T_c} \sim 1.1$ only. Figure and caption are from [40]

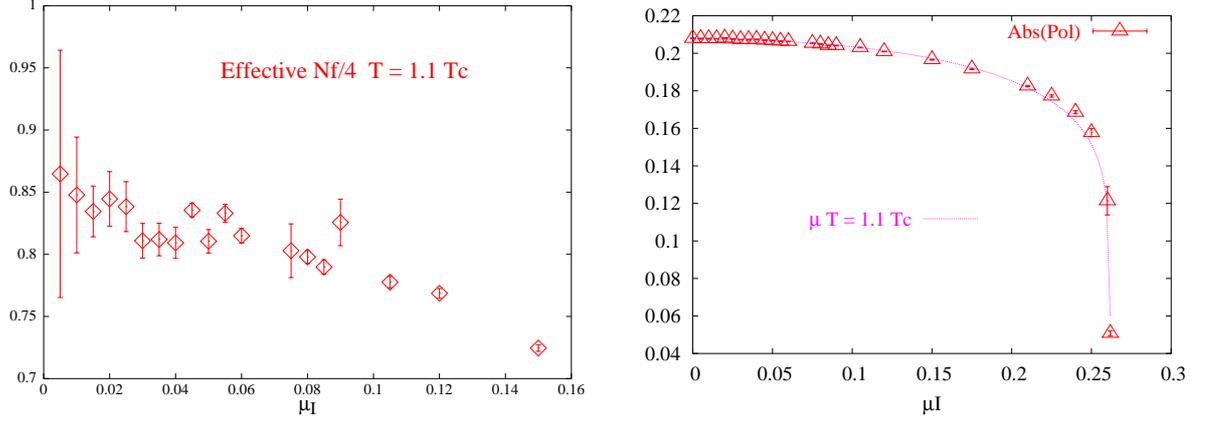


Figure 16: $T = 1.1T_c$ $n(\mu_I)/n(\mu_I)_{free}$ as a function of μ_I , showing a very clear evidence of a deviation from a free field behaviour (left diagram). Average Polyakov loop fitted to the form predicted by a simple critical behaviour $f_c(\mu_I)$ [41, 87]

Next, we checked our data against the form proposed in ref. [79], which was motivated by the hadron resonance gas parametrisation. Going imaginary (and setting $\mu_{isospin} = 0$) the proposal of ref. [79] yields:

$$\frac{P}{T^4} = A_q(T) \cos(\mu/T) + B_{qq}(T) \cos(2\mu/T) + C_{qqq}(3\mu/T) \quad (11.1)$$

giving in turn:

$$n(i\mu, T) = A_q(T) \sin(\mu/T) + 2B_{qq}(T) \sin(2\mu/T) + 3C_{qqq}(3\mu/T) \quad (11.2)$$

We have performed different fits to the above form, none of them really satisfactory. More terms improve of course the fit, but we do not get any close to the nice agreement with HRG observed in the hadronic phase, and in particular it is difficult to capture the behaviour close to the RW singularity.

We propose to use a power law fit derived from a singular behaviour of the free energy:

$$\log Z \propto \frac{1}{(\mu_I^{c^2} - \mu_I^2)^\alpha} \quad (11.3)$$

This in turn gives

$$Pol(\mu_I) \propto (\mu_I^{c^2} - \mu_I^2)^{(\beta)} \quad (11.4)$$

. In Figure 16, right we show the results of the fit, which looks indeed satisfactory [87].

From the results above, we conclude that the data in the candidate region for a strongly coupled QCD are very well accounted for by a conventional critical behaviour: clearly, a free field behaviour would have been incompatible with it. In other words, the nonperturbative features of the plasma are closely related with the occurrence of the critical line at negative μ^2 !

We can then go back and analyse the behaviour of $n(\mu)$: the results for the Polyakov loop validates the simple form for the free energy, which, in turn, gives for $N(\mu_I)$

$$n(\mu_I) \propto \mu_I (\mu_I^{c^2} - \mu_I^2)^{(\alpha-1)} \quad (11.5)$$

and we checked that the behaviour of $n(\mu)$ is reasonably well reproduced by our fit.

In turn, it is now possible to analytically continue from imaginary to real chemical potential, yielding:

$$n(\mu) = K(\mu(\mu_I^2 + \mu^2)^{(\alpha-1)}) \quad (11.6)$$

where $\alpha = 1.2$.

It is then amusing to notice that by using the simple arguments from the theory of critical phenomena we arrive at a modified Stefan-Boltzmann law, which would correspond to $\alpha = 2$. (modulo coefficients).

