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Gravitational Droplets and Open–Closed String Duality

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This talk deals with the possibility of describing closed strings via an open string language. I start from an example taken from the AdS/CFT correspondence, in which 1/2 BPS LLM geometries are shown to correspond to open string field theory projectors. Following this example a dictionary is then proposed which translates (off-shell level matched) closed strings into open string field theory (star algebra) projectors. It is shown in particular that under this correspondence a boundary state of closed string theory is mapped into the identity state in open string field theory.

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1. Introduction

Duality between open and closed strings has been a well–known problem since the inception of string theory. The AdS/CFT correspondence has revived the interest on this subject and has offered the possibility to work on a concrete case: AdS/CFT is a kind of limiting case of such duality in which the open string side of the correspondence is represented by a conformal gauge theory. More recently A. Sen, [1], has extended the scope of such duality by suggesting that open string theory might be able to describe all the closed string physics, at least in a background where D–branes are present.

This talk is a review of some research work done in this direction. The idea behind it is to start from some basic clue from AdS/CFT to learn more about the duality suggested by Sen. Witten's open string field theory should be a privileged ground to test this idea. For open string field theory is of course formulated in terms of open strings degrees of freedom, but there is ample evidence that tachyon condensation leads to a new vacuum and that this new vacuum is the closed string one.

The first thing we discuss is in fact (see [5]) a remarkable correspondence (in the context of the AdS/CFT) between N=4 SYM states in 4D and star algebra projectors, or, more appropriately, family of them and star algebra projectors in SFT. After taking a coarse graining limit, the former give rise to the geometry of supergravity solutions (the 1/2 BPS solutions of [7]). Although the correspondence is imperfect due to the lack of supersymmetry on the SFT side, it is very suggestive, because it implies that supergravity solutions can be constructed out of open string bricks. One would therefore expect that closed string modes should be expressible in terms of open string degrees of freedom. Following this logical implication we turn to the task of establishing an explicit relation between open and closed string modes, i.e. a dictionary to translate from the open string to the closed string language.

The result of the analysis carried out in [14] can be summarized as follows: (perturbative) closed string modes are string field theory (SFT) projectors. More precisely: momentum and level–matched off–shell closed string states are in one–to–one correspondence with star algebra projectors in SFT. One very interesting outcome of this proposal is that a boundary state (describing a D–brane in the closed string language) under this correspondence gets translated into the open string identity state.

Recently M. Schnabl [13] has found an exact analytic solution to the SFT equation of motion, which corresponds to a vacuum without perturbative open strings modes and provide a proof of the first two Sen's conjectures. The existence of lower dimensional solutions (third Sen's conjecture) was shown in the past in the context of the vacuum SFT (VSFT) [16], a simplified (and singular) version of Witten's open SFT. We recall that the solutions to the VSFT equation of motion are star algebra projectors (at least for the matter part). In the sequel the basic objects are precisely star algebra projectors. Since the star product is the same in SFT and in VSFT, star algebra projectors are well defined objects in SFT, even without reference to VSFT. This is the sense in which they will be considered here, namely as objects pertaining to SFT. However it is useful to remember that VSFT solutions were interpreted as D–branes and it is not excluded that to any such projector there correspond a SFT solution á la Schnabl.

The talk is organized as follows. Section 2 is devoted to 1/2 BPS droplets. Section 3 is an SFT reminder. Section 4 discusses the correspondence between droplets and SFT projectors. In section 5 I will introduce closed string oscillators in terms of open string ones. In section 6 the zero momentum off–shell closed string states will be introduced. While in section 7 I will show how to endow them with a nonzero momentum. In section 8 I will show a remarkable consistency check of the proposed translation dictionary. Finally, section 9 is devoted to a discussion of the results and of the questions they raise.

2. Half–BPS solutions

In the field theory side of the AdS/CFT correspondence, half–BPS multiplets of $\mathcal{N}=4$ Yang–Mills theory fall into $(0,l,0)$ representations of the $SO(6)$ R–symmetry group. Highest weight states can be constructed as gauge invariant polynomials of a complex scalar field X. The conformal dimension of the latter is $\Delta=1$ and the $U(1)$ R–charge $J=1$, where $U(1)\in SO(6)$. A highest weight therefore satisfies $\Delta=J$. The most general state of this type of charge n takes the form

$$(\mathrm{tr}(X^{l_1}))^{k_1}(\mathrm{tr}(X^{l_2}))^{k_2}\dots(\mathrm{tr}(X^{l_p}))^{k_p} \tag{2.1}$$

where the integers l_i,k_i form a partition of n: $\sum_{i=1}^p l_ik_i=n$. A basis for these states is given by the degree n Schur polynomials of the group $U(N)$. These in turn correspond to Young tableaux of maximal column length N, [2, 8]. Therefore one can classify these highest weight states (chiral primaries) by means of Young diagrams.

It can be shown that they can be represented in another useful way, as a system of N fermionic harmonic oscillators, whose energy levels are $E_i=n_i+\frac{1}{2}$ (with $\hbar=1$), where n_i are nonnegative integers. The ground state corresponds to $n_1=0,n_2=1,\dots n_N=N-1$. Therefore the generic excited state can be represented by means of a Young diagram with rows $(r_1,r_2,\dots,r_N)$, with $r_i=n_i-i+1$ not all vanishing natural numbers in decreasing order. The energy of the state above the Fermi sea is $E=J=\sum_i r_i$, which is the total number of boxes in the Young diagram.

Here is a short list a few states which will be considered in the sequel by means of their Young diagram representation. A giant graviton is represented by a single column Young diagram, whose maximum length is of course N. A giant graviton, [6] is a half–BPS state which can be described as a D3–brane wrapping around an S^3 cycles in the S^5 factor of $AdS_5\times S^5$. A dual giant graviton, i.e. a D3–brane wrapping around an S^3 cycle in AdS_5, is represented by a one–row Young diagram of arbitrary length. A black ring is represented by a large rectangular diagram of size N (see below). A superstar [9] is represented by a large triangular diagram of size $\sim N$. It represents a stack of giant gravitons located at the origin of AdS_5. From the supergravity viewpoint, it is a singular solution in that it has a naked singularity.

In the last two cases the energy of the states is proportional to the area of the Young tableau and therefore $\sim N^2$. Following in particular [3], these are the states we will be mostly interested in in the following.

