PROCEEDINGS OF SCIENCE

Quantum Symmetries of $sl(2)$ and $sl(3)$ graphs

R. Coquereaux[*†‡]

G. Schieber[§¶]

After summarizing the description, in terms of action of categories, of the algebraic structures underlying $sl(2)$ and $sl(3)$ boundary conformal field theories on a torus, in particular the associated quantum groupoid together with its fusion algebra and algebra of quantum symmetries, we provide tables of dimensions describing the semisimple and co-semisimple blocks of the weak bialgebras, tables of quantum dimensions and tables describing induction - restriction. For reasons of size, the $sl(3)$ tables of induction are only given for theories with self-fusion (existence of a monoidal structure).

Fifth International Conference on Mathematical Methods in Physics — IC2006
April 24-28 2006
Centro Brasilerio de Pesquisas Fisicas, Rio de Janeiro, Brazil

[*]Speaker.

[†] Email: Robert.Coquereaux@cpt.univ-mrs.fr

[‡]*Centre de Physique Théorique, Luminy, Marseille.* UMR 6207, du CNRS et des Universités Aix-Marseille I, II, et du Sud Toulon-Var.

[§] Email: schieber@cbpf.br

[¶]*CBPF – Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro.*

1. Introduction

Our main and original purpose was to gather tables giving characteristic numbers for $sl(3)$ boundary conformal field theories on a torus: dimensions of the blocks of the associated weak bialgebras, global quantum dimensions ("quantum mass"), induction tables, etc. a material that is hitherto scattered in a number of publications or unavailable. Many formulae given here are new. The corresponding data for $sl(2)$ does not use much space, and we could easily summarize it, but the situation is different with $sl(3)$: to keep the size of this paper below the prescribed limits, we had sometimes to restrict ourselves to the case of Di Francesco - Zuber graphs with self-fusion (other cases will be described in [17]) and give only partial results for induction tables. Because we also needed a short introductory section discussing the underlying algebraic structures and giving our notations, we decided to describe boundary conformal field theories on a torus in terms of module categories (action of a monoidal category on a category) mostly extracting the relevant material from [36], while adding few things like the construction of weak bialgebras in terms of Hom spaces, or the description of the bimodule structure $\mathscr{A} \times \mathscr{O} \times \mathscr{A} \mapsto \mathscr{O}$, where $\mathscr{A}$ is the fusion algebra and $\mathscr{O}$ is the Ocneanu algebra of quantum symmetries. Independently of the interest of the tables of results, we hope that our presentation will provide a bridge between several mathematical or physical communities interested in those topics.

2. The stage

In this paper $\mathscr{A}_k$ is the fusion category of the affine algebra $\widehat{sl}(2)$, or $\widehat{sl}(3)$, at level k, or equivalently, the category of irreducible representations with non-zero q-dimension for the quantum groups $sl(2)$ or $sl(3)$ at roots of unity (set $q = exp(i\pi/\kappa)$, with $\kappa = k+2$ for $sl(2)$ and $\kappa = k+3$ for $sl(3)$. This category is additive (existence of $\oplus$), monoidal (existence of $\otimes : \mathscr{A}_k \times \mathscr{A}_k \mapsto \mathscr{A}_k$, with associativity constraints, unit object, etc.), tensorial ($\otimes$ is a bifunctor), complex-linear, rigid (existence of duals), finite (finitely many irreducible objects), and semisimple, with irreducible unit object. It is also modular (braided, balanced, with invertible S-matrix) and ribbon (or tortile). We refer to the literature [28], [26], [1] for a detailed description of these structures. The Grothendieck ring of this monoidal category comes with a special basis (corresponding to simple objects), it is usually called the fusion ring, or the Verlinde algebra. The corresponding structure constants, encoded by the so - called fusion matrices $(N_n)_q^p$, are therefore non - negative integers: NIM-reps in CFT terminology. The rigidity property of the category implies that $(N_{\overline{n}})_{pq} = (N_n)_{qp}$, where $\overline{n}$ refers to the dual object i.e., , in our case, to the conjugate representation, so that the fusion ring is automatically a based $\mathbb{Z}_+$ ring in the sense of [36] (maybe it would be better to call it “rigid”). In the case of $sl(2)$, this is a ring with one generator (corresponding to the fundamental representation), and fusion matrices are symmetric, because $\overline{n} = n$. In the case of $sl(3)$, it has two generators (corresponding to the two fundamental representations) that are conjugate to one another. Multiplication by a generator (choose one of the possible two in the $sl(3)$ case) is encoded by a particular fusion matrix; it is a finite size matrix of dimension $r \times r$, with $r = k+1$ for $sl(2)$ and $r = k(k+1)/2$ for $sl(3)$. Since its elements are non negative integers, it can be interpreted as the adjacency matrix of a graph, which is the Cayley graph of multiplication by this generator, that we call the McKay graph of the category. Edges are non oriented in the case of $sl(2)$ (rather, they carry both orientations) and are oriented in the case of $sl(3)$. Irreducible representations are denoted by λ_p, with $p \geq 0$, in the first case, and λ_{pq}, with $p,q \geq 0$ in the next. Notice the shift of indices: the unit $\mathbb{1}$ of the category corresponds to λ_0 or to λ_{00}. One should certainly keep in mind the distinction between the monoidal category, with its objects and morphisms, and its Grothendieck ring, but they will often be denoted by the same symbol. Actually, the Mckay graph itself is also denoted $\mathscr{A}_k$. In the $sl(2)$ case, it can be identified with the Coxeter - Dynkin diagram A_r, with $r = k+1$. In both $sl(2)$ and $sl(3)$ cases, it is a truncated Weyl chamber at level k (a Weyl alcove). It is often useful to think of $\mathscr{A}_k$ as a category of representations of a would-be quantum object, that can be also denoted by the same symbol, although this can be quite misleading.

The next ingredient is a category $\mathscr{E}$, not necessarily monoidal, but we suppose that it is additive, semisimple and indecomposable, on which the previous one $\mathscr{A}_k$ (which is monoidal) acts. Action of a monoidal category on a category has been described, under the name "module categories" by [19], and, in our context, by [36]. Using a slightly shorter description, we may say that we have such an action when we are given a (monoidal) functor from $\mathscr{A}_k$ to the (monoidal) category of endofunctors of $\mathscr{E}$. The reader can think of this situation as being an analogue of the action of a group on a given space. Actually, it may be sometimes interesting to think that $\mathscr{E}$ can be acted upon in more than one way, so that we can think of the action of $\mathscr{A}_k$ as a particular "enrichment" of $\mathscr{E}$. The word "module" being used in so many different ways, we prefer to say that we have an action, or that $\mathscr{E}$ is an actegory (another nice substantive coined by R. Street), and we shall freely use both terminologies. Irreducible objects of $\mathscr{E}$ are boundary conditions for the corresponding Conformal Field Theory specified by $\mathscr{A}_k$. It is useful to assume, from now on, that the category $\mathscr{E}$ is indecomposable (it is not equivalent to the direct sum of two non trivial categories with $\mathscr{A}_k$ action). Since $\mathscr{E}$ is additive, we have a Grothendieck group, also denoted by the same symbol. Because of the existence of an action, this (abelian) group has to be a module over the Grothendieck ring of $\mathscr{A}_k$, and it is automatically a $\mathbb{Z}_+$ module: the structure constants of the module, usually called annulus coefficients in string theory articles [20], and described by (annular) matrices $F_n = (F_n)_{ab}$, are non negative integers. Let us consider the class of a particular simple object of $\mathscr{A}_k$, namely the generator $n = 1$, for $sl(2)$ or $n = (1,0)$ for $sl(3)$ (one of the two conjugated fundamental irreps). We interpret F_1 (or $F_{(1,0)}$) as the adjacency matrix of a graph, called the McKay graph of the category $\mathscr{E}$. The rigidity property of $\mathscr{A}_k$ implies that the module $\mathscr{E}$ is rigid (or based [36]). In other words: $(F_{\overline{n}})_{ab} = (F_n)_{ba}$. In the case of $sl(2)$, $\mathbb{Z}_+$ modules for fusion rings at level k have been classified by [16] and [18]; McKay graphs are all the Coxeter - Dynkin diagram, plus some diagrams with loops (tadpoles), and $F_1 = 2\,\text{Cartan matrix} - \mathbb{1}$. The rigidity condition in the case of $sl(2)$ implies that the matrix F_1 is symmetric; this condition implies that non simply laced diagrams B_r, C_r, F_4 and G_2 should be rejected (this only means that they do not fit in the presented framework: see also [24] Êand references therein): we are left with the ADE diagrams and the tadpoles. A detailed analysis of the situation ([34], [36]) shows that the tadpole graphs do not give rise to any category endowed with an action of the monoidal categories of type $sl(2)$. As already mentioned, the category $\mathscr{E}$ is not required to be monoidal, but there are cases where it is, so that it has a tensor product, compatible with the $\mathscr{A}_k$ action. In another terminology, one says that the corresponding graphs have self - fusion (this is also related to the concept of flatness), or that they define "quantum subgroups" of $sl(2)$, whereas the others are only "quantum modules". Like in the classical situation, we have a restriction functor $\mathscr{A}_k \mapsto \mathscr{E}$ and an induction functor $\mathscr{E} \mapsto \mathscr{A}_k$. The cases where $\mathscr{E}$ is monoidal correspond to the graphs A_r (with $k=r-1$), D_{even}, with $k=0\,\mathrm{mod}4$, E_6, with $k=10$, and E_8, with $k=28$. This was already known, at the level of rings, in [37], [16] and was proved, at the categorical level, by [27]. The cases D_{odd}, with level $k=2\,\mathrm{mod}\ 4$, and E_7, at level 16, are non monoidal actegories. $\mathscr{A}_k$ is always modular, but the corresponding actegories are not, even when they happen to be monoidal; however, they always contain a subcategory which is modular (we shall come back to it). At the level of graphs, the D diagrams (even of odd) are $\mathbb{Z}_2$ orbifolds of the A diagrams at the same level. In the case of $sl(3)$, the classification of $\mathbb{Z}_+$ modules over the corresponding fusion rings at level k is not tractable (or not useful), however there is another route stemming from the classification of $sl(3)$ modular invariants [21]. The graphs encoding all $sl(3)$ module categories are called the di Francesco - Zuber diagrams [16]. Existence of the corresponding categories was shown by A. Ocneanu [33], actually one of the candidates had to be discarded, very much like the tadpole graphs of $sl(2)$. Several $sl(3)$ actegories have monoidal structure (graphs with self - fusion), namely: $\mathscr{A}_k$ itself, the $\mathscr{D}_k$, whose McKay diagrams are $\mathbb{Z}_3$ orbifolds of those of $\mathscr{A}_k$, when k is divisible by 3, and three exceptional cases called $\mathscr{E}_5$, $\mathscr{E}_9$ and $\mathscr{E}_{21}$, at levels 5, 9 and 21. The other actegories (not monoidal) are: the series $\mathscr{A}_k^*$, for which the number of simple objects is equal to the number of self dual simple objects in $\mathscr{A}_k$, the $\mathscr{D}_k$ series, when $k=1$ or 2 mod 3, the series $\mathscr{D}_k^*$, for all k, and several modules of exceptionals called $\mathscr{E}_5^*$, $\mathscr{E}_9^*$, $\mathscr{D}_9^t$ (a generalization of E_7) and $\mathscr{D}_9{}^{t*}$. Some of the graphs of that system have double lines, like $\mathscr{E}_9$, so that it is not appropriate to say that Di Francesco - Zuber diagrams are the "simply laced" diagrams of type $sl(3)$: better to call them "higher ADE". In all cases however, with self-fusion or not, the rigidity property implied by $\mathscr{A}_k$ holds (the condition $(F_{\overline{n}})_{ab}=(F_n)_{ba}$ does not forbid double lines). Taking quotients of the above diagrams by discrete groups gives higher analogues of the non *ADE* Dynkin diagrams which define modules over the Grothendieck ring of $\mathscr{A}_k$, but the rigidity condition is not satisfied and the corresponding category should not exist (see however the previous footnote). Classification of $sl(4)$ module categories is also claimed to be completed [33].