Obviously, this modified form accounts for a slower increase of the particle density closer to T_c than in the free case, as well as for the behaviour observed in fig.16, left.

From a more mathematical point of view, the proposed parametrisation is a Pade' approximant of order [2,1], as appropriate in the standard application to critical phenomena. The results thus obtained can be analytically continued within the entire analyticity domain.

We underscore that our results do not rule out the HRG-type parametrisation proposed in ref. [79], and, in turn, the occurrence of coloured states: the HRG form might still be valid, once a dependence on the chemical potential of $A(T), B(T), C(T)$ is considered.

On the other hand, if this is the case, the simple interpretation of susceptibility ratios as probes of degrees of freedom has to be revised.

12. Summary

It is possible to study QCD for imaginary values of the baryochemical potential, and infer properties of the physical region. From a technical point of view, these simulations are no more expensive than ordinary QCD, and, in particular, the infinite volume limit is well defined. On the other hand, we should be aware that we are extrapolating results: one has to pay attention to the fact that, by modifying the fitting function (whatever it might be) at imaginary chemical potential by a non-leading term, the difference in physical quantities is still non leading. This can be complicated, and often mathematical arguments need to be supplemented by some physical insight.

We have seen that by use of different techniques and lattice discretization we can gain a reasonable control on the shape of the critical line of two, three, two plus one, and four flavor QCD, as well as on thermodynamics observables in different phases. The results can be pushed beyond the radius of convergence of the Taylor expansion, and the density of states method or canonical approach, even if more preliminary, offer some guidance from the low temperature side.

A cross check with analytic models is particularly simple, as one can analytically continue them from real to imaginary chemical potential. Results obtained in this way include a verification of the validity of the hadron gas model, and the approach to a free gas. In the hot phase close to T_c the critical line at imaginary chemical potential suggests an alternative interpretation of the non-perturbative features of the strongly interactive quark gluon plasma, which is consistent with the numerical results.

References

- [1] J. B. Kogut, H. Matsuoka, M. Stone, H. W. Wyld, S. H. Shenker, J. Shigemitsu and D. K. Sinclair, Nucl. Phys. B **225**, 93 (1983); P. Hasenfratz and F. Karsch, Phys. Lett. **B125**, (1983) 308.
- [2] S. Gottlieb, W. Liu, D. Toussaint, R. L. Renken and R. L. Sugar, Phys. Rev. D **38** (1988) 2888;
- [3] C. Bernard *et al.* [MILC Collaboration], Nucl. Phys. Proc. Suppl. **119** (2003) 523 [arXiv:hep-lat/0209079].
- [4] R. V. Gavai and S. Gupta, Phys. Rev. D **68**, 034506 (2003);
- [5] S. Choe *et al.*, Phys. Rev. D **65** (2002) 054501.
- [6] R. V. Gavai and S. Gupta, Phys. Rev. D **68**, 034506 (2003)
- [7] C. R. Allton *et al.*, Phys. Rev. D **66**, 074507 (2002) [arXiv:hep-lat/0204010].
- [8] C. R. Allton *et al.*, Phys. Rev. D **68**, 014507 (2003).
- [9] Z. Fodor and S. D. Katz, Phys. Lett. B **534**, 87 (2002) [arXiv:hep-lat/0104001].
- [10] Z. Fodor and S. D. Katz, JHEP **0203**, 014 (2002) [arXiv:hep-lat/0106002].
- [11] Z. Fodor, S. D. Katz and K. K. Szabo, Phys. Lett. B **568**, 73 (2003) [arXiv:hep-lat/0208078].
- [12] Z. Fodor and S. D. Katz, JHEP **0404**, 50 (2004) [arXiv:hep-lat/0402006].
- [13] E. Dagotto, A. Moreo, R. L. Sugar and D. Toussaint, Phys. Rev. B **41** (1990) 811.
- [14] A. Hasenfratz and D. Toussaint, Nucl. Phys. B **371** (1992) 539.
- [15] M. G. Alford, A. Kapustin and F. Wilczek, Physical Review **D59** (1999) 054502.
- [16] M.-P. Lombardo, Nuclear Physics B (Proc. Suppl.) **83**, (2000) 375.
- [17] A. Hart, M. Laine and O. Philipsen, Nucl. Phys. **B586**, (2000) 443; Phys. Lett. **B505**, (2001) 141.
- [18] Ph. de Forcrand and O. Philipsen, Nucl. Phys. B **642**, 290 (2002) [arXiv:hep-lat/0205016].
- [19] M. D'Elia and M. P. Lombardo, Phys. Rev. D **67**, 014505 (2003) [arXiv:hep-lat/0209146].
- [20] Ph. de Forcrand and O. Philipsen, Nucl. Phys. B **673**, 170 (2003) [arXiv:hep-lat/0307020].
- [21] M. D'Elia and M. P. Lombardo, Phys. Rev. D **70** (2004) 074509 [arXiv:hep-lat/0406012].
- [22] E. Laermann and O. Philipsen, Ann. Rev. Nucl. Part. Sci. **53**, 163 (2003) [arXiv:hep-ph/0303042].
- [23] S. Muroya, A. Nakamura, C. Nonaka and T. Takaishi, Prog. Theor. Phys. **110**, 615 (2003) [arXiv:hep-lat/0306031].
- [24] M. P. Lombardo, Prog. Theor. Phys. Suppl. **153**, 26 (2004) [arXiv:hep-lat/0401021].
- [25] O. Philipsen, PoS **LAT2005**, 016 (2006) [PoS **JHW2005**, 012 (2006)] [arXiv:hep-lat/0510077].
- [26] C. Schmidt, arXiv:hep-lat/0610116.
- [27] A. Roberge and N. Weiss, Nucl. Phys. B **275**, 734 (1986).
- [28] A. Dumitru, R. D. Pisarski and D. Zschiesche, Phys. Rev. D **72** (2005) 065008 [arXiv:hep-ph/0505256].
- [29] B. Alles and E. M. Moroni, arXiv:hep-lat/0206028.