2.1 1/2–BPS states as supergravity solutions

In [7] a beautiful characterization of 1/2–BPS states in type IIB supergravity was found. Regular 1/2–BPS solutions with a geometry invariant under $SO(4)\times SO(4)\times R$ correspond to the following ansatz

$$ds^2=-h^{-2}(dt+V_idx^i)^2+h^2(dy^2+dx^idx^i)+ye^Gd\Omega_3^2+ye^{-G}d\tilde{\Omega}_3^2$$

$$\begin{aligned} h^{-2} &= 2y\cosh G \\ y\partial_y V_i &= \varepsilon_{ij}\,\partial_j z, \qquad y(\partial_i V_j - \partial_j V_i) = \varepsilon_{ij}\,\partial_y z \\ z &= \frac{1}{2}\tanh G \end{aligned} \tag{2.2}$$

where $i,j = 1,2$ and ε_{ij} is the antisymmetric symbol. There are also N units of 5–form flux, with

$$F_{(5)} = F_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\Omega_3 + \tilde{F}_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\tilde{\Omega}_3$$

where $\mu,\nu = 0,...,3$ refer to t,x^1,x^2,y. As for the ansatz for F and $\tilde{F}$, see [7]. The full solution is determined in terms of a single function z, which must satisfy the equation

$$\partial_i\partial_i z + y\partial_y\left(\frac{\partial_y z}{y}\right) = 0 \tag{2.3}$$

Regular solutions can exist only if at the boundary $y = 0$ the function $z(0,x_1,x_2)$ takes the values $\pm\frac{1}{2}$. Therefore regular solutions correspond to boundary functions $z(0,x_1,x_2)$ that are locally constant in the x_1,x_2 plane. The region of this plane where $z = -1/2$ are called droplets and denoted by $\mathscr{D}$. Now let us insert $\hbar$ back into the game and make the identification $\hbar \leftrightarrow 2\pi\ell_p^4$. It is useful to introduce the new notation $u(0;x_1,x_2) = \frac{1}{2} - z(0;x_1,x_2)$; u is the characteristic function of the droplet, since it equals 1 inside the droplet and 0 outside. Solutions with such (sharp) characteristic functions are regular since the boundary conditions are satisfied. Solutions characterized by a function u which is not exactly 1 or 0, are singular [12]. This is the case of the superstar solution [9].

The area of the droplet must equal N:

$$N = \int \frac{d^2x}{2\pi\hbar}u(0;x_1,x_2) \tag{2.4}$$

while the conformal dimension of the state corresponding to the droplet $\mathscr{D}$ is

$$\Delta = \int \frac{d^2x}{2\pi\hbar}\,\frac{1}{2}\,\frac{x_1^2+x_2^2}{\hbar}u(0;x_1,x_2) - \frac{1}{2}\left(\int \frac{d^2x}{2\pi\hbar}u(0;x_1,x_2)\right)^2 \tag{2.5}$$

In conclusion, the information about the solution is encoded in the droplet. For instance, if the droplet is a disk of radius r_0 we recover the $AdS_5 \times S^5$ solution; if the droplet is the upper half plane one gets the plane wave solution. In general if the droplet is compact the solution is asymptotically $AdS_5 \times S^5$.

2.2 The Wigner distribution

It is clearly of upmost importance to establish a dictionary between the 1/2–BPS states introduced at the beginning of this section starting from $N = 4$ SYM and the droplet solutions. This will give us a recipe to recognize the geometry emerging from a given gauge field theory state. The clue is the free fermion representation introduced above: any state represented by a Young diagram can be interpreted as a system of N fermions with energies above the Fermi sea. The formulas (2.4,2.5) suggest that u be identified with the semiclassical limit of the quantum one-particle

(q,p) phase–space distributions of the free dual fermions after the identification $(x_1,x_2) \leftrightarrow (q,p)$. A well–known distribution is the Wigner one [10]:

$$W(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dy \langle q-y|\hat{\rho}|q+y\rangle e^{2ipy/\hbar} \tag{2.6}$$

where $\hat{\rho}$ is the density matrix. In the case of a pure state ψ, $\langle q'|\hat{\rho}|q''\rangle = \psi(q')\,\psi^\star(q'')$, therefore

$$W(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dy \psi^\star(q+y)\psi(q-y)e^{2ipy/\hbar} \tag{2.7}$$

In general $\hat{\rho}$ will take the form of

$$\hat{\rho}(q',q'') = \sum_{f\in\mathscr{F}} \psi_f(q')\psi_f^\star(q'')$$

$\mathscr{F}$ being a given family of pure states. We will consider family of pure states representing excited states of N (fermionic) harmonic oscillators $f_n = r_n + n - 1$, with $n = 1,\dots,N$ (where we have dropped $\hbar$). In this case $\mathscr{F}$ will be a subset of the natural numbers and

$$\psi_{f_n} = A(f_n)H_{f_n}(q/\sqrt{\hbar})e^{-q^2/2\hbar}$$

where $A(n)$ is a normalization constant and H_n are the Hermite polynomials. Using a well–known integration formula for Hermite polynomials one gets, [10],

$$W(q,p) = \sum_{f_n\in\mathscr{F}} W_{f_n}(q,p) = \frac{1}{2\pi\hbar}e^{-(q^2+p^2)/\hbar}\sum_{f_n\in\mathscr{F}}(-1)^{f_n}L_{f_n}\left(2\frac{q^2+p^2}{\hbar}\right) \tag{2.8}$$

Here we are interested in Wigner distributions because they represent a precise recipe to bosonize associated fermion systems: from the fermion system we easily get the Wigner distribution and from the latter we can reconstruct the former. In the sequel we will use Wigner distributions in this sense, and will be concerned specifically with distributions relative to ensembles, in which N is supposed to be very large. The semiclassical limit will correspond to $\hbar \to 0$ while keeping $\hbar N$ finite. We will use such distributions to make a comparison with the u droplet functions, [3], and with space profiles of SFT projectors.

Let us consider a few significant cases. The first concerns the Fermi sea. The relevant distribution is

$$2\pi\hbar W_{FS} = 2\pi\hbar\sum_{n=0}^{N-1} W_n(q,p) \tag{2.9}$$

A numerical analysis shows that the limit $\hbar \to 0$ with $\hbar N$ fixed reproduces the finite disk characteristic of the latter solution (see, for instance, [11]).