Let us pause to develop a tentative pedagogical analogy that should make the next result look natural. Consider a finite group, a subgroup, and the corresponding homogenous space (the space of right cosets, for example). The group can be fibered as a principal bundle over the coset space, the structure group being the chosen subgroup. This subgroup has representations, in particular irreducible ones. For any such, let us say a, one can build an associated vector bundle, and consider the space Γ_a of its sections. It carries a representation of the big group, although not irreducible (theory of induced representations). For the particular choice of the trivial representation of the small group, call it 0, the space of sections Γ_0 is an algebra, namely the space of functions $\mathscr{F}$ on the coset. Moreover every space of sections, say Γ_a, is a module over the algebra $\mathscr{F}$. In our case we have a non commutative geometry which is still, in a sense, finite, but the situation is similar. Simple objects, labelled by a, of the module category $\mathscr{E}$ can be thought as points of a graph, as irreducible representations of a would-be quantum subgroup of $su(2)$ or $su(3)$ at some root of unity, or as spaces of sections Γ_a above a quantum space determined by the pair $(\mathscr{A}_k,\mathscr{E})$, i.e., as modules over some particular algebra $\mathscr{F}$ which is an algebra in a monoidal category ($\mathscr{A}_k$ in our case), and right modules over $\mathscr{F}$ form an additive category $Mod_{\mathscr{A}_k}(\mathscr{F})$. Reciprocally, we have the following theorem proved in [36] under actually weaker assumptions than those listed previously for the action of ($\mathscr{A}_k$ on $\mathscr{E}$: there exists a semisimple indecomposable algebra $\mathscr{F}$, belonging to the set of objects of $\mathscr{A}_k$ such that the module categories $\mathscr{E}$ and $Mod_{\mathscr{A}_k}(\mathscr{F})$ are equivalent. It is shown in [27] that $\mathscr{E}$ is monoidal (self-fusion) if and only if $\mathscr{F}$ is commutative. In the next section we shall describe $\mathscr{F}$ as an object in $\mathscr{A}$ for all $sl(2)$ and $sl(3)$ actegories, in particular for those that are monoidal, but we shall not use the above characterization. Its simple summands play the role of quantum Klein invariants. The same algebra $\mathscr{F}$, called Frobenius algebra, plays a prominent role in the approach of [20]. Notice that module categories associated with *ADE* Dynkin diagrams for $sl(2)$, or with Di Francesco - Zuber diagrams for $sl(3)$ are never modular (unless $\mathscr{E} = \mathscr{A}$ itself) but they contained a subcategory J, also denoted $Mod^0_{\mathscr{A}_k}(\mathscr{F})$ which is modular. When $\mathscr{E}$ is monoidal, the subring of its Grothendieck ring associated with this modular subcategory is called the modular subring (or subalgebra) and is also denoted J.

The third and final needed ingredient is the centralizer category of $\mathscr{E}$ with respect to the action of $\mathscr{A}_k$. It is sometimes called the "dual category" (not a very good name) and is defined as the category of module functors from $\mathscr{E}$ to itself: these endofunctors should be functors F "commuting" with the action of $\mathscr{A}_k$, i.e., such that $F(\lambda_n \otimes \lambda_a)$ is isomorphic with $\lambda_n \otimes F(\lambda_a)$, for $\lambda_n \in Ob(\mathscr{A}_k)$ and $\lambda_a \in Ob(\mathscr{E})$, via a family of morphisms c_{λ_n,λ_m} obeying triangular and pentagonal constraints. We simply call $\mathscr{O} = Fun_{\mathscr{A}_k}(\mathscr{E},\mathscr{E})$ this centralizer category[1], but one should remember that its definition involves both $\mathscr{A}_k$ and $\mathscr{E}$. Because of the previous compatibility property, if $\mathscr{E}$ is a left actegory over $\mathscr{A}_k$, it is probably better to consider it as a right actegory over $\mathscr{O}$ (actually over its opposite, since we are permuting the two factors). The category $\mathscr{O}$ is additive, semi-simple in our case, and monoidal (use composition of functors as tensor product). $\mathscr{E}$ is therefore both a module category over $\mathscr{A}_k$ and over $\mathscr{O}$. The Grothendieck group of $\mathscr{E}$ is therefore not only a $\mathbb{Z}_+$ module over the fusion ring, but also a $\mathbb{Z}_+$ module over the Grothendieck ring of $\mathscr{O}$, called the Ocneanu ring (or algebra) of quantum symmetries and denoted by the same symbol. Structure constants of the ring of quantum symmetries are encoded by matrices O_x, called "matrices of quantum symmetries"; structure constants of the module, with respect to the action of quantum symmetries, are encoded by the so called "dual annular matrices" S_x. The next problem is to find a way to describe explicitly this centralizer category. The solution lies in the construction of a finite dimensional weak bialgebra $\mathscr{B}$, which is going to be such that the monoidal category $\mathscr{A}_k$ can be realized as $Rep(\mathscr{B})$, and also such that the monoidal category $\mathscr{O}$ can be realized as $Rep(\widehat{\mathscr{B}})$ where $\widehat{\mathscr{B}}$ is the dual of $\mathscr{B}$. These two algebras are finite dimensional (actually semisimple in our case) and one algebra structure (say $\widehat{\mathscr{B}}$) can be traded against a coalgebra structure on its dual. $\mathscr{B}$ is a weak bialgebra, not a bialgebra, because $\Delta \mathbb{1} \neq \mathbb{1} \otimes \mathbb{1}$, where Δ is the coproduct in $\mathscr{B}$, and $\mathbb{1}$ is its unit. Actually, in our cases, it is not only a weak bialgebra but a weak Hopf algebra (we can define an antipode, with the expected properties [2], [31], [32]). One categorical construction of $\mathscr{B}$ is given in [36]. We propose another one that should lead to the same bialgebra, and may be simpler. Label irreducible objects λ_{-} of categories $\mathscr{A}_k$ by $m,n,\ldots$, of $\mathscr{E}$ by $a,b,\ldots$, and of $\mathscr{O}$ by $x,y,\ldots$. Call $H^m_{ab} = Hom(\lambda_n \otimes \lambda_a, \lambda_b)$, the "horizontal space of type n from a to b" (also called space of essential paths of type n from a to b, space of admissible triples, or triangles...) Call $V^x_{ab} = Hom(\lambda_a \otimes \lambda_x, \lambda_b)$ the "vertical space of type x from a to b". We just take these horizontal and vertical spaces as vector spaces and consider the graded sums $H^m = \sum_{ab} H^m_{ab}$ and $V^x = \sum_{ab} V^x_{ab}$. To construct the weak bialgebra, we take the (graded) endomorphism algebras $\mathscr{B} = \sum_m End(H^m)$ and $\widehat{\mathscr{B}} = \sum_x End(V^x)$. For obvious reasons, $\mathscr{B}$ and $\widehat{\mathscr{B}}$ are sometimes called "algebra of double triangles". Existence of the bialgebra struc-

[1] For $sl(2)$, the structure of $Fun_{\mathscr{A}_k}(\mathscr{E}_1,\mathscr{E}_2)$, where $\mathscr{E}_{1,2}$ can be distinct module categories was obtained by [34].

ture (compatibility) rests on the properties of the pairing, or, equivalently, on the properties of the coefficients[2] (Ocneanu cells) obtained by pairing two bases of matrix units[3] for the two products. Being obtained by pairing double triangles, Ocneanu cells (generalized $6J$ symbols) are naturally associated with tetrahedra with two types (black "b", or white "w") of vertices, so that edges *bb*, *bw* or *ww* refer to labels n, a, x of $\mathscr{A}$, $\mathscr{E}$ and $\mathscr{O}$. Using the two product and coproduct structures[4] one can obtain representation categories $Rep(\mathscr{B})$ and $Rep(\widehat{\mathscr{B}})$ respectively equivalent to $\mathscr{A}_k$ and $\mathscr{O}$. As already discussed, the character ring of the first (the fusion algebra) is always described by a (generalized) Dynkin diagram of type $\mathscr{A}$. The character ring of the next (the algebra of quantum symmetries) is not always commutative, it has two generators, called chiral, in the case of $sl(2)$, together with their complex conjugated in the case of $sl(3)$. The Cayley graph of multiplication by the two chiral generators (two types of lines), called the Ocneanu graph of $\mathscr{E}$, encodes the structure. As already mentioned, $\mathscr{B}$ is weak. This should not be a surprise since not any monoidal category arises as representation category of a bialgebra, however (at least when the category has finitely many simple objects, and this is our case), it can be realized as the category of representations of a weak bialgebra. In the next paragraph, we shall see that the knowledge of a modular invariant is sufficient to reconstruct the character rings of $\mathscr{A}_k$ (that we already know), of $\mathscr{O}$, and the semisimple and co semisimple structure of $\mathscr{B}$. The $\mathscr{A}_k \times \mathscr{O}$ module corresponding to $\mathscr{E}$ itself can be recovered from the study of the source and target subalgebras of $\mathscr{B}$. Results obtained in operator algebra by [34] and [4, 5, 6] have been translated to a categorical language by [36]: choice of a braiding or of the opposite braiding in the category $\mathscr{A}_k$ can be used to construct two tensor functors from $(\mathscr{A}_k)$ to $\mathscr{O}$, called $\alpha^{L,R}$ ("alpha induction" in the language of [5]). Here our presentation differs from [36], because we find easier to think that there exists a functor $\mathscr{A}_k \times \mathscr{O} \times \mathscr{A}_k \mapsto \mathscr{O}$, so that the previous $\alpha^{L,R}$ are obtained as particular cases ($\mathbb{1}_A \times \mathbb{1}_O \times \mathscr{A}_k$ or to $\mathscr{A}_k \times \mathbb{1}_O \times \mathbb{1}_A$). At the level of Grothendieck rings, we have a bimodule property, that reads (we only use labels to denote the corresponding irreducible objects): $m\,x\,n = \sum_y (W_{xy})_{mn}\, y$, where m,n refer to irreducible objects of $\mathscr{A}_k$, x,y to irreducible objects of $\mathscr{O}$, and where W_{xy} constitute a family of so - called toric matrices, with matrix elements $(W_{xy})_{mn}$, again non negative integers. When both x and y refer to the unit object (that we label 0), one recovers the modular invariant partition function $Z = W_{00}$ of conformal field theory. As explained in [38], when one or two indices x and y are non trivial, toric matrices are interpreted as partition functions on a torus, in a conformal theory of type $\mathscr{A}_k$, with boundary conditions specified by $\mathscr{E}$, but with defects (one or two) specified by x and y. Only Z is modular invariant (it commutes with the generator S and T of $SL(2,\mathbb{Z})$ in the Hurwitz - Verlinde representation). Toric matrices were first introduced and calculated by Ocneanu (unpublished). Various methods to compute or define them can be found in [8], [38]. Ref. [11] gives explicit expressions for all W_{x0}, for all members of the $sl(2)$ family. The modular invariant partition functions themselves have been known for many years: we have the *ADE* classification of [7] for $sl(2)$, and the classification [21] for $sl(3)$, encoded by Di Francesco - Zuber diagrams. Left and right associativity constraints $(m(nxp)q) = (mn)x(pq)$ for the $\mathscr{A} \times \mathscr{A}$ bimodule structure of $\mathscr{O}$ can be written in terms of fu-

[2]Constructions of $\mathscr{B}$, inspired from [35], and using these properties, were given in [38] and [13].