- [30] J. Ambjorn, K. N. Anagnostopoulos, J. Nishimura and J. J. M. Verbaarschot, *JHEP* **0210**, 062 (2002) [arXiv:hep-lat/0208025].
- [31] K. Splittorff, arXiv:hep-lat/0610072.
- [32] B.P. Dolan, W. Janke, D.A. Johnston and M. Stathakopolous, *J. Phys.* **A34** (2001) 6211 [arXiv:cond-mat/01053317].
- [33] I wish to thank Ph. de Forcrand for pointing out a mistake in a previous version of this plot.
- [34] S.Y. Kim, *Nucl.Phys.* **B637** (2002) 409, [arXiv:cond-mat/0205451].
- [35] S. Kratochvila and P. de Forcrand, *PoS LAT2005* (2006) 167 [arXiv:hep-lat/0509143].
- [36] A. Alexandru, M. Faber, I. Horvath and K. F. Liu, *Nucl. Phys. Proc. Suppl.* **140** (2005) 517 [arXiv:hep-lat/0410002].
- [37] P. de Forcrand and S. Kratochvila, *Nucl. Phys. Proc. Suppl.* **153** (2006) 62
- [38] R. V. Gavai and S. Gupta, *Phys. Rev. D* **71** (2005) 114014 [arXiv:hep-lat/0412035].
- [39] F. Karbstein and M. Thies, arXiv:hep-th/0610243.
- [40] S. Kratochvila and P. de Forcrand, *Phys. Rev. D* **73** (2006) 114512 [arXiv:hep-lat/0602005].
- [41] M. D'Elia, F. Di Renzo and M. P. Lombardo, *AIP Conf. Proc.* **806** (2006) 245 [arXiv:hep-lat/0511029].
- [42] M. P. Lombardo, *PoS LAT2005* (2006) 168 [arXiv:hep-lat/0509181].
- [43] A. Papa, P. Cea, L. Cosmai and M. D'Elia, *PoS LAT2006* (19??)
- [44] M. Golterman, Y. Shamir and B. Svetitsky, *Phys. Rev. D* **74** (2006) 071501 [arXiv:hep-lat/0602026].
- [45] V. Azcoiti, G. Di Carlo, A. Galante and V. Laliena, *Nucl. Phys. B* **723** (2005) 77 [arXiv:hep-lat/0503010].
- [46] H. S. Chen and X. Q. Luo, *Phys. Rev. D* **72** (2005) 034504 [arXiv:hep-lat/0411023].
- [47] L. K. Wu, X. Q. Luo and H. S. Chen, arXiv:hep-lat/0611035.
- [48] R. Casalbuoni, arXiv:hep-ph/0610179.
- [49] S. Hands, *Nucl. Phys. Proc. Suppl.* **106** (2002) 142.
- [50] S. Hands, *Nucl. Phys.* **A 642** (1998) 228.
- [51] A. S. Vshivtsev, B. V. Magnitsky, V. C. Zhukovsky and K. G. Klimenko, *Phys. Part. Nucl.* **29** (1998) 523
- [52] K. G. Klimenko, *Z. Phys.* **C37** (1988)457; S. Hands, A. Kocic and J. B. Kogut, *Nucl. Phys.* **B 390** (1993) 355.
- [53] C. Schmidt, Z. Fodor and S. D. Katz, *PoS LAT2005* (2006) 163 [arXiv:hep-lat/0510087].
- [54] Ph. de Forcrand, S. Kim and T. Takaishi, *Nucl. Phys. Proc. Suppl.* **119**, 541 (2003) [arXiv:hep-lat/0209126].
- [55] M. A. Stephanov, *Phys. Rev. D* **73** (2006) 094508 [arXiv:hep-lat/0603014].
- [56] P. de Forcrand and O. Philipsen, arXiv:hep-lat/0611027.
- [57] M.P. Lombardo, talk at *Extreme QCD 2006*, to appear. I wish to thank Francesco Di Renzo and Elisabetta Pallante for discussions on this point.