The second example is the case corresponding to a rectangular Young diagram of row length K. It represents N fermions all excited above the sea by the same amount K. We are interested in the limit of large N and K such that $\hbar K$ as well as $\hbar N$ are finite. The Wigner distribution is

$$2\pi\hbar W_{\rm rect} = 2\pi\hbar\sum_{n=K}^{N+K-1} W_{K+n-1}(q,p) \tag{2.10}$$

Setting $u(0,x_1,x_2) = 2\pi\hbar W_{\rm rect}$ identifies a characteristic function which is (approximately) 1 in the ring $\hbar K \leq \frac{q^2+p^2}{2} \leq \hbar(N+K)$ and 0 outside, in the large N and K limit. This corresponds to the 1/2–BPS called "black ring" in [7]. It has conformal dimension $\Delta = NK \sim N^2$, since K must be some multiple of N.

The last example concerns Young diagrams which are approximately triangular with $\Delta = NN_c/2$ and so correspond to superstar ensembles. In this case we have $f_n = (n-1)\delta_n$, with δ_n an integer $\sim \frac{N_c}{N}+1$. For illustrative purposes let us set $\delta_n = \delta = \frac{N_c}{N}+1$. Then

$$2\pi\hbar W_{\rm triangle} = 2\pi\hbar \sum_{n=0}^{N-1} W_{n\delta}(q,p) = 2e^{-\frac{2H}{\hbar}} \sum_{n=0} (-1)^{n\delta} L_{n\delta}(4H/\hbar) \tag{2.11}$$

where $H = (q^2+p^2)/2$. In the large N limit

$$2\pi\hbar W^{\infty}_{\rm triangle} = \frac{1}{\delta} + oscillations\, at\, scale\, \Delta H = \hbar$$

Therefore identifying once again $2\pi\hbar W^{\infty}_{\rm triangle}$ with $u(0;x_1,x_2)$ we get approximately $u(0,x_1,x_2) = 1/\delta$ within a finite radius disk. This corresponds to a fractionally filled droplet and represents the superstar solution, which, as we pointed out, is singular.

3. SFT: a reminder

In this section we will introduce string fields in SFT that mimic the behavior of the fermionic systems and the relevant half-BPS Wigner distributions discussed in the previous section.

The SFT action is

$$\mathscr{S}(\Psi) = -\left(\frac{1}{2}\langle\Psi|Q_B|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi*\Psi\rangle\right) \tag{3.1}$$

where Q_B is the open string BRST charge.

The string state we are looking for are star–algebra projectors, that is states that satisfy the equation

$$\Psi * \Psi = \Psi$$

Since the star product factorizes into matter and ghost part it is natural to make for projectors the following factorized ansatz

$$\Psi = \Psi_m \otimes \Psi_g \tag{3.2}$$

where Ψ_g and Ψ_m depend purely on ghost and matter degrees of freedom, respectively. The projector equation splits into

$$\Psi_g = \Psi_g *_g \Psi_g \tag{3.3}$$

$$\Psi_m = \Psi_m *_m \Psi_m \tag{3.4}$$

where $*_g$ and $*_m$ refer to the star product involving only the ghost and matter part. Solutions to (3.4) have been studied in the past as VSFT solutions (and interpreted as D–branes).

In the rest of the paper we will concentrate on the matter part, eq.(3.4), and forget about the ghost part (see [14]). The $*_m$ product in the operator formalism is defined as follows

$$ {}_{123}\langle V_3|\Psi_1\rangle_1|\Psi_2\rangle_2 = {}_3\langle \Psi_1 *_m \Psi_2|, \tag{3.5} $$

where ${}_{123}\langle V_3|$ is the three strings vertex. The basic ingredient in this definition are the matrices of vertex coefficients V^{rs}_{nm}, $r,s = 1,2,3,\ n,m = 1,\ldots,\infty$.

As it turns out the projectors we need must be superpositions of matter projectors (stacks of VSFT solutions) with the following characteristics: they must cover the ordinary 4D Minkowski space (parallel directions), be, in the low energy limit ($\alpha' \to 0$), delta–function like in 16 directions and have some width in the remaining 6 directions (these 22 directions will be referred to as the transversal ones). Out of the latter two will have a special status, in that a constant B field will be switched on along them. We can imagine that all the internal dimensions are compactified on tori, but this is not strictly necessary for our argument. In the remaining part of this section we will review the construction of such projectors.

In the following we need both translationally invariant (in the parallel directions) and non-translationally invariant projectors (in the transverse directions).

Although there is a great variety of such projectors we will stick to those introduced in [16], i.e. the sliver and the lump. The former is translationally invariant and is defined by

$$ |\Xi\rangle = \mathcal{N} e^{-\frac{1}{2}a^\dagger\cdot S\cdot a^\dagger}|0\rangle, \qquad a^\dagger\cdot S\cdot a^\dagger = \sum_{n,m=1}^{\infty} a_n^{\mu\dagger} S_{nm} a_m^{\nu\dagger}\eta_{\mu\nu} \tag{3.6} $$

where $S = CT$ and

$$ T = \frac{1}{2X}\left(1+X-\sqrt{(1+3X)(1-X)}\right) \tag{3.7} $$

with $X = CV^{11}$, where $C_{nm} = (-1)^n\delta_{nm}$ is the so–called twist matrix. $\mathcal{N}$ is a normalization constant.

The lump projector was engineered to represent a lower dimensional brane (Dk-brane) in VSFT, therefore it will have $(25-k)$ transverse space directions along which translational invariance is broken. Accordingly we split the three string vertex into the tensor product of the perpendicular part and the parallel part

$$ |V_3\rangle = |V_{3,\perp}\rangle \otimes |V_{3,_\parallel}\rangle \tag{3.8} $$