[3]Definition of cells involve normalization choices: the spaces H_{ab}^{m} are not always one-dimensional, moreover one may decide to use bases made of vector proportional to matrix units rather than matrix units themselves.

[4]In the operator algebra community, one would usually define a star operation and a scalar product on $\mathscr{B}$, so that both products could be defined on the same underlying vector space[35].

PoS(IC2006)001

sion and toric matrices; a particular case of this equation reads $\sum_x (W_{0x})_{\lambda\mu} W_{x0} = N_\lambda Z N_\mu^{tr}$, called "equation of modular splitting", was presented by A.Ocneanu in Bariloche (2000). Given fusion matrices N_p (known in general) and a modular invariant matrix $Z = W_{00}$, solving this equation, i.e., finding the W_{x0}, allows one to reconstruct the character ring of $\mathscr{O}$. A practical method to solve this equation is given in [24], with several $sl(2)$ and $sl(3)$ examples. Left and right chiral categories $\mathscr{O}_{L,R}$ are defined, using alpha-induction functors, as additive and monoidal subcategories of $\mathscr{O}$ whose objects are direct summands of $\alpha_{LR}(\lambda)$, for all λ in $\mathscr{A}_k$. They are not braided but their intersection, the ambichiral subcategory $\mathscr{J}$ is. When $\mathscr{E}$ is monoidal, the Grothendieck ring of $\mathscr{J}$, called ambichiral, is isomorphic with the modular subalgebra J already defined.

3. Notations and miscellaneous results

3.1 Notations (summary)

From now on we shall work at the level of Grothendieck groups, or rings, but use for them the same notation as for the categories themselves. So, we have a commutative and associative algebra $\mathscr{A}$ with[5] a base λ_n, structure constants $(N_n)_{pq}$, an associative algebra $\mathscr{O}$, with a base o_x, structure constants $(O_x)_{yz}$, a vector space $\mathscr{E}$ with a base σ_a which is a module over $\mathscr{A}$ and $\mathscr{O}$, with structure constants $(F_n)_{ab}$ and $(S_x)_{ab}$. When $\mathscr{E}$ has self fusion, its structure constants are $(E_a)_{bc}$. The ring $\mathscr{O}$ is a $\mathscr{A}$ bimodule with structure coefficients $(W_{xy})_{m,n}$. The modular invariant partition function is $Z = W_{00}$. Like before, the notation $\mathscr{E}$ refers to a generic example, unless it denotes an exceptional case (the context should be clear). $\mathscr{E}$ being chosen, the numbers of irreducible objects in categories $\mathscr{A}$, $\mathscr{E}$ and $\mathscr{O}$ are respectively denoted r_A, $r = r_E$ and r_O. They are the number of vertices of the associated graphs. In the case of $\widehat{sl}(2)_k$, $r_A = k+1$. In the case of $\widehat{sl}(3)_k$, $r_A = (k+1)(k+2)/2$. The script notation using the level k as an index can be generalized to all $sl(N)$ theories but it is incompatible with the traditional notation for $sl(2)$ (the Dynkin diagrams), where the index refers to the rank. We have the identifications: $E_6 = \mathscr{E}_{10}, E_7 = \mathscr{E}_{16}, E_8 = \mathscr{E}_{28}, A_r = \mathscr{A}_{k=r-1}, D_{s+2} = \mathscr{D}_{k=2s}$. There are no $\mathscr{D}$ cases with odd level in the $sl(2)$ family. Notations for higher ADE diagrams of type $sl(3)$ were given in the previous section. Generalized Coxeter numbers are $\kappa = k+N$ for members of $sl(N)$ families.

3.2 Modular blocks

Call $Z = W_{00}$ the modular invariant. It is a matrix Z_{mn} indexed by (classes of) irreducible objects $m,n \in Irr(\mathscr{A})$. It can also be written as a sesquilinear quadratic form $\sum_{mn} \chi_{\overline{m}} Z_{\overline{m}n} \chi_n$ (the partition function). The following results are attributed to [34] and [4, 5, 6]: $r_E = Tr(Z)$, and $r_O = Tr(ZZ^t)$; the Grothendieck ring of $\mathscr{A}$ is commutative but the one of $\mathscr{O}$, which is not necessarily commutative, is isomorphic to the direct sum of matrix algebras of sizes Z_{mn}. For example, take $sl(2)$ at level 8; if we are given $Z = |\chi_0+\chi_8|^2 + |\chi_2+\chi_6|^2 + 2\,|\chi_4|^2$, we know a priori that $r_A = 9$, $r_E = 6$, $r_O = 12$, and that the algebra of quantum symmetries is isomorphic with $\bigoplus_{x=1}^{x=8} \mathbb{C}_x \oplus M(2,\mathbb{C})$. Actually this is the D_6 module of A_9. The collection K of those irreducible objects λ_n of $\mathscr{A}$ that appear on the diagonal of Z, with multiplicity Z_{nn}, is called the multiset[6] of exponents

[5] In the previous section, we used the notation $\lambda_n, \lambda_a, \lambda_x$ for λ_n, σ_a, o_x.

[6] Exponents are conventionally defined as indices n shifted by $+1$ for $sl(2)$, or by $+(1,1)$ for $sl(3)$.

of Z (or exponents of $\mathscr{E}$). Forgetting multiplicities, the matrix Z defines a partition on this set: two exponents m,n are in the same modular block iff $Z_{mn} \neq 0$. For instance, at level 10, in the case of E_6, we have $Z = |\chi_0+\chi_6|^2+|\chi_3+\chi_7|^2+|\chi_4+\chi_{10}|^2$ and therefore a partition of $K = \{0,6,3,7,4,10\}$ into three subsets $\{0,6\}$, $\{3,7\}$, $\{4,10\}$. At level 17, in the case of E_7, we have $Z = |\chi_0+\chi_{16}|^2+|\chi_4+\chi_{12}|^2+|\chi_6+\chi_{10}|^2+|\chi_8|^2+(\overline{\chi}_8)(\chi_2+\chi_{14})+(\overline{\chi}_2+\overline{\chi}_{14})\chi_8$, so we have a partition of $K = \{0,16,4,12,6,10,8\}$ into four subsets $\{0,16\}$, $\{4,12\}$, $\{6,10\}$, $\{8\}$. The modular block of the origin (containing 0) is denoted K_0. We observe that when $\mathscr{E}$ is monoidal (self-fusion), Z is a sum of squares and there is a one to one correspondence between modular blocks K_a and the modules Γ_a sitting above the irreducible objects a belonging to the modular subalgebra J of the Grothendieck ring of $\mathscr{E}$. For instance the first modular block $K_0 = \{0,6\}$ of E_6 corresponds to the algebra $\mathscr{F} = \Gamma_0 = \lambda_0 \oplus \lambda_6$. This is not so when there is no self-fusion: the first modular block K_0 of E_7 is $\{0,16\}$ although $\mathscr{F} = \Gamma_0 = \lambda_0 \oplus \lambda_8 \oplus \lambda_{16}$.

3.3 Dimensions of horizontal and vertical spaces H and V, dimension of the weak Hopf algebra $\mathscr{B}$

The horizontal space $H = \bigoplus_n H_n$, coming in the construction of the first algebra structure on the weak Hopf algebra $\mathscr{B}$ was defined before, in terms of categorical data. In the $sl(2)$ case H can be realized as the (Ocneanu) vector space of essential paths on ADE graphs but also as the vector space underlying the Gelfand-Ponomarev preprojective algebra associated with the corresponding unoriented quiver. In the first realization H_n is defined as a particular subspace of the space *Paths* of all paths on the graph $\mathscr{E}$, whereas in the second construction it is defined as a quotient. Identification stems, for instance, from the fact that dimensions of these finite dimensional vector spaces, calculated according to the two definitions, are equal. In the case of $sl(3)$, the grading label of the horizontal space H_n refers to a pair of integers (n_1,n_2), specifying an irreducible representation (it can also be seen as a Young tableau), this suggests a generalization of the notion of preprojective algebras associated with quivers.

In the case of $sl(2)$, dimensions $d_n = dim H_n$, $d_H = \sum_n d_n$ and $d_{\mathscr{B}} = \sum_n d_n^2$, where n runs in the set of irreducible objects of $\mathscr{A}_k$, and $d_x = dim V_x$, $d_V = \sum_x d_x$ and $d_{\widehat{\mathscr{B}}} = \sum_x d_x^2$, where x runs in the set of irreducible objects of $\mathscr{O}$ have been calculated first by [34], then by [8], [38], [11]. In the case of $sl(3)$, they have been calculated by [34], [12], [39] and [22]. One check, of course, that $d_{\mathscr{B}} = d_{\widehat{\mathscr{B}}}$ in all cases, since the underlying vector space is the same. Surprisingly, one also observes[7] that $d_H = d_V$ in most cases. The collection of known results giving d_H for all $sl(2)$ cases can be condensed into a closed formula by using a recent result obtained by [30] for the dimensions of preprojective algebras associated with *ADE* quivers[8]. To compute d_n, the pedestrian approach, that works in all cases, is to calculate the annular matrices F_n, describing the module action of $\mathscr{A}_k$ on $\mathscr{E}$, using recursion formulae giving irreps of $sl(2)$ or $sl(3)$, then to sum over all matrix elements, since $H_{ab}^m = Hom(\lambda_n \otimes \lambda_a, \lambda_b)$. To compute d_x, one has first to determine $\mathscr{O}$, for instance by solving the modular splitting equation, then the dual annular matrices S_x describing the module action of $\mathscr{O}$ on $\mathscr{E}$, and finally to sum over all matrix elements, since $V_{ab}^x = Hom(\lambda_a \otimes \lambda_x, \lambda_b)$.

[7]When it is not so, in particular when the graph $\mathscr{E}$ is a $\mathbb{Z}_2$ or $\mathbb{Z}_3$ orbifold, one knows how to "correct" this curious linear sum rule, which was first observed in the case of $sl(2)$ by [38].

[8]Warning: the preprojective algebra is a multiplicative structure on H, at least for $sl(2)$ cases, it cannot be identified with either $\mathscr{B}$ or $\widehat{\mathscr{B}}$, see also [10].

3.3.1 $sl(2)$

We shall give the values of d_n, d_x, d_H, d_V and $d_{\mathscr{B}}$ in tables, but thanks to the above identification with the vector space underlying the preprojective algebra of quiver theories, we have a closed formula for d_H that work for all $sl(2)$ cases, namely $d_H = \frac{\kappa(\kappa+1)r}{6}$. Recall that k is the level, $r = k+1$ is the rank (the number of vertices), and $\kappa = k+2$ is the Coxeter number. In terms of the dimension $dim(E)$ of the Lie group corresponding to the chosen ADE diagram, and using the Kostant formula $dim(E) = (\kappa+1)r$, we can also write $d_H = \frac{\kappa\, dim(E)}{6}$. For instance, $E_6 \to 156 = 78\frac{12}{6}$ $\quad E_7 \to 399 = 133\frac{18}{6}$ $\quad E_8 \to 1240 = 248\frac{30}{6}$. For A_r graphs, the rank $r = k+1 = \kappa - 1$ so that $d_H(A_r) = (\kappa-1)\kappa(\kappa+1)/6$, this can be obtained directly from the fact that $d_n = (n+1)(k+1-n)$, the trivial representation being labelled $n = 0$. Notice that 2κ is the period (in n) of matrices F_n. It is interesting to summarize how the general formula for d_H is obtained in quiver theories [30]: one constructs a generating function for the d_n (matrix Hilbert series) and obtains d_H as twice the sum of matrix elements of the inverse of the Cartan matrix of the chosen graph ($2\,\mathbb{1} - F_1$); this, in turn, is given by the Freudenthal - de Vries strange formula, which, using Kostant relation, can be written $\kappa(\kappa+1)r/12)$, hence the result.