- [58] D. K. Sinclair and J. B. Kogut, arXiv:hep-lat/0609041.
- [59] M. I. Stephanov, plenary talk Lattice06, to appear in the Proceedings.
- [60] P. de Forcrand and O. Philipsen, arXiv:hep-lat/0607017.
- [61] J. B. Kogut and D. K. Sinclair, arXiv:hep-lat/0608017.
- [62] A. Mocsy, F. Sannino and K. Tuominen, Phys. Rev. Lett. **92**, 182302 (2004) [arXiv:hep-ph/0308135].
- [63] C. Ratti, S. Roessner, M. A. Thaler and W. Weise, arXiv:hep-ph/0609218.
- [64] B. Alles, M. D'Elia and M. P. Lombardo, Nucl. Phys. B **752** (2006)
- [65] M. D'Elia, S. Conradi and A. D'Alessandro, arXiv:hep-lat/0609057.
- [66] P.H. Damgaard, U.M. Heller, R. Niclausen and B. Svetitsky Nucl. Phys. B **633** (2002) 97; P.M. Damgaard, arXiv:hep-lat/0110192.
- [67] M.A. Halasz, A.D. Jackson, R.E. Shrock, M.A. Stephanov and J.J.M Verbaarschot Phys. Rev. D **58** (1998) 096007.
- [68] N. Bilic and K. Demeterfi, Phys. Lett. B **212** (1988) 83.
- [69] V. Azcoiti, G. Di Carlo, A. Galante and V. Laliena, Phys. Lett. B **563** (2003) 117 [arXiv:hep-lat/0305005].
- [70] P. Giudice and A. Papa, Phys. Rev. D **69**, 094509 (2004) [arXiv:hep-lat/0401024].
- [71] C. R. Allton *et al.*, Phys. Rev. D **71** (2005) 054508 [arXiv:hep-lat/0501030].
- [72] D. Toublan and J. B. Kogut, Phys. Lett. B **605** (2005) 129 [arXiv:hep-ph/0409310].
- [73] F. Karsch, K. Redlich and A. Tawfik, Phys. Lett. B **571**, 67 (2003). [arXiv:hep-ph/0306208].
- [74] I wish to thank Francesco Becattini for discussions on this point.
- [75] E. V. Shuryak, arXiv:hep-ph/0608177.
- [76] J. P. Blaizot, arXiv:nucl-th/0611104.
- [77] E. V. Shuryak and I. Zahed, Phys. Rev. D **70** (2004) 054507 [arXiv:hep-ph/0403127].
- [78] M. Bluhm and B. Kampfer, arXiv:hep-ph/0611083.
- [79] S. Ejiri, F. Karsch and K. Redlich, Phys. Lett. B **633** (2006) 275 [arXiv:hep-ph/0509051].
- [80] K. K. Szabo and A. I. Toth, JHEP **0306**, 008 (2003) [arXiv:hep-ph/0302255].
- [81] F. Csikor, G. I. Egri, Z. Fodor, S. D. Katz, K. K. Szabo and A. I. Toth, arXiv:hep-lat/0401022.
- [82] J. Letessier and J. Rafelski, Phys. Rev. C **67**, 031902 (2003). [arXiv:hep-ph/0301099].
- [83] M. D'Elia and M. P. Lombardo, arXiv:hep-lat/0409010, SEWM2004, Helsinki.
- [84] A. Vuorinen, Phys. Rev. D **68**, 054017 (2003) [arXiv:hep-ph/0305183].
- [85] A. Ipp, A. Rebhan and A. Vuorinen, Phys. Rev. D **69**, 077901 (2004) [arXiv:hep-ph/0311200].
- [86] A. Ipp, K. Kajantie, A. Rebhan and A. Vuorinen, Phys. Rev. D **74** (2006) 045016 [arXiv:hep-ph/0604060].
- [87] M. D'Elia, F. Di Renzo and M.P. Lombardo, to appear.