The parallel part is the same as in the sliver case while the perpendicular part is modified as follows. Following [16], we denote by $x^{\bar\mu}, p^{\bar\mu}$, $\bar\mu = 1,\ldots,k$ the coordinates and momenta in the transverse directions and introduce the canonical zero modes oscillators

$$ a_0^{(r)\bar\mu} = \frac{1}{2}\sqrt{b}\hat p^{(r)\bar\mu} - i\frac{1}{\sqrt{b}}\hat x^{(r)\bar\mu}, \qquad a_0^{(r)\bar\mu\dagger} = \frac{1}{2}\sqrt{b}\hat p^{(r)\bar\mu} + i\frac{1}{\sqrt{b}}\hat x^{(r)\bar\mu}, \tag{3.9} $$

where b is a free parameter. Denoting by $|\Omega_b\rangle$ the oscillator vacuum ($a_0^{\bar\mu}|\Omega_b\rangle = 0$), in this new basis the three strings vertex is given by

$$ |V_{3,\perp}\rangle' = K e^{-E'}|\Omega_b\rangle \tag{3.10} $$

K being a suitable constant and

$$E' = \frac{1}{2}\sum_{r,s=1}^{3}\sum_{M,N\geq 0} a_M^{(r)\bar{\mu}\dagger} V'^{rs}_{MN} a_N^{(s)\bar{\nu}\dagger}\eta_{\bar{\mu}\bar{\nu}} \tag{3.11}$$

where M,N denote the couple of indexes $\{0,m\}$ and $\{0,n\}$, respectively. The coefficients V'^{rs}_{MN} are given in Appendix B of [16]. The new Neumann coefficients matrices V'^{rs} satisfy the same relations as the V^{rs} ones. In particular one can introduce the matrices $X'^{rs} = CV'^{rs}$, where $C_{NM} = (-1)^N\delta_{NM}$. The lump solution $|\Xi'_k\rangle$ has the form (3.6) with S along the parallel directions, while $|0\rangle$ is replaced by $|\Omega_b\rangle$ and S is replaced by S' along the perpendicular ones. Here $S' = CT'$ and T' has the same form as T eq.(3.7) with X replaced by X'. It can be verified that the ratio of tensions for such projectors is the appropriate one for Dk–branes. For our basic projector we will choose $k = 22$.

As said above, two of the transverse directions are special, in that a constant background B field is switched on there. We denote these two directions by the labels α and β (for instance $\alpha,\beta = 24,25$) and denote them simply by y_1,y_2; we take for B the explicit form

$$B_{\alpha\beta} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} \tag{3.12}$$

Then, as is well–known, in these two directions we have a new effective metric $G_{\alpha\beta}$, the open string metric, as well as an effective antisymmetric parameter $\theta_{\alpha\beta}$, given by

$$G_{\alpha\beta} = \sqrt{\mathrm{Det}G}\,\delta_{\alpha\beta}, \qquad \theta^{\alpha\beta} = \varepsilon^{\alpha\beta}\theta \tag{3.13}$$

where until further notice we set $\alpha' = 1$ and $\mathrm{Det}G = \big(1+(2\pi B)^2\big)^2$. In (3.13) $\varepsilon^{\alpha\beta}$ represents the 2×2 antisymmetric symbol with $\varepsilon^1{}_2 = 1$. The transverse vertex $|V_{3,\perp}\rangle$ will become in this case $|V'_{3,\perp}\rangle$

$$|V'_{3,\perp}\rangle = |V_{3,\perp,\theta}\rangle \otimes |V_{3,\perp}\rangle \tag{3.14}$$

where

$$|V'_{3,\perp,\theta}\rangle = K_\theta\, e^{-E_\theta}|\Omega_b\rangle \tag{3.15}$$

K_θ is a suitable constant and, [22, 15],

$$E_\theta = \frac{1}{2}\sum_{r,s=1}^{3}\sum_{M,N\geq 0} a_M^{(r)\alpha\dagger}\mathcal{V}^{rs}_{\alpha\beta,MN} a_N^{(s)\beta\dagger} \tag{3.16}$$

The coefficients $\mathcal{V}^{\alpha\beta,rs}_{MN}$ are given in [22]. The new Neumann coefficients matrices $\mathcal{V}^{rs}$ satisfy the same relations as the V^{rs} ones. If one introduces the matrices $\mathcal{X}^{rs} = C\mathcal{V}^{rs}$, then the lump solution $|\mathcal{S}\rangle$ along α and β has the form (3.6) with $|0\rangle$ replaced by $|\Omega_b\rangle$ and S replaced by $\mathcal{S}$, where $\mathcal{S} = C\mathcal{T}$ and $\mathcal{T}$ has the same form as T in eq.(3.7) with X replaced by $\mathcal{X}$. It can be verified that the ratio of tensions for such solutions is the appropriate one for D–branes in a magnetic field, [22].

It is possible to construct a full family of such solutions which are $*$– and bpz–orthonormal. This goes as follows, [22, 15]. First we introduce two 'vectors' $\xi = \{\xi_{N\alpha}\}$ and $\zeta = \{\zeta_{N\alpha}\}$, which are chosen to satisfy the conditions

$$\rho_R\xi = 0, \qquad \rho_L\xi = \xi, \qquad \text{and} \quad \rho_R\zeta = 0, \qquad \rho_L\zeta = \zeta, \tag{3.17}$$

where ρ_L, ρ_R are the half–string projectors [17]. Moreover we define the matrix τ by means of $\tau = \{\tau_\alpha{}^\beta\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Next we set

$$\mathrm{x} = (a^\dagger \tau \xi)(a^\dagger C \zeta) = (a_N^{\alpha\dagger} \tau_\alpha{}^\beta \xi_{N\beta})(a_N^{\alpha\dagger} C_{NM} \zeta_{M\alpha}) \tag{3.18}$$

Finally we introduce the Laguerre polynomials $L_n(z)$, of the generic variable z, and define the sequence of states

$$|\Lambda_n\rangle = (-\kappa)^n L_n\left(\frac{\mathrm{x}}{\kappa}\right)|\mathcal{S}_{\perp\theta}\rangle \tag{3.19}$$

where, for simplicity, we have written down the tensorial factor involving the the y_1, y_2 directions only and understood the other directions. As part of the definition of $|\Lambda_n\rangle$ we require the two following conditions to be satisfied

$$\xi^T \tau \frac{1}{\mathbb{I} - \mathcal{T}^2} \zeta = 1, \qquad \xi^T \tau \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} \zeta = \kappa \tag{3.20}$$

where κ is a real number. To guarantee Hermiticity for $|\Lambda_n\rangle$, we require $\zeta = \tau \xi^*$.

The states $|\Lambda_n\rangle$ satisfy the remarkable property

$$|\Lambda_n\rangle * |\Lambda_m\rangle = \delta_{n,m}|\Lambda_n\rangle \tag{3.21}$$

$$\langle\Lambda_n|\Lambda_m\rangle = \delta_{n,m}\langle\Lambda_0|\Lambda_0\rangle \tag{3.22}$$

Therefore each Λ_n, as well as any combination of Λ_n with unit coefficients, are lump projectors.