3.3.2 $sl(3)$

Dimensions are later given in tables. In the case of d_H, there is no known formula that works in all cases. We however obtained the following closed result for $\mathscr{A}_k$ diagrams: $d_H = (k+1)(k+2)(k+3)(k+4)(k+5)(k^2+6k+14)/1680$ or, using $\kappa = k+3$, $d_H = (\kappa-2)(\kappa-1)\kappa(\kappa+1)(\kappa+2)(\kappa^2+5)/1680$. There is no Lie group theory associated with the Di Francesco Zuber diagrams, nevertheless it is natural to call rank the quantity $r = (k+1)(k+2)/2$, since it is the number of vertices of the $\mathscr{A}_k$ Weyl alcove. Our previous result for d_H uses the fact that the sum of matrix elements of the inverse of $(3\,\mathbb{1} - (F_1 + F_1^t)/2)$ can be shown to be equal to $r(\kappa+1)(\kappa+2)/60$.

3.4 Quantum dimensions

As usual, q-integers are denoted $[n] = (q^n - q^{-n})/(q - q^{-1})$ with $q = exp[i\pi/\kappa]$. One can define quantum dimensions μ of the vertices of $\mathscr{E}$ from the Perron Frobenius eigenvector of its adjacency matrix. If the graph $\mathscr{E}$ has self - fusion, they can be calculated from the quantum dimensions of the unit (which is 1) and of the generator (which is $[2] = 2\cos(\pi/\kappa)$ for $sl(2)$ and $[3] = 1 + 2\cos 2\pi/\kappa$ for $sl(3)$), by using the character property of μ. Let $\mu_a = dim_{\mathscr{E}}(\sigma_a)$ the quantum dimension of the vertex σ_a. Call also $|\mathscr{E}| = \sum_a \mu_a^2$. Graphs $\mathscr{A}$ and $\mathscr{O}$ describing fusion and quantum symmetries have self - multiplication, and quantum dimensions $\mu_n = dim_{\mathscr{A}}(\lambda_n)$ and $\mu_x = dim_{\mathscr{O}}(o_x)$ for their vertices can be obtained in a similar way. In the particular case $\mathscr{E} = \mathscr{A}$, the μ_n can be obtained from the modular generator S (a unitarizing matrix for the adjacency matrix of the graph): $\mu_n = S_{0n}/S_{00}$. For $sl(2)$, $\mu_n = [n+1]$. For $sl(3)$, $\mu_{p,q} = [p+1][q+1][p+q+2]/[2]$. Unitarity of S implies $|\mathscr{A}| = \sum_n \mu_n^2 = 1/S_{00}^2$ and several explicit expressions given at the end of this section.

Call $\mathfrak{e}$ the intertwiner describing induction - restriction between $\mathscr{A}$ and $\mathscr{E}$ (also called essential matrix relative to the unit vertex). It is a rectangular matrix with r_A lines and r_E columns. Essential matrices $\mathfrak{e}_a$ are obtained from annular matrices F^n as follows: $(\mathfrak{e}_a)_{nb} = (F^n)_{ab}$. The intertwiner is $\mathfrak{e} = \mathfrak{e}_0$. From induction, one obtains $dim(\Gamma_a) = \sum_n (\mathfrak{e})_{na}\,\mu_n$. It is convenient to write $\lambda_n \uparrow \Gamma_a$

when the space of sections Γ_a contains λ_n in its reduction, i.e., when the integer $(\mathfrak{e})_{na} \geq 1$. Possible multiplicities being understood, we shall write

$$dim(\Gamma_a) = \sum_{\lambda_n \uparrow \Gamma_a} \mu_n$$

In particular, for the space of functions $\mathscr{F}$ over the "discrete non commutative space": $\mathscr{A}/\mathscr{E}$, we have $dim(\Gamma_0) = \sum_n (\mathfrak{e})_{n0}\, \mu_n$ and we shall write (see also [9]):

$$|\mathscr{A}/\mathscr{E}| = dim(\Gamma_0) = dim(\mathscr{F}) = \sum_{\lambda_n \uparrow \Gamma_0} \mu_n$$

Then, we have the following result :

$$dim_{\mathscr{E}}(a) = \frac{dim\,\Gamma_a}{dim\,\Gamma_0} = \frac{dim\,\Gamma_a}{|\mathscr{A}/\mathscr{E}|}$$

Moreover, $|\mathscr{A}/\mathscr{E}| = |\mathscr{A}|/|\mathscr{E}|$. To obtain the quantum dimensions μ_a of vertices of $\mathscr{E}$, and therefore the order $|\mathscr{E}|$, it is therefore enough to know the μ_n for $\mathscr{A}$ diagrams and the induction rules.

Example: take $\mathscr{E} = E_6$, and consider the central vertex σ_2 (the triple point). Its quantum dimension can be calculated using Perron Frobenius, or directly, using $\sigma_2 = \sigma_1^2 - \sigma_0$:

$$dim_E(\sigma_2) = dim_E(\sigma_1)^2 - dim_E(\sigma_0) = (2cos(\pi/12))^2 - 1 = 1 + \sqrt{3}$$

but one can also obtain from A_{11}: induction from σ_0 gives $\Gamma_0 = \lambda_0 \oplus \lambda_6$, so that $|A_{11}/E_6| = [1] + [7] = 3 + \sqrt{3}$, and induction from σ_2 gives $\Gamma_2 = \lambda_2 \oplus \lambda_4 \oplus \lambda_6 \oplus \lambda_8$, so

$$dim_E(\sigma_2) = \frac{[3]+[5]+[7]+[9]}{[1]+[7]} = \frac{6+4\sqrt{3}}{3+\sqrt{3}} = 1+\sqrt{3},.$$

Call J the modular subalgebra of $\mathscr{E}$. For graphs with self - fusion, we have both $|\mathscr{O}| = |\mathscr{E}| \times |\mathscr{E}|/|J|$ and $|\mathscr{O}| = |\mathscr{A}|$, so that we have also $|\mathscr{A}|/|\mathscr{E}| = |\mathscr{E}|/|J|$.

Using quantum dimensions of $\mathscr{E}$, one can compute $|J| = \sum_{\sigma_c \in J \subset \mathscr{E}} \mu_c^2 = \sum_{\sigma_c \in J} |\Gamma_c|^2/|\Gamma_0|^2$, but the relation for $|\mathscr{O}|$ can be written $|J| = |\mathscr{A}|/|\Gamma_0|^2$, so that comparing the two expressions implies $|A| = \sum_{c \in J} |\Gamma_c|^2$. Actually the following *ADE* or generalized *ADE* trigonometric identities, which seem to belong to the folklore of CFT (we shall write them later in a non standard but suggestive way), imply the previous result. Let Z be a modular invariant matrix of an $sl(N)$ system at level k, with or without self-fusion. Its matrix elements $(Z)_{m,n}$ are indexed by a pair of irreps λ_m and λ_n. For $sl(N)$, these are of course multi-indices. As before, call μ_m the quantum dimension of λ_m. Then we have the following identity :

$$\sum_{m,n} \mu_m\, (Z)_{m,n}\, \mu_n = \sum_m \mu_m^2$$

The right hand side of this identity is $|\mathscr{A}_k|$ and can be calculated from the modular matrix S.

$$|\mathscr{A}| = \frac{\kappa}{2} \frac{1}{\sin^2\left(\frac{\pi}{\kappa}\right)} \quad \text{for } sl(2), \text{ with } \kappa = k+2$$

$$|\mathscr{A}| = \frac{3}{256} \kappa^2 \frac{1}{\sin^6\left(\frac{\pi}{\kappa}\right)} \frac{1}{\cos^2\left(\frac{\pi}{\kappa}\right)} \quad \text{for } sl(3), \text{ with } \kappa = k+3$$

After simplification by a common denominator the *ADE* trigonometric identities read[9] explicitly

$$\sum_{m,n=1}^{r} (Z)_{m,n} \sin(m\pi/\kappa)\,\sin(n\pi/\kappa) = \kappa/2$$

For instance in the E_8 case, $\kappa = 30$, we have

$$\left\{sin(7\frac{\pi}{30})+sin(13\frac{\pi}{30})+sin(17\frac{\pi}{30})+sin(23\frac{\pi}{30})\right\}^2 + \left\{sin(1\frac{\pi}{30})+sin(11\frac{\pi}{30})+sin(19\frac{\pi}{30})+sin(29\frac{\pi}{30})\right\}^2 = 15$$

After simplification, the higher *ADE* trigonometric identities of $sl(3)$ type read[9]:

$$\sum_{\substack{n=(n_1,n_2)\\ m=(m_1,m_2)}} (Z)_{m,n}\, sin(m_1\frac{\pi}{\kappa})\, sin(m_2\frac{\pi}{\kappa})\, sin((m_1+m_2)\frac{\pi}{\kappa})\, sin(n_1\frac{\pi}{\kappa})\, sin(n_2\frac{\pi}{\kappa})\, sin((n_1+n_2)\frac{\pi}{\kappa}) = 3\kappa^2/64$$

For instance in the $\mathscr{E}_{21}$ case, $\kappa = 24$, setting $s_n = sin(n\pi/24)$, we have

$$(2s_1s_7s_8+2s_5s_8s_{13}+2s_5s_{11}s_{16}+2s_1s_{16}s_{17}+2s_8s_{11}s_{19}+2s_7s_{16}s_{23})^2 +$$
$$+(s_1^2s_2+s_5^2s_{10}+2s_2s_{11}s_{13}+s_7^2s_{14}+2s_7s_{10}s_{17}+2s_5s_{14}s_{19}+s_{11}^2s_{22}+2s_1s_{22}s_{23})^2 = 27$$

Product of quantum dimensions and the discriminant formula:
Define $D = (\sum_{\lambda_n\in\mathscr{A}}\mu_n^2)^r/\Pi_{\lambda_n\in\mathscr{A}}\mu_n^2$, where r is the total number of irreps of the fusion algebra, $r = k+1$ for $sl(2)$, $r = (k+1)(k+2)/2!$ for $sl(3)$, $r = (k+1)(k+2)(k+3)/3!$ for $sl(4)$. In the case of $sl(2)$, using previous results for $\sum_{\lambda_n\in\mathscr{A}}\mu_n^2$ and a classical trigonometric identity giving $\Pi_{\lambda_n\in\mathscr{A}}\,\mu_n^2 = [2^{-(\kappa-1)}\kappa\,(Sin(\pi/\kappa))^{-(\kappa-1)}]^2$ where results are written in terms of the Coxeter number κ, one finds $D = 2^{\kappa-1}\kappa^{\kappa-3}$. More generally, it was shown recently in [25], that D is the square of the determinant of the matrix S_{mn}/S_{m0} (this is the matrix of "quantum conjugacy classes") which, in the case of $sl(2)$ could also be defined as the discriminant of the characteristic polynomial of the adjacency matrix of the graph A_r. Example: Take A_{11} of the $sl(2)$ system, the characteristic polynomial of the adjacency matrix is $-s^{11}+10s^9-36s^7+56s^5-35s^3+6s$ and its discriminant is 10567230160896, indeed equal to $2^{11}\,12^9$. D is an integer and general expressions for this quantity are obtained in the quoted reference. In the case of $sl(2)$, one recovers the expression calculated previously, whereas [25] gives $D = 3^{(\kappa-2)(\kappa-1)/2}\,\kappa^{(\kappa-4)(\kappa-2)}$ for $sl(3)$, where $\kappa = k+3$. Since $|\mathscr{A}_k|$ is already known, one can use the Gepner formula for D to provide an efficient way for obtaining the square product of quantum dimensions. For $sl(3)$ we find:

$$\Pi_{\lambda_n\in\mathscr{A}}\,\mu_n^2 = 16^{-(\kappa-2)(\kappa-1)}\kappa^{3(\kappa-2)}\left(\cos\left(\frac{\pi}{\kappa}\right)\sin^3\left(\frac{\pi}{\kappa}\right)\right)^{-(\kappa-2)(\kappa-1)}$$

3.5 Tables

3.5.1 The $sl(2)$ family

There are two tables. Notations should be clear. Most results are hardly new. In particular, the dimensions d_n and d_v given in table 1 were obtained already in [34], [8], [38] and [11]. The compact expression for d_H can be seen in [30], and several quantum dimensions or masses given in table 2 can also be found in [11] and [39]. Nevertheless, both tables contain new entries and explicit formulae.