So far we have set $\alpha' = 1$. It is easy to insert back α'. In order to evaluate the low energy profile of $|\Lambda_n\rangle$ we first contract it with the eigenstate of the position operators with eigenvalues y^α: $\langle y|\Lambda_n\rangle$, and then take the limit $\alpha' \to 0$, [22]. The leading term in the α' expansion turns out to be

$$\langle y|\Lambda_n\rangle = \frac{1}{\pi}(-1)^n L_n\left(\frac{2r^2}{\theta}\right) e^{-\frac{r^2}{\theta}}|\Xi\rangle + \mathscr{O}(\sqrt{\alpha'}) \tag{3.23}$$

where $r^2 = y^\alpha y^\beta \delta_{\alpha\beta}$ and $|\Xi\rangle$ is the sliver solution.

The projectors we need in the following have this $\alpha' \to 0$ limit in the y^α directions; as for the remaining directions, they have the form of the sliver in the parallel directions and, finally, they become delta–like functions multiplied by the sliver in the remaining transverse directions, i.e. they are localized at the origin of the latter. This can be easily seen by taking the limit $\theta \to 0$ in the case $n = 0$ in (3.23).

4. Wigner distributions–projectors correspondence

Looking at eqs.(2.9,2.10,2.11) of section 2, one immediately notices that they can be seen (up to an overall normalization constant) as the low energy limit space profiles of combinations of the string states Λ_n introduced in the previous section, with unit coefficients. Since combinations of Λ_n with unit coefficients are also projectors, one can view the above Wigner distributions as the low energy profile of VSFT solutions (up to the common $|\Xi\rangle$ factor). It is therefore tantalizing to make the following association

Wigner distribution for an N fermion system $\leftrightarrow$ SFT projector

For this to work we must require the correspondence $\hbar \leftrightarrow \theta$ and that the coordinates x_1, x_2 be identified with y_1, y_2.

Let us delve into this correspondence. It can be read in two directions. First: one can say that to any 1/2–BPS state to which we can associate a Wigner distribution of the type (2.6), there corresponds a SFT projector given by a combination

$$|W\rangle = \sum_{f_n \in \mathscr{F}} |\Lambda_{f_n}\rangle, \qquad 2\hbar W(q,p)|\Xi\rangle = \langle y|W\rangle \tag{4.1}$$

where (p,q) is identified with (y_1, y_2) and the latter are the eigenvalues of $|y\rangle$. Vice versa: to any SFT projector of the type (4.1) we can associate a Wigner distribution $W(q,p)$ according to (2.6). In this way we can associate to $|W\rangle$ a Young tableau and therefore a 1/2–BPS state in the $\mathscr{N}=4$ superconformal field theory (before taking the large N limit) and we can associate a geometry (after taking it[1]). The latter point implies that we may be able to associate a geometry to a given SFT projector. This immediately leads to open–closed string duality, since we see geometry emerging from a SFT projector which is entirely expressed in terms of open string creation operators.

In the following we would like to list some arguments in support of this correspondence.

1) With the above association we connect a microstate corresponding to a given geometry, i.e. a given supergravity solution, to a string state which is a SFT projector or a solution to the VSFT equation of motion. The correspondence (4.1) is one–to–one.

2) The droplet geometry lives in a (x_1, x_2) plane which lies in the internal (compactified) dimensions. In the same way the plane (y_1, y_2) lies in the compactified part of the bosonic target space. As pointed out above, we identify the two planes. One could phrase it by saying that the two space coordinates x_1, x_2, which had been replaced by two phase–space coordinates q, p in the intermediate argument, have returned to their natural role via the identification with y_1, y_2.

3) The correspondence (4.1) tells us how the Pauli principle gets incorporated into a bosonic setting. The numbers f_n in the LHS of (4.1) correspond to the fermion energy levels in the original fermion system. Therefore, due to the Pauli exclusion principle, each f_n can appear only once in the family $\mathscr{F}$. Therefore in the summation each $|\Lambda_{f_n}\rangle$ appears only once. This guarantees that $|W\rangle$ is a SFT projector since, for any $*$–projector$|\Lambda\rangle$, $n|\Lambda\rangle$ is a $*$–projector if and only if $n = 0, 1$. On the other hand any SFT projector that can be written in the form $\sum_{f_n \in \mathscr{F}} |\Lambda_{f_n}\rangle$ tells us that the numbers $f_n \in \mathscr{F}$ can be interpreted as energy levels of a fermionic harmonic oscillator system, since each appears only once.

4) The VSFT solution corresponding to the Fermi sea (2.9) is represented by a stack of N projectors. The giant graviton solution is represented by one missing from the stack. The superstar solution (2.11) is represented by a stack of such missing projectors. This is in keeping with the interpretation of superstars as stack of giant gravitons, [9].

5) There is an algebra isomorphism between Wigner distributions of the type (2.6) and SFT projectors like $|W\rangle$, an isomorphism that was pointed out in [26, 27, 22]. It is a well–known fact that any classical function $f(q,p)$ in a (q,p) phase space can be mapped to a quantum operator $\hat{O}_f$ via the Weyl transform, and that the product for quantum operators $\hat{O}_f \hat{O}_g$ is mapped into the Moyal product $f \star g$ for functions. Under this correspondence the $(x_1, x_2) \leftrightarrow (q,p)$ plane becomes

[1] In the process of taking the large N limit one smears out many details, so that multiple states are mapped to the same geometry

noncommutative. It is a well–known fact that, under this correspondence the classical Wigner distributions like (2.9,2.10,2.11) are mapped into projectors of the Moyal star–algebra:

$$(2\pi\hbar W)\star(2\pi\hbar W)=2\pi\hbar W. \tag{4.2}$$

Now, it was shown in [26, 27] that in the low energy limit, the SFT star product factorizes into Witten's star product and the Moyal $\star$ product. Due to this factorization, the correspondence (4.1) is in fact a star–algebra isomorphism

$$|W\rangle * |W\rangle \longleftrightarrow (2\pi\hbar W)\star(2\pi\hbar W) \tag{4.3}$$

This remark should be adequately appreciated. Let us consider $2\pi\hbar W$ and suppose it is such that we can ignore its derivatives with respect to p and q. Then eq.(4.2) becomes $(2\pi\hbar W)^2=2\pi\hbar W$, which is the equation of a characteristic function (it can only be either 0 or 1). This is indeed what happens in the case of the vacuum and the black ring solutions, see [3, 4]. It is not the case of the superstar distribution because in that case we cannot ignore derivatives. But this remark suggests that the property of being Moyal projectors is basic for Wigner distributions to represent 1/2–BPS states. The string state $|W\rangle$ 'inherits' this property. It is the 'continuation' of the space profile to the whole string theory. In this sense it is natural that $|W\rangle$ correspond to a string field theory projector. It should be remarked that on the SFT side noncommutativity is produced by the B field, which mimics the role of the five–form flux (the latter being of course absent in the bosonic case).

The correspondence (4.1) must be thought as embedded in the following table:

$$\begin{array}{ccc}
\boxed{\begin{array}{l}\mathcal{N}=4\ \text{U(N) SYM}\\ \text{chiral primaries,}\\ \text{Young tableaux}\end{array}} & & \\
\Updownarrow & & \\
\boxed{\begin{array}{l}\text{N fermion systems}\\ \text{of harmonic oscillators,}\\ \text{Young tableaux}\end{array}} & & \\
\Updownarrow & & \\
\boxed{\begin{array}{l}\text{Wigner distributions}\\ \text{Young tableaux}\end{array}} & \Longleftrightarrow & \boxed{\begin{array}{l}\text{VSFT solutions:}\\ \text{sum of } *\text{–projectors,}\\ \text{Young tableaux}\end{array}} \\
\downarrow & & \downarrow \\
\boxed{\begin{array}{l}\text{Half BPS IIB}\\ \text{SUGRA solutions}\end{array}} & \longleftrightarrow & \boxed{\text{Singular gravity solutions (?)}}
\end{array}$$

where double–line arrows represent one–to–one correspondences, simple down arrows represent the large N limit and the question mark indicates the conjectural part of our proposed correspondence.

The fact that $|W\rangle$ is a SFT projector is the strongest support of our conjectured correspondence. The weak point is that we know it is a solution of bosonic SFT projector but we do not know whether it corresponds to a supersymmetric vacuum string field theory. However it is not unconceivable that the bosonic part of 1/2–BPS states be well described by bosonic string field theory projectors. Unfortunately the study of the tachyon condensation in superstring field theory has not progressed much, see [20, 21]. From what we know nowadays it is possible that the bosonic parts of some supersymmetric SFT projectors take a form like $|W\rangle$, although a satisfactory answer is not yet at hand.

5. Closed string theory oscillators

Although the correspondence outlined in the previous sections is imperfect due to the lack of supersymmetry on the SFT side, it is very suggestive, because it implies that supergravity solutions can be constructed out of open string bricks. One is thus led to expect that closed string modes are expressible in terms of open string degrees of freedom. In the rest of this paper we would like to show how this conjecture could come true. The construction that follows is based on split string field theory and the so–called comma vertex algebra, see [18] and references therein.

Let us consider the translationally invariant sliver projector, represented by the matrix S along all directions. and introduce, as above, the projectors ρ_L and ρ_R:

$$\rho_L^2 = \rho_L, \qquad \rho_R^2 = \rho_R, \qquad \rho_L + \rho_R = 1 \tag{5.1}$$

which project onto the left and right hand part of the string, respectively. Then we define the operators

$$s^\mu = \omega(a^\mu + Sa^{\mu\,\dagger}) = (a^\mu + Sa^{\mu\,\dagger})\omega, \qquad \omega = \frac{1}{\sqrt{1-S^2}} \tag{5.2}$$

and the conjugate ones, where the labels n,m running from 1 to $+\infty$ are understood. Using the algebra of open string creation and annihilation operators these operators can be shown to satisfy

$$[s^\mu_m, s^{\nu^\dagger}_n] = \delta_{nm}\eta^{\mu\nu} \tag{5.3}$$

while the other commutators vanish. Moreover, understanding the Lorentz indexes,

$$s_n|\Xi\rangle = \mathcal{N} e^{-\frac{1}{2}a^\dagger S a^\dagger}\omega(a - Sa^\dagger + Sa^\dagger)|0\rangle = 0 \tag{5.4}$$

Therefore the combinations s_n represent Bogoliubov transformations, which map the Fock space based on the initial vacuum $|0\rangle$ to a new Fock space in which the role of vacuum is played by the sliver string field.

Now we introduce the vector ξ such that

$$\rho_L\xi = \xi, \qquad \rho_R\xi = 0 \tag{5.5}$$

As a consequence

$$\rho_R C\xi = C\xi, \qquad \rho_L C\xi = 0$$

There exists a complete basis ξ_n ($n = 1,2,...$), which satisfy these conditions and are orthonormal in the sense that

$$\langle\xi_n|\frac{1}{1-T^2}|\xi_m\rangle = \delta_{nm} \tag{5.6}$$

see, for instance, [24]. We can project ξ_n on the continuous basis $|k\rangle$, [19], and get the complete basis of functions $\xi_n(k) = \langle \xi_n | k\rangle$,

Let us define, for any ξ,

$$\xi^L = \frac{1}{\sqrt{1-S^2}}\xi, \qquad \xi^R = -\frac{1}{\sqrt{1-S^2}}C\xi \tag{5.7}$$

In this way we have two complementary bases ξ_n^L and ξ_n^R. They are complementary in the sense that

$$\sum_{n=1}^{\infty}\left(\xi_n^L(k)\xi_n^L(k') + \xi_n^R(k)\xi_n^R(k')\right) = \delta(k,k') \tag{5.8}$$

Notice that

$$\xi_n^R(k) = \xi_n^L(-k), \qquad \text{while} \qquad \xi_n^R(-k) = \xi_n^L(k) = 0. \tag{5.9}$$

We can project ξ_n^L and ξ_n^R on the ordinary $\mathrm{v}_n(k)$ basis of eigenvectors of the continuous spectrum, of X, [19], and define the coefficients

$$b_{nl} = \langle \xi_n^L | \mathrm{v}_l\rangle, \qquad \tilde{b}_{nl} = \langle \xi_n^R | \mathrm{v}_l\rangle$$

Using the latter we can introduce

$$\beta_m^\mu = \sum_{l=1}^{\infty} b_{ml} s_l^\mu, \qquad \tilde{\beta}_m^\mu = -\sum_{l=1}^{\infty} \tilde{b}_{ml} s_l^\mu \tag{5.10}$$

with the respective hermitian conjugates. The reason for the minus sign in the second definition above will become clear shortly. These operators satisfy the algebra

$$[\beta_m^\mu, \beta_n^{\nu\dagger}] = \delta_{m,n}\eta^{\mu\nu}$$
$$[\tilde{\beta}_m^\mu, \tilde{\beta}_n^{\nu\dagger}] = \delta_{m,n}\eta^{\mu\nu}$$

while all the other commutators vanish. It must be remarked that the definition of $\beta_n, \tilde{\beta}_n$ depends on the ξ_n basis we use. This entails an $O(\infty)$ 'gauge' freedom in the choice of these operators.