[9]Warning: In this expression, we shifted by $+1$ the indices labelling irreps.

3.5.2 The $sl(3)$ family

Many results about the $sl(3)$ graphs can be found in the original article [16] and in the book [3], chap.17. Other aspects of members of this system, in particular their quantum symmetries, are described in [33], [23], [14], [24] and [22]. Here we are only interested in giving tables describing the block structure of the quantum groupoid $\mathscr{B}$, i.e., dimensions d_n, d_x, the quantum dimensions and the quantum masses. Many such numbers were certainly obtained by [34], several examples are studied in [12] and [39]. Generic formulae are ours. Results presented here are complete only for the cases with self - fusion, for which we also give tables describing induction rules (actually the induction tables for $\mathscr{E}_{21}$ could not fit in this publication and will appear in a unabridged version of our work). A description of the cases without self-fusion, along the same lines, should be made available in [17] and [22]. Some of the results presented here already appeared in [12].

Acknowledgments

One of us (R.C.) is grateful to A. Andruskiewitsch, A. Davydov , H. Montani and R. Street for many discussions and patient explanations about categories. We also thank CONICET, CBPF, IMPA and Macquarie University for partial support. G.S. was supported by a fellowship of FAPERJ.

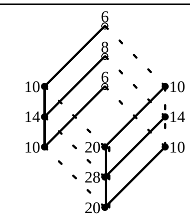
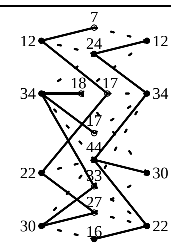
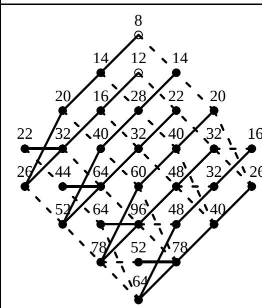
Graph	$\kappa=k+2$	d_n, Z and exponents, $\mathscr{F}$	d_x	$d_H=\kappa(\kappa+1)r/6$	d_V-d_H	$d_{\mathscr{B}}=d_{\hat{\mathscr{B}}}$
$\mathscr{A}_1=A_2$	3	$(2,2)$	$(2,2)$	4	0	$8=2^3$
$\mathscr{A}_2=A_3$	4	$(3,4,3)$	$(3,4,3)$	10	0	$34=2^1 17^1$
$\mathscr{A}_3=A_4$	5	$(4,6,6,4)$	$(4,6,6,4)$	20	0	$104=2^3 13^1$
$\mathscr{A}_4=A_5$	6	$(5,8,9,8,5)$	$(5,8,9,8,5)$	35	0	$259=7^1 37^1$
$\mathscr{A}_5=A_6$	7	$(6,10,12,12,10,6)$	$(6,10,12,12,10,6)$	56	0	$560=2^4 5^1 7^1$
$\mathscr{A}_{10}=A_{11}$	12	$(11,20,27,32,35,36,35,32,27,20,11)$	$\ldots$	$286=2^1 11^1 13^1$	0	$8294=2^1 11^1 13^1 29^1$
$\mathscr{A}_{16}=A_{17}$	18	$(17,32,45,56,65,72,77,80,81,80,77,72,65,56,45,32,17)$	$\ldots$	$3^1 17^1 19^1$	0	$3^1 5^1 13^1 17^1 19^1$
$\mathscr{A}_{28}=A_{29}$	30	$(29,56,81,104,125,144,161,176,189,200,209,216,221,224;225;\text{sym.})$	$\ldots$	$5^1 29^1 31^1$	0	$17^1 29^1 31^1 53^1$
$\mathscr{A}_k=A_{r=k+1}$	$k+2$	$d_n=(n+1)(k+1-n),\ n=0,...,k$	$d_x=d_n$	$(\kappa-1)\kappa(\kappa+1)/6$	0	$\kappa(\kappa^4-1)/30$
$\mathscr{D}_4=D_4$	6	$(4,6;8;6,4)$	$(4,6;4,4)\,(4,6;4,4)$	$28=4^1 7^1$	$(4+4)$	$168=2^3 3^1 7^1$
$\mathscr{D}_8=D_6$	10	$(6,10,14,16;18;16,14,10,6)$	$(6,10,14,16;9,9)\,(6,10,14,16;9,9)$	$110=2^1 5^1 11^1$	$(9+9)$	$1500=2^2 3^1 5^3$
$\mathscr{D}_{16}=D_{10}$	18	$(10,18,26,32,38,42,46,48;50;\text{sym.})$	$(10,18,26,32,38,42,46,48;25,25)_2$	$570=2^1 3^1 5^1 19^1$	$(25+25)$	$2^2 5501^1$
$\mathscr{D}_{28}=D_{16}$	30	(16, 30, 44, 56, 68, 78, 88, 96, 104, 110, 116, 120, 124, 126 ; 128 ; sym.)	$\ldots\ldots$	$2480=2^4 5^1 31^1$	$(64+64)$	$2^5 7757^1$
$\mathscr{D}_k=D_{r_e=\frac{k}{2}+2}$	$2r-2$	$(r,\kappa,\ldots;\frac{1}{2}(1+\frac{\kappa}{2})^2;\ldots,\kappa,r)$, $\mathscr{F}=\lambda_0\oplus\lambda_k$	$(r,\kappa,\ldots;\frac{1}{4}(1+\frac{\kappa}{2})^2,\frac{1}{4}(1+\frac{\kappa}{2})^2)_2$	$\kappa(\kappa+1)(\kappa+2)/12$	$\frac{1}{2}(1+\frac{\kappa}{2})^2$	$\frac{(2+\kappa)(120+\kappa(28+\kappa(26+\kappa(17+4\kappa))))}{480}$
$\mathscr{D}_6=D_5$	8	$(5,8,11,12,11,8,5)$	$(5,8,11,12,11,8,5)$	$60=2^2 3^1 5^1$	0	$564=2^2 3^1 47^1$
$\mathscr{D}_{10}=D_7$	12	$(7,12,17,20,23,24,23,20,17,12,7)$	$(7,12,17,20,23,24,23,20,17,12,7)$	$182=2^1 7^1 13^1$	0	$3398=2^1 1699^1$
$\mathscr{D}_k=D_{r_o=\frac{k}{2}+2}$	$2r-2$	$(r,\kappa,\ldots,\kappa,r)$, $\mathscr{F}=\lambda_0\oplus\lambda_k$	$(r,k+2,\ldots,k+2,r)$	$\kappa(\kappa+1)(\kappa+2)/12$	0	$\frac{\kappa(176+\kappa(80+\kappa(60+\kappa(25+4\kappa))))}{480}$
$\mathscr{E}_{10}=E_6$	12	(6,10,14,18,20,20,20,18,14,10,6) $Z=\|1+7\|^2+\|4+8\|^2+\|5+11\|^2$ $\mathscr{F}=\lambda_0\oplus\lambda_6$		$156=2^2 3^1 13^1$	0	$2512=2^4 157^1$
$\mathscr{E}_{16}=E_7$	18	(7,12,17,22,27,30,33,34,35,34,33,30,27,22,17,12,7) $Z=\|1+17\|^2+\|5+13\|^2+\|7+11\|^2+\|9\|^2+((3+15)\ \overline{9}+h.c.)$ $\mathscr{F}=\lambda_0\oplus\lambda_8\oplus\lambda_{16}$		$399=3^1 7^1 19^1$	0	$10905=3^1 5^1 727^1$
$\mathscr{E}_{28}=E_8$	30	(8,14,20,26,32,38,44,48,52,56,60,62,64,64,64; sym.) $Z=\|1+11+19+29\|^2+\|7+13+17+23\|^2$ $\mathscr{F}=\lambda_0\oplus\lambda_{10}\oplus\lambda_{18}\oplus\lambda_{28}$		$1240=2^3 5^1 31$	0	$63136=2^5 1973^1$

Table 1: Horizontal dimensions, vertical dimensions, and bialgebras dimensions for $sl(2)$ cases.