These $\beta, \tilde{\beta}$ operators are natural candidates as closed string creation and annihilation operators. For the same reason it is natural to interpret the sliver $|\Xi\rangle$ as the closed string vacuum $|0_c\rangle$.

6. Zero momentum closed string states

Let us consider the states we can construct by operating with $\beta^\dagger$ and $\tilde{\beta}^\dagger$ on $|\Xi\rangle$. The most general ones are defined as follows. Let us define sequences of natural numbers $\mathbf{n} \equiv (n_1, n_2, \ldots)$, where the label $l = 1, 2, \ldots$ in n_l corresponds to the oscillator type. For every type l half string oscillator we will have a collection of symmetric Lorentz indexes $\mu_1^l, \mu_2^l, \ldots, \mu_{n_l}^l$. Then for any two sequences $\mathbf{n}$ and $\mathbf{m}$ we define the states:

$$\Lambda^{\{\mu_1\ldots\mu_{\mathbf{n}}\},\{\nu_1\ldots\nu_{\mathbf{m}}\}} = \prod_{l,r=1}^{\infty}\frac{(-1)^{m_r}}{\sqrt{n_l! m_r!}}\beta_l^{\mu_1^l\dagger}\ldots\beta_l^{\mu_{n_l}^l\dagger}\tilde{\beta}_r^{\nu_1^r\dagger}\ldots\tilde{\beta}_r^{\nu_{m_r}^r\dagger}|\Xi\rangle \tag{6.1}$$

Their star product is given by

$$\Lambda^{\{\mu_1^l\ldots\mu_{n_l}^l\},\{\nu_1^l\ldots\nu_{m_l}^l\}} * \Lambda_{\{\rho_1^l\ldots\rho_{p_l}^l\}}^{\{\sigma_1^l\ldots\sigma_{q_l}^l\}} = \prod_l \delta_{m_l,p_l}\, \hat{\delta}_{\rho_1^l\ldots\rho_{p_l}^l}^{\nu_1^l\ldots\nu_{m_l}^l}\, \Lambda^{\{\mu_1^l\ldots\mu_{n_l}^l\},\{\sigma_1^l\ldots\sigma_{q_l}^l\}} \tag{6.2}$$

where we have used the symmetrized delta

$$\hat{\delta}_{\nu_1\ldots\nu_n}^{\mu_1\ldots\mu_n} = \frac{1}{n!}\sum_{\sigma(1\ldots n)} \delta_{\nu_1}^{\mu_{\sigma(1)}}\ldots\,\delta_{\nu_n}^{\mu_{\sigma(n)}}$$

Among these states there are some that are already familiar to us. Let us consider, for instance,

$$\frac{1}{\sqrt{n!m!}}(\beta_k^\dagger)^n(-\tilde{\beta}_l^\dagger)^m|0_c\rangle$$

where, for simplicity, we have dropped the Lorentz index μ. Written out explicitly they take the form

$$(\beta_k^\dagger)^n(-\tilde{\beta}_l^\dagger)^m|0_c\rangle = \langle\xi_l C\omega^2(a^\dagger + Sa)\rangle^m\langle\xi_k\omega^2(a^\dagger + Sa)\rangle^n e^{-\frac{1}{2}a^\dagger S a^\dagger}|0\rangle$$

After some algebra they can be shown to give rise to the identities

$$\frac{1}{\sqrt{n!m!}}(\beta_k^\dagger)^n(-\tilde{\beta}_l^\dagger)^m|0_c\rangle = \sqrt{\frac{n!}{m!}}(\kappa_{kl})^n Y_l^{m-n} L_n^{m-n}(-\frac{X_k Y_l}{\kappa_{kl}})|\Xi\rangle \tag{6.3}$$

where

$$\kappa_{kl} = \langle\xi_k\frac{T}{1-T^2}\xi_l\rangle \tag{6.4}$$

For $n = m$ and $k = l$ this type of states have already appeared in section 3. In the literature they have been interpreted as D–brane solutions of vacuum SFT, [22, 23].

A very interesting property of this class of states is that they give rise to the identity

$$\sum_n \beta_n^{\mu\dagger}\tilde{\beta}_n^{\nu\dagger}\eta_{\mu\nu} = \sum_n\langle s^{\mu\dagger}|\xi_n^L\rangle\langle\xi_n^L|Cs^{\nu\dagger}\rangle\eta_{\mu\nu} = \frac{1}{2}\sum_{k=1}^{\infty} s_k^{\mu\dagger}C_{kl}s_l^{\nu\dagger}\eta_{\mu\nu}$$

The factor of $\frac{1}{2}$ comes from the fact that ξ_n is a complete basis for the left ξ's. We have to consider also the other half made of $C\xi_n$, which gives the same contribution, see (5.8). Hence the factor of $\frac{1}{2}$. The - signs come from the definition (5.10) and from the property (5.9).

Exponentiating the above identity and applying it to the closed string vacuum we get

$$e^{-\sum_n\beta_n^{\mu\dagger}\tilde{\beta}_n^{\nu\dagger}\eta_{\mu\nu}}|0_c\rangle = e^{-\frac{1}{2}\sum_{k=1}^{\infty}s_k^{\mu\dagger}C_{kl}s_l^{\nu\dagger}\eta_{\mu\nu}}|\Xi\rangle \sim e^{-\frac{1}{2}\sum_{k=1}^{\infty}a_k^{\mu\dagger}C_{kl}a_l^{\nu\dagger}\eta_{\mu\nu}}|0\rangle \tag{6.5}$$

where $|0\rangle$ is the original open string vacuum. The last step of the proof can be found for instance in [20], the equality holds up to a constant.

The LHS is proportional to the nonzero mode part of the boundary state in closed string theory, the right hand side is the identity state in open string field theory. An interpretation of this identity will be presented below.