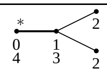
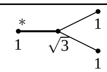
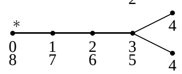
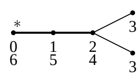
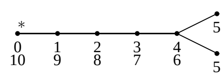
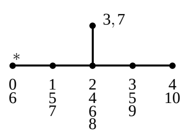
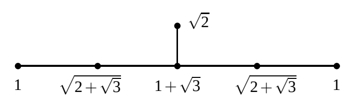
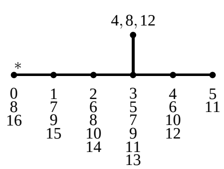
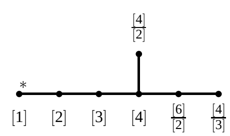
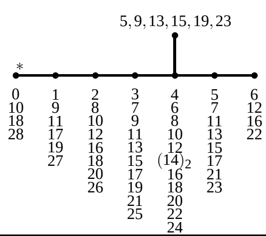
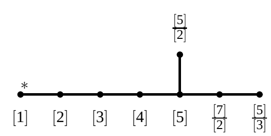
Graph	induction	$q-dim$	$\lvert\mathscr{E}\rvert$	$\lvert\mathscr{A}/\mathscr{E}\rvert$	$\lvert\mathscr{J}\rvert$
$\mathscr{A}_2 = A_3$	.	$1, \sqrt{2}, 1$	4	1	$= \lvert\mathscr{A}\rvert$
$\mathscr{A}_3 = A_4$	.	$1, \frac{1}{2}(1+\sqrt{5}), \frac{1}{2}(1+\sqrt{5}), 1$	$5+\sqrt{5}$	1	$= \lvert\mathscr{A}\rvert$
$\mathscr{A}_4 = A_5$	.	$1, \sqrt{3}, 2, \sqrt{3}, 1$	12	1	$= \lvert\mathscr{A}\rvert$
$\mathscr{A}_5 = A_6$	.	$1, 2\cos(\frac{\pi}{7}), 1+2\cos(\frac{\pi}{7}), 1+2\cos(\frac{\pi}{7}), 1$	18.59	1	$= \lvert\mathscr{A}\rvert$
$\mathscr{A}_k = A_{r=k+1}$	.	$[1], [2], \ldots, [2], [1]$	$(\kappa/2)\csc^2(\pi/\kappa)$	1	$= \lvert\mathscr{A}\rvert$
$\mathscr{A}_{10} = A_{11}$	.	$1, \sqrt{2+\sqrt{3}}, 1+\sqrt{3}, \sqrt{3(2+\sqrt{3})}, 2+\sqrt{3}; 2\sqrt{2+\sqrt{3}};$ + sym.	$24(2+\sqrt{3})$	1	$= \lvert\mathscr{A}\rvert$
$\mathscr{A}_{16} = A_{17}$	.	...	$9\csc^2(\pi/18) = 298.471$	1	$= \lvert\mathscr{A}\rvert$
$\mathscr{A}_{28} = A_{29}$	.	...	$30\left(12+5\sqrt{5}+\sqrt{3\left(85+38\sqrt{5}\right)}\right)$	1	$= \lvert\mathscr{A}\rvert$
$\mathscr{D}_4 = D_4$			$\frac{\lvert\mathscr{A}_5\rvert}{2} = 6$	2	$\frac{\lvert\mathscr{A}_5\rvert}{4}$
$\mathscr{D}_8 = D_6$		$\sqrt{\frac{1}{2}(5+\sqrt{5})}, \frac{1}{2}(3+\sqrt{5}), \sqrt{5+2\sqrt{5}}; \frac{1}{2}(1+\sqrt{5}), \frac{1}{2}(1+\sqrt{5})$	$\frac{\lvert\mathscr{A}_9\rvert}{2} = 5(3+\sqrt{5})$	2	$\frac{\lvert\mathscr{A}_9\rvert}{4}$
$\mathscr{D}_{16} = D_{10}$	...	$[1], [2], [3], [4], [5], [6], [7], [8]; [9]/2, [9]/2$	$\frac{\lvert\mathscr{A}_{17}\rvert}{2} = 149.235$	2	$\frac{\lvert\mathscr{A}_{17}\rvert}{4}$
$\mathscr{D}_{28} = D_{16}$	...	......	$\frac{\lvert\mathscr{A}_{29}\rvert}{2}$	2	$\frac{\lvert\mathscr{A}_{29}\rvert}{4}$
$\mathscr{D}_6 = D_5$		$1, \sqrt{2+\sqrt{2}}, 1+\sqrt{2}; \sqrt{1+\frac{1}{\sqrt{2}}}, \sqrt{1+\frac{1}{\sqrt{2}}}$	$\lvert A_7\rvert/2 = 2csc^2(\pi/8) = 13.65$	?	.
$\mathscr{D}_{10} = D_7$		$1, \sqrt{2+\sqrt{3}}, 1+\sqrt{3}, \sqrt{3(2+\sqrt{3})}, 2+\sqrt{3}; \sqrt{2+\sqrt{3}}, \sqrt{2+\sqrt{3}}$	$\lvert A_{11}\rvert/2 = 3csc^2(\pi/12)$	.	.
$\mathscr{E}_{10} = E_6$			$4(3+\sqrt{3})$	$3+\sqrt{3}$	4
$\mathscr{E}_{16} = E_7$		 $q = \exp(\frac{i\pi}{18})$	$\mathscr{E}\rvert = 2([2]^2+[4]^2+[4]^2/[3]^2)$ $= 38.468$; $\lvert A_{17}\rvert = \lvert D_{10}\rvert\,\lvert D_{10}\rvert/\lvert J\rvert$; $\lvert D_{10}\rvert = \lvert A_{17}\rvert/2$	$[1]+[9]+[17] = 7.758$	$\lvert J\rvert = \lvert D_{10}\rvert/2 = \lvert A_{17}\rvert/4$; $\lvert J\rvert = [1]^2+[3]^2+[5]^2+[7]^2+[9]^2/4+[9]^2/4$
$\mathscr{E}_{28} = E_8$		 $q = \exp(\frac{i\pi}{30})$	$\frac{1}{2}\left(15\left(3+\sqrt{5}\right)+\sqrt{30\left(65+29\sqrt{5}\right)}\right)$	$\frac{1}{2}\left(3\left(5+\sqrt{5}\right)+\sqrt{150+66\sqrt{5}}\right)$	$\frac{1}{2}\left(5+\sqrt{5}\right)$

Table 2: Quantum dimensions for $sl(2)$ cases.

Graph	κ	r_E, r_A, r_O	d_H	$d_V - d_H$	$d_{\mathscr{B}} = d_{\hat{\mathscr{B}}}$	$\lvert\mathscr{E}\rvert$	$\lvert\mathscr{A}/\mathscr{E}\rvert$	$\lvert J\rvert$
$\underline{\mathscr{A}_1}$	4	$3,3,3$	9	0	27	3	1	$=\lvert\mathscr{E}\rvert$
$\underline{\mathscr{A}_2}$	5	$6,6,6$	45	0	351	$\frac{3}{2}(5+\sqrt{5})$	1	$=\lvert\mathscr{E}\rvert$
$\underline{\mathscr{A}_3}$	6	$10,10,10$	164	0	2920	36	1	$=\lvert\mathscr{E}\rvert$
$\underline{\mathscr{A}_4}$	7	$15,15,15$	486	0	17766	106.027	1	$=\lvert\mathscr{E}\rvert$
$\underline{\mathscr{A}_5}$	8	$21,21,21$	1242	0	85644	$48(3+2\sqrt{2})$	1	$=\lvert\mathscr{E}\rvert$
$\underline{\mathscr{A}_9}$	12	$55,55,55$	21307	0	10517299	$432(7+4\sqrt{3})$	1	$=\lvert\mathscr{E}\rvert$
$\underline{\mathscr{A}_{21}}$	24	$253,253,253$	2729870	0	41644127980	$288\left(18+10\sqrt{3}+\sqrt{6\left(97+56\sqrt{3}\right)^2}\right)$	1	$=\lvert\mathscr{E}\rvert$
$\underline{\mathscr{A}_k}$	$k+3$	$r_E = r_A = r_O = \frac{(k+1)(k+2)}{2}$	$\frac{(\kappa-2)(\kappa-1)\kappa(\kappa+1)(\kappa+2)(\kappa^2+5)}{1680}$	0	$d_{\mathscr{B}}(\mathscr{A}_k)$	$3\,\frac{\kappa^2 csc^6(\pi/\kappa) sec^2(\pi/\kappa)}{256}$	1	$=\lvert\mathscr{E}\rvert$
$\mathscr{A}_3^*$	6	$2,10,10$	36	0	144	2	4	
$\mathscr{A}_4^*$	7	$3,15,15$	102	0	798	2.86294	3	
$\mathscr{A}_5^*$	8	$3,21,21$	.	0	.	.	.	
$\mathscr{A}_{k\geq 1}^*$	$k+3$	$.,r_A,r_A$	.	0	.	.	.	
$\underline{\mathscr{D}_3}$	6	$6,10,18$	96	30	1032	$12 = \frac{1}{3}\lvert\mathscr{A}_3\rvert$	3	4
$\underline{\mathscr{D}_6}$	9	$12,28,36$	1218	.	64698	$671.56 = \frac{1}{3}\lvert\mathscr{A}_6\rvert$	3	223.853
$\underline{\mathscr{D}_9}$	12	$21,55,63$	8193	622	1573275	$144(7+4\sqrt{3}) = \frac{1}{3}\lvert\mathscr{A}_9\rvert$	3	$48(7+4\sqrt{3})$
$\underline{\mathscr{D}_{k=0 \bmod 3}}$	$k+3$	$\frac{r_A-1}{3}+3, r_A, 3r_E$	.	.	.	$= \frac{1}{3}\lvert\mathscr{A}_k\rvert$	3	$\frac{1}{3}\lvert\mathscr{E}\rvert$
$\mathscr{D}_4$	7	$5,15,15$	$\frac{1}{3}d_H(\mathscr{A}_4) = 162$	0	$\frac{1}{9}d_{\mathscr{B}(\mathscr{A}_4)} = 1974$	$\frac{1}{3}\lvert\mathscr{A}_4\rvert = 35.3424$	3	
$\mathscr{D}_5$	8	$7,21,21$	$\frac{1}{3}d_H(\mathscr{A}_5) = 414$	0	$\frac{1}{9}d_{\mathscr{B}(\mathscr{A}_5)} = 9516$	$\frac{1}{3}\lvert\mathscr{A}_5\rvert = 16(3+2\sqrt{2})$	3	
$\mathscr{D}_{k=1,2 \bmod 3}$	$k+3$	$\frac{1}{3}r_A, r_A, r_A$	$\frac{1}{3}d_H(\mathscr{A}_k)$	0	$\frac{1}{9}d_{\mathscr{B}(\mathscr{A}_k)}$	$\frac{1}{3}\lvert\mathscr{A}_k\rvert$	3	
$\mathscr{D}_3^*$	6	$6,10,18$	$3d_H(\mathscr{A}_3^*) = 108$	.	$9d_{\mathscr{B}}(\mathscr{A}_3^*) = 1296$	.	.	.
$\mathscr{D}_9^*$	12	$18,55,54$	$3d_H(\mathscr{A}_9^*)$	.	$9d_{\mathscr{B}}(\mathscr{A}_9^*)$	.	.	.
$\mathscr{D}_{k=0 \bmod 3}^*$	$k+3$	$3r_{A*}, r_A, r_O(\mathscr{D}_k)$	$3d_H(\mathscr{A}_k^*)$	.	$9d_{\mathscr{B}}(\mathscr{A}_k^*)$	.	.	.
$\mathscr{D}_4^*$	7	$9,10,10$	$3d_H(\mathscr{A}_4^*) = 306$	0	$9d_{\mathscr{B}}(\mathscr{A}_4^*) = 7182$	.	.	.
$\mathscr{D}_5^*$	8	$9,21,21$	$3d_H(\mathscr{A}_5^*)$	0	$9d_{\mathscr{B}}(\mathscr{A}_5^*)$	.	.	.
$\mathscr{D}_{k=1,2 \bmod 3}^*$	$k+3$	$3r_{A*}, r_A, r_A$	$3d_H(\mathscr{A}_k^*)$	0	$9d_{\mathscr{B}}(\mathscr{A}_k^*)$	.	.	
$\mathscr{D}_9^t$	12	$17,55,63$	7001	.	1167355	$72(2+\sqrt{3})$	$6(2+\sqrt{3})$	.
$\mathscr{D}_9^{t*}$	12	$11,55,63$	.	.	.	.	.	
$\underline{\mathscr{E}_5}$	8	$12,21,24$	720	0	29376	$12(2+\sqrt{2})$	$2(2+\sqrt{2})$	6
$\mathscr{E}_5/3$	8	$4,21,24$	$\frac{1}{3}d_H(\mathscr{E}_5) = 240$	0	$\frac{1}{9}d_{\mathscr{B}}(\mathscr{E}_5) = 3264$	$\frac{1}{3}\lvert\mathscr{E}_5\rvert = 4(2+\sqrt{2})$	$6(2+\sqrt{2})$	.
$\underline{\mathscr{E}_9}$	12	$12,55,72$	4656	792	518976	$36(2+\sqrt{3})$	$12(2+\sqrt{3})$	3
$\mathscr{E}_9/3$	12	$12,55,72$	5616	936	754272	$\frac{1}{3}\lvert\mathscr{E}_9\rvert = 12(2+\sqrt{3})$	$36(2+\sqrt{3})$	.
$\underline{\mathscr{E}_{21}}$	24	$24,253,288$	288576	0	480701952	$24\left(18+10\sqrt{3}+\sqrt{6\left(97+56\sqrt{3}\right)}\right)$	$12\left(18+10\sqrt{3}+\sqrt{6\left(97+56\sqrt{3}\right)}\right)$	2

Table 3: Dimensions and quantum masses for $sl(3)$ cases. $d_{\mathscr{B}}(\mathscr{A}_k) = \frac{(\kappa-2)(\kappa-1)\kappa^2(\kappa+1)(\kappa+2)(1052+325\kappa^2+58\kappa^4+5\kappa^6)}{4435200}$

References

[1] B. Bakalov, A. Kirillov Jr., *Lectures on tensor categories and modular functors,* AMS, Univ. Lect. Notes Series Vol.21(2001).

[2] G. Böhm and K. Szlachányi; *A coassociative C^*-quantum group with non-integral dimensions,* math.QA/9509008 (1995), Lett. Math. Phys. 38 (1996), no. 4, 437–456.

[3] P.Di Francesco, P. Matthieu, D. Senechal; *Conformal Field Theory* (Springer, 1997).

[4] J. Böckenhauer and D. Evans; *Modular invariants from subfactors: Type I coupling matrices and intermediate subfactors*, Commun. Math. Phys., 213, Issue 2, 267-289 (2000), math.OA/9911239.

[5] J. Böckenhauer and D. Evans; *Modular invariants, graphs and α induction for nets of subfactors II.* Commun. Math. Phys. 200, 57-103 (1999).

[6] J. Böckenhauer, D. Evans and Y. Kawahigashi; *Chiral structure of modular invariants for subfactors* Commun. Math. Phys. 210, 733-784 (2000).

[7] A. Cappelli, C. Itzykson and J.-B. Zuber; *The ADE classification of minimal and $A_1^{(1)}$ conformal invariant theories.* Commun. Math. Phys. 13,1 (1987).

[8] R. Coquereaux; *Notes on the quantum tetrahedron*, Moscow Math. J. vol2, n1(2002), 1-40, hep-th/0011006.

[9] R. Coquereaux; *Classical and quantum polyhedra: A fusion graph algebra point of view* AIP Conference Proceedings 589 (2001), 37th Karpacz Winter School of Theor. Phys., J. Lukierski and J. Rembielinski eds, 181-203, hep-th/0105239.

[10] R. Coquereaux and A. Garcia; *On bialgebras associated with paths and essential paths on ADE graphs*, math-ph/0411048, Intern. J. of Geom. Meth. in Mod. Phys., Vol 2, No 3, (2005) 1-26.

[11] R. Coquereaux, G. Schieber; *Twisted partition functions for ADE boundary conformal field theories and Ocneanu algebras of quantum symmetries*, J. of Geom. and Phys. 781 (2002), 1-43, hep-th/0107001.

[12] R. Coquereaux, G. Schieber; *Determination of quantum symmetries for higher ADE systems from the modular T matrix*, J. of Math. Physics **44** (2003) 3809–3837, hep-th/0203242.

[13] R. Coquereaux and R. Trinchero; *On quantum symmetries of ADE graphs*, Advances in Theor. and Math. Phys. , volume 8, issue 1, 2004, hep-th/0401140.

[14] R. Coquereaux, D. Hammaoui, G. Schieber, E. H. Tahri, *Comments about quantum symmetries of $SU(3)$ graphs*, J. of Geom. and Phys., in press, math-ph/0508002.

[15] P. Di Francesco and J.-B. Zuber, in *Recent developments in Conformal Field Theory*, Trieste Conference, 1989, S. Randjbar-Daemi, E. Sezgin and J.-B. Zuber eds, World Scientific, 1990; P. Di Francesco, Int. J. of Mod. Phys. A7, 92, 407.

[16] P. Di Francesco and J.-B. Zuber; *$SU(N)$ - lattice integrable models associated with graphs*, Nucl. Phys. B338, 602 (1990).

[17] D. Hammaoui and G. Schieber;*In preparation*

[18] P. Etingof, M. Khovanov, *Representations of tensor categories and Dynkin diagrams*, IMRN, No. 5 (1995), 235-247.

[19] P.Etingof, V. Ostrik, *Finite tensor categories*, math.QA/0301027.

[20] J. Fuchs, I. Runkel, C. Schweigert; *TFT construction of RCFT correlators I: Partition functions*, Journal-ref: Nucl.Phys. B646 (2002) 353-497, hep-th/0204148.

[21] T. Gannon; *The classification of affine su(3) modular invariant partition functions*, Commun. Math. Phys. **161** (1994) 233–263, hep-th/9212060; *The classification of SU(3) modular invariants revisited*, Annales de l'Institut Poincaré, Phys. Theor. **65** (1996) 15–55, hep-th/9404185.

[22] D. Hammaoui; *PhD thesis, in preparation, Dec. 2006* , LPTP, Université Mohamed I, Oujda, Maroc.

[23] D. Hammaoui, G. Schieber, E.H. Tahri, *Higher Coxeter graphs associated to affine su*(3) *modular invariants*, J. Phys **A38** (2005) 8259–8286, hep-th/0412102.

[24] E. Isasi and G. schieber; *From modular invariants to graphs: the modular splitting method*, math-ph/0609064.

[25] D. Gepner; *Galois groups in Rational Conformal Field Theory II. The Discriminant*, hepth-th/0608140.

[26] A. Joyal and R. Street; *Braided tensor categories*, Adv. in Maths, Vol. 102, No1, 1993. *Braided monoidal categories*, Macquarie Math. Reports No. 850067, Dec. 1985.

[27] A. Kirillov and V. Ostrik ; *On q-analog of McKay correspondence and ADE classification of SL2 conformal field theories* math.QA/0101219

[28] S. Mac Lane; *Categories for the Working Mathematician*, Graduate Texts in Maths, vol 5, 1971, Springer-Verlag

[29] J. McKay; *Graphs, singularities and finite groups* Proc. Symp. Pure Math. 37, 183 (1980)

[30] A. Malkin, V. Ostrik and M. Vybornov; *Quiver varieties and Lusztig's algebra*, math.RT/0403222

[31] D. Nikshych and L. Vainerman;*Finite quantum groupoïds and their applications*, mathQA/0006057 (2000), in New directions in Hopf algebras, 211–262, Math. Sci. Res. Inst. Publ., 43, Cambridge Univ. Press, Cambridge, 2002.

[32] D. Nikshych, V. Turaev and L. Vainerman; *Invariant of knots and 3-manifolds from quantum groupoids*, Topology and its Applications 127 (2003), no. 1-2, 91–123, math.QA/0006078

[33] A. Ocneanu; *The Classification of subgroups of quantum SU(N)*, Lectures at Bariloche Summer School, Argentina, Jan.2000, AMS Contemporary Mathematics 294, R. Coquereaux, A. García and R. Trinchero eds.

[34] A. Ocneanu; *Unpublished*: *circa* 1996.

[35] A. Ocneanu; *Paths on Coxeter diagrams: from Platonic solids and singularities to minimal models and subfactors.* Notes taken by S. Goto, Fields Institute Monographs, AMS 1999, Rajarama Bhat et al, eds.

[36] V. Ostrik, *Module categories, weak Hopf algebras and modular invariants*, math.QA/0111139. Transform. groups, 2003, vol. 8, no2, pp. 177-206.

[37] V. Pasquier; *Two-dimensional critical systems labelled by Dynkin diagrams* Nucl.Phys. B285, 162 (1987).

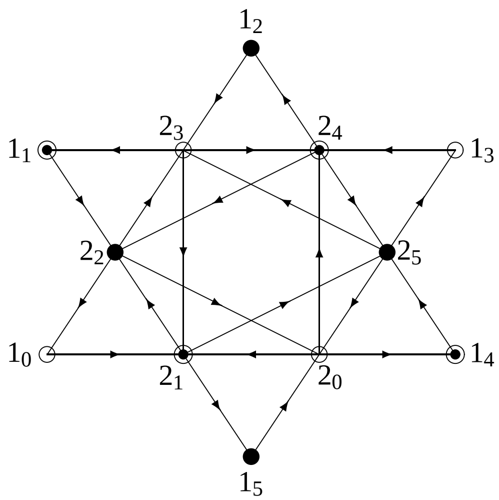
[38] V.B. Petkova and J.B. Zuber; *The many faces of Ocneanu cells*, Nucl. Phys. **B603** (2001) 449–496, hep-th/0101151.

[39] G. Schieber; *L'algèbre des symétries quantiques d'Ocneanu et la classification des systèmes conformes à 2D*, PhD thesis (available in French and in Portuguese), UP (Marseille) and UFRJ (Rio de Janeiro). Sept. 2003.

$$\mathscr{Z} = |[1,1]+[3,3]|^2 + |[1,3]+[4,3]|^2 + |[2,3]+[6,1]|^2 + |[4,1]+[1,4]|^2 + |[3,2]+[1,6]|^2 + |[3,1]+[3,4]|^2$$

with $[a,b] = (a-1, b-1)$.

$$\mathscr{F} = \lambda_{(0,0)} \oplus \lambda_{2,2}$$



q-dim	$\mathscr{E}_5 \hookleftarrow \mathscr{A}_5$
$[1] = 1$	$1_0 \hookleftarrow (0,0),(2,2)$
$[1]$	$1_1 \hookleftarrow (0,2),(3,2)$
$[1]$	$1_2 \hookleftarrow (1,2),(5,0)$
$[1]$	$1_3 \hookleftarrow (3,0),(0,3)$
$[1]$	$1_4 \hookleftarrow (2,1),(0,5)$
$[1]$	$1_5 \hookleftarrow (2,0),(2,3)$
$[3] = 1+\sqrt{2}$	$2_0 \hookleftarrow (1,1),(3,0),(2,2),(1,4)$
$[3]$	$2_1 \hookleftarrow (1,0),(2,1),(1,3),(3,2)$
$[3]$	$2_2 \hookleftarrow (0,1),(1,2),(3,1),(2,3)$
$[3]$	$2_3 \hookleftarrow (1,1),(0,3),(2,2),(4,1)$
$[3]$	$2_4 \hookleftarrow (0,2),(2,1),(4,0),(1,3)$
$[3]$	$2_5 \hookleftarrow (2,0),(1,2),(3,1),(0,4)$

Figure 1: The $\mathscr{E}_5$ graph, quantum dimensions and $\mathscr{E}_5 \hookleftarrow \mathscr{A}_5$ induction rules.

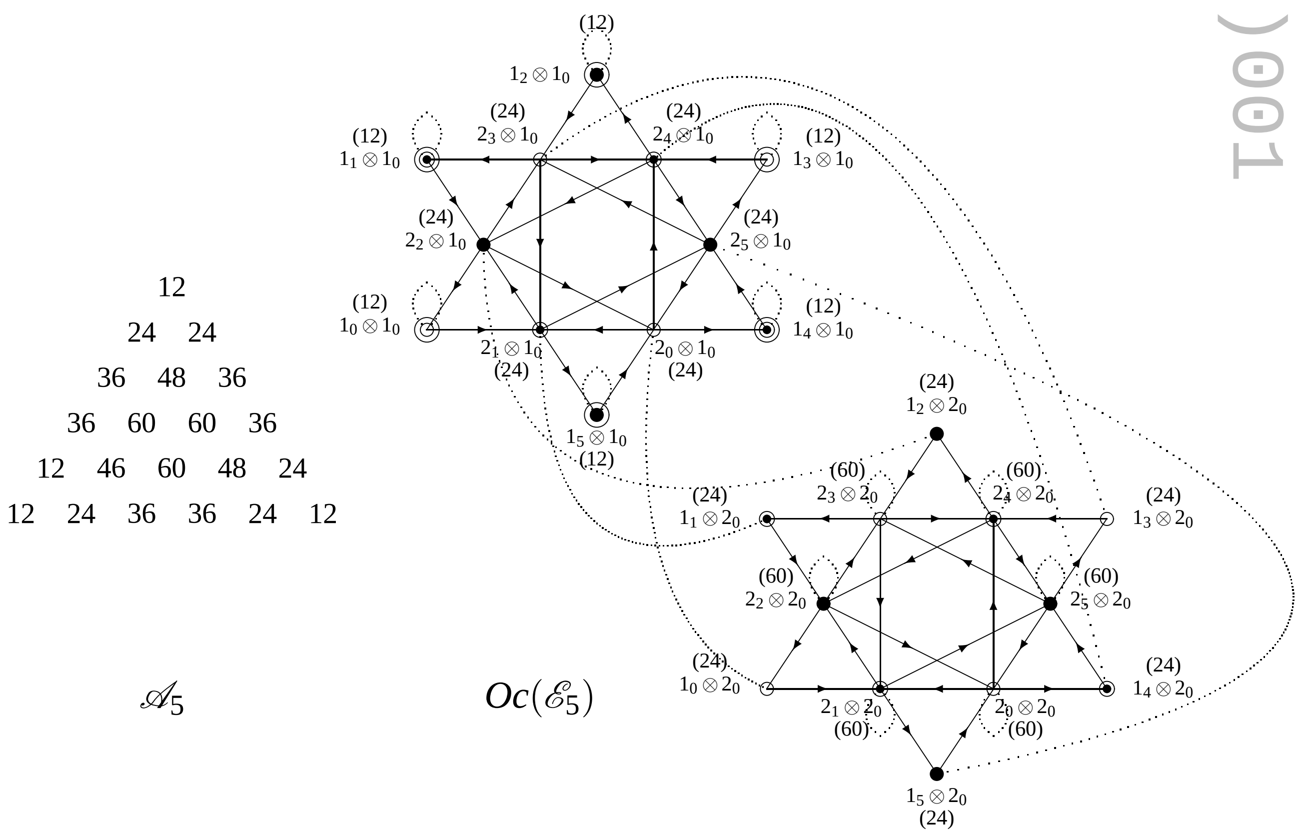
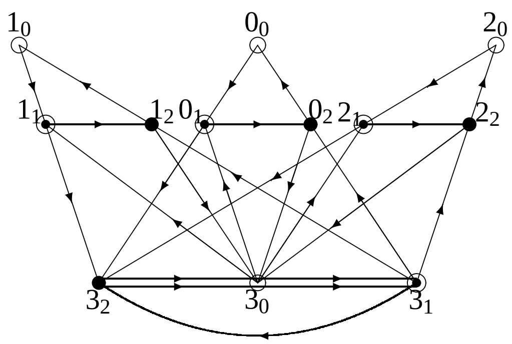


Figure 2: Dimension d_n and d_x of the blocks for $\mathscr{E}_5$.



q-dim	$\mathcal{E}_9$		$\mathcal{A}_9$
$[1]=1$	0_0	$\hookleftarrow$	$(0,0),(4,1),(1,4),(4,4),(9,0),(0,9)$
$[1]$	1_0	$\hookleftarrow$	$(2,2),(5,2),(2,5)$
$[1]$	2_0	$\hookleftarrow$	$(2,2),(5,2),(2,5)$
$3+2\sqrt{3}$	3_0	$\hookleftarrow$	$(1,1),(3,0),(0,3),(6,0),(0,6),(7,1),(1,7),(6,3),(3,6),(2,2)_2,(4,1)_2,(1,4)_2,(5,2)_2,(2,5)_2,(4,4)_2,(3,3)_3$
$[3]=1+\sqrt{3}$	0_1	$\hookleftarrow$	$(1,0),(4,0),(1,3),(3,2),(0,5),(5,1),(2,4),(4,3),(3,5),(0,8),(8,1),(5,4)$
$[3]$	1_1	$\hookleftarrow$	$(2,1),(1,3),(3,2),(5,1),(2,4),(4,3),(1,6),(6,2),(3,5)$
$[3]$	2_1	$\hookleftarrow$	$(2,1),(1,3),(3,2),(5,1),(2,4),(4,3),(1,6),(6,2),(3,5)$
$3+\sqrt{3}$	3_1	$\hookleftarrow$	$(0,2),(2,1),(4,0),(1,3),(0,5),(5,1),(7,0),(1,6),(6,2),(3,5),(5,4),(2,7),(3,2)_2,(2,4)_2,(4,3)_2$
$[3]$	0_2	$\hookleftarrow$	$(0,1),(3,1),(0,4),(5,0),(2,3),(4,2),(1,5),(3,4),(8,0),(5,3),(4,5),(1,8)$
$[3]$	1_2	$\hookleftarrow$	$(1,2),(3,1),(2,3),(4,2),(1,5),(6,1),(3,4),(5,3),(2,6)$
$[3]$	2_2	$\hookleftarrow$	$(1,2),(3,1),(2,3),(4,2),(1,5),(6,1),(3,4),(5,3),(2,6)$
$3+\sqrt{3}$	3_2	$\hookleftarrow$	$(2,0),(1,2),(3,1),(0,4),(5,0),(1,5),(6,1),(0,7),(5,3),(2,6),(7,2),(4,5),(2,3)_2,(4,2)_2,(3,4)_2$

Figure 3: The $\mathcal{E}_9$ graph, quantum dimensions and $\mathcal{E}_9 \hookleftarrow \mathcal{A}_9$ induction rules.

12
26 26
42 60 42
60 94 94 60
68 120 144 120 68
68 132 162 162 132 68
60 120 162 180 162 120 60
42 94 144 162 162 144 94 42
26 60 94 120 132 120 94 60 26
12 26 42 60 68 68 60 42 26 12

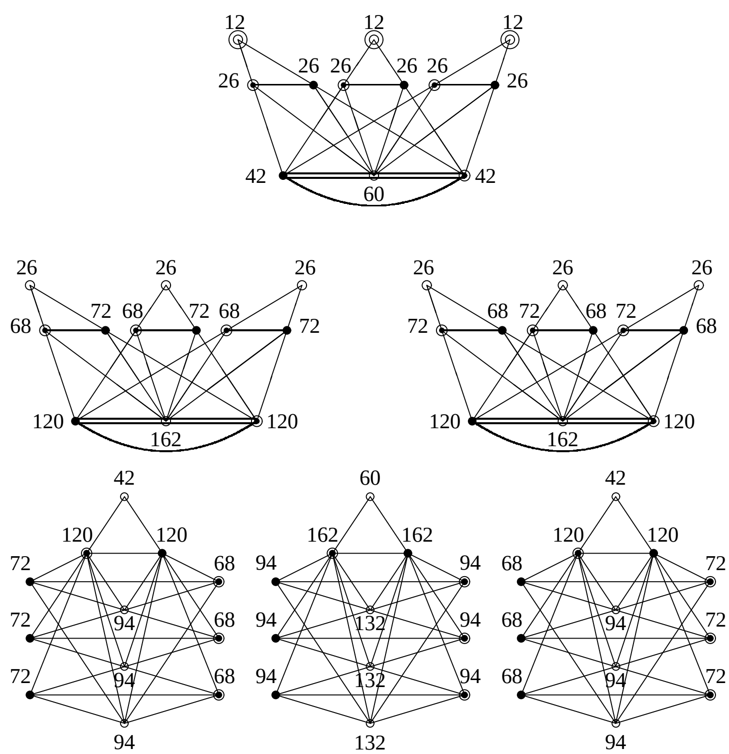
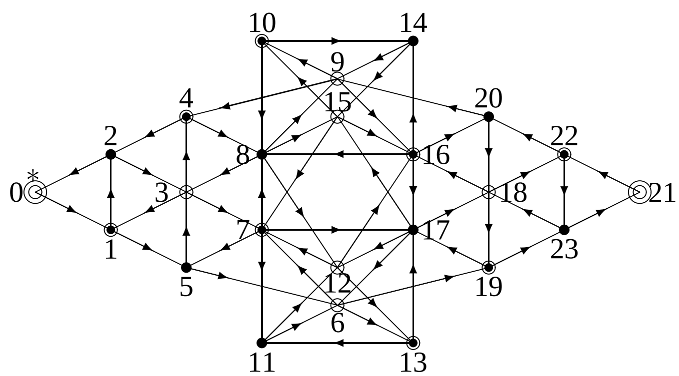


Figure 4: Dimensions d_n and d_x for the $\mathcal{E}_9$ graph.

$$\mathscr{Z} = |[1,1]+[5,5]+[7,7]+[10,10]+[1,22]+[22,1]+[2,11]+[11,2]+[5,14]+[14,5]+[7,10]+[10,7]|^2$$
$$+ \ |[1,7]+[7,1]+[1,16]+[16,1]+[5,8]+[8,5]+[5,11]+[11,5]+[7,16]+[16,7]+[8,11]+[11,8]|^2$$

with [a,b] = (a-1,b-1)



$$\mathscr{F} = \lambda_{(0,0)} \oplus \lambda_{(4,4)} \oplus \lambda_{(6,6)} \oplus \lambda_{(9,9)} \oplus \lambda_{(0,21)} \oplus \lambda_{(21,0)}$$
$$\oplus \ \lambda_{(1,10)} \oplus \lambda_{(10,1)} \oplus \lambda_{(4,13)} \oplus \lambda_{(13,4)} \oplus \lambda_{(6,9)} \oplus \lambda_{(9,6)}$$

$\mathscr{E}_{21} \hookleftarrow \mathscr{A}_{21}$	
$0 \hookleftarrow$	$(0,0),(4,4),(10,1),(1,10),(6,6),(9,6),$ $(6,9),(13,4),(4,13),(10,10),(21,0),(0,21)$
$21 \hookleftarrow$	$(6,0),(0,6),(7,4),(4,7),(10,4),(4,10),$ $(15,0),(0,15),(10,7),(7,10),(15,6),(6,15)$

Figure 5: The $\mathscr{E}_{21}$ graph and $\mathscr{E}_{21} \hookleftarrow \mathscr{A}_{21}$ induction rules (for vertices $\in J$).


```
24
60 60
108 144 108
168 252 252 168
240 384 432 384 240
312 528 636 636 528 312
384 672 852 912 852 672 384
444 804 1056 1188 1188 1056 804 444
492 912 1236 1440 1512 1440 1236 912 492
528 996 1380 1656 1800 1800 1656 1380 996 528
552 1056 1488 1824 2040 2112 2040 1824 1488 1056 552
552 1080 1548 1932 2208 2352 2352 2208 1932 1548 1080 552
528 1056 1548 1968 2292 2496 2568 2496 2292 1968 1548 1056 528
492 996 1488 1932 2292 2544 2676 2676 2544 2292 1932 1488 996 492
444 912 1380 1824 2208 2496 2676 2736 2676 2496 2208 1824 1380 912 444
384 804 1236 1656 2040 2352 2568 2676 2676 2568 2352 2040 1656 1236 804 384
312 672 1056 1440 1800 2112 2352 2496 2544 2496 2352 2112 1800 1440 1056 672 312
240 528 852 1188 1512 1800 2040 2208 2292 2292 2208 2040 1800 1512 1188 852 528 240
168 384 636 912 1188 1440 1656 1824 1932 1968 1932 1824 1656 1440 1188 912 636 384 168
108 252 432 636 852 1056 1236 1380 1488 1548 1548 1488 1380 1236 1056 852 636 432 252 108
60 144 252 384 528 672 804 912 996 1056 1080 1056 996 912 804 672 528 384 252 144 60
24 60 108 168 240 312 384 444 492 528 552 552 528 492 444 384 312 240 168 108 60 24
```

Figure 6: Dimension of the blocks labelled by vertices of the $\mathscr{A}_{21}$ graph for $\mathscr{E}_{21}$.