What we have seen so far is enough to motivate our interest in the $\beta, \tilde{\beta}$ operators and the states we can construct with them. Starting from the isomorphism between star–algebra operators and closed string creation and annihilation operators, the relevant question is now: what are the (open) string fields that correspond to closed string Fock states created under the above correspondence? By closed string states we mean both off-shell and on-shell states. For instance a graviton state with momentum k in closed string theory is given by

$$\theta_{\mu\nu}\alpha_1^{\mu\dagger}\alpha_1^{\nu\dagger}|0_c,k\rangle \tag{6.6}$$

where $|0_c,k\rangle$ is the closed string vacuum with momentum k, and $\theta_{\mu\nu}$ is the polarization. This state is on-shell when $k^2=0$ and $\theta_{\mu\nu}k^\nu=\theta_{\mu\nu}k^\mu=0$. When the latter conditions are not satisfied the graviton is off-shell. Off-shell states must have definite momentum (i.e. the left and right momenta must be equal) and they must be level–matched. Usually in dealing with closed strings, these two conditions are so obvious that they are understood, but, as we shall see, under the correspondence with open strings, they become significant and select a very precise class of string fields, the projectors. In the present section, to start with, we consider only zero momentum *off–shell* states. Non-zero momentum states will be introduced in the next section.

It is evident from the above that there is a correspondence between (zero momentum) states in the Fock space of the closed string theory and open string fields of the type (6.1). The question is: what are the string fields that correspond to off–shell states in the closed string theory?

Let us (operationally) define Virasoro generators $L_n,\tilde{L}_n$ using the $\beta,\tilde{\beta}$ operators in the usual way. Then using L_0 and $\tilde{L}_0$ we define the mass operator and the level matching condition by means of

$$N_L=\sum_{n=1}^{\infty} n\,\beta_n^\dagger\cdot\beta_n, \qquad N_R=\sum_{n=1}^{\infty} n\,\tilde{\beta}_n^\dagger\cdot\tilde{\beta}_n, \tag{6.7}$$

Off-shell states are characterized in particular by the condition $N_R=N_L=N$, where the number N specifies the level of the state. They are in general combinations of monomials of $\beta^\dagger$ and $\tilde{\beta}^\dagger$ applied to the vacuum with arbitrary coefficients. The statement one can prove is the following:

Closed string Fock space states of given level, satisfying the level matching condition, can always be decomposed into combinations with arbitrary coefficients of states of the type (6.1) that are $*$-algebra projectors. Loosely speaking, level–matched states of the closed string Fock space come from star–algebra projectors.

The proof is not difficult but rather involved, see [14]. The basic idea is the following. Any level–matched closed string state is a combination of states of the form (6.1). Among them there always is a state $-\beta_n^\mu\,\tilde{\beta}_n^\nu|0_c\rangle$, with highest n. On the basis of (6.2), the states $-\beta_n^\mu\,\tilde{\beta}_n^\mu|0_c\rangle$ (i.e. with $\mu=\nu$) are star–algebra projectors. The state $-\beta_n^\mu\,\tilde{\beta}_n^\nu|0_c\rangle$, for $\mu\neq\nu$ is *not* a projector, but $-\beta_n^\mu\,\tilde{\beta}_n^\mu|0_c\rangle+a\,\beta_n^\mu\,\tilde{\beta}_n^\nu|0_c\rangle$, for an arbitrary constant a and $\mu\neq\nu$, is. By generalizing this example it is possible to prove the theorem.

7. Closed string states with nonzero momentum

Every closed string state is constructed by tensoring a Fock space state with a momentum eigenfunction, which, in the coordinate representation, is the plane wave e^{ikx}. The momentum k comes in equal parts from the left and the right-handed sectors. The purpose of this section is to explain where this factor comes from in the open-closed correspondence of the previous sections. Once more we shall see that the origin of this factor is a star algebra projector.

To start with we remark that in the previous sections all developments were based on the sliver projector, which is translationally invariant in all directions. If we want to find a momentum dependence we have therefore to start from projectors that are not translationally invariant. To this end we will use the *lump* projector, see section 3.

Next we repeat the same steps as in the previous section in order to define the operators β_M and $\tilde{\beta}_M$. We are of course interested in particular in the zero mode. Let us consider a lump projector $|\Xi'\rangle$ and concentrate on a transverse direction, say μ. We introduce, in a way analogous to the previous section, left and right Fock space projectors ρ'_L and ρ'_R, with the same properties as ρ_L and ρ_R. These operators can be diagonalized (see [25]). Differently from the sliver case here we have both a continuous and discrete spectrum. The continuous spectrum is spanned by a real number k, $-\infty < k < +\infty$. The discrete spectrum can be written in terms of a positive real number η and by $-\eta$ (η is related to the parameter b). The corresponding eigenvectors are denoted $V_N(k), V_N(\eta), V_N(-\eta)$.

Using this basis, S' and $\omega' = 1/\sqrt{1-T'^2}$, we write down the analog of formula (5.2). The operators s'^{μ}_M satisfy the Heisenberg algebra

$$[s'^{\mu}_M, s'^{\nu\dagger}_N] = \delta_{MN}\eta^{\mu\nu} \tag{7.1}$$

and annihilate the lump projector $|\Xi'\rangle$.

In analogy with what we did in the previous section, we define now vectors ξ' such that $\rho'_L\xi' = \xi'$ and $\rho'_R\xi' = 0$. There exists a complete basis of ξ'_N ($N = 0,1,2,\dots$) that satisfy these conditions and are orthonormal in the sense that

$$\langle\xi'_N|\frac{1}{1-T'^2}|\xi'_M\rangle = \delta_{NM} \tag{7.2}$$

Then we define

$$\xi'^{L}_N = \omega'\xi'_N, \qquad \xi'^{R}_N = \omega' C\xi'_N \tag{7.3}$$

When projected onto the continuous basis $|k\rangle$ and the discrete one $|\eta\rangle$, $|-\eta\rangle$, they give rise to a vector of functions and numbers $\xi'^{L}_N(k)$, $\xi'^{L}_N(-\eta)$ and $\xi'^{R}_N(k)$, respectively, $\xi'^{R}_N(\eta)$, which satisfy the orthogonality relations

$$\sum_{N=0}^{\infty}\Big(\xi'^{L}_N(k)\xi'^{L}_N(k') + \xi'^{R}_N(k)\xi'^{R}_N(k')\Big) = \delta(k,k') \tag{7.4}$$

$$\sum_{N=0}^{\infty}\Big(\xi'^{L}_N(\eta)\xi'^{L}_N(\eta) + \xi'^{R}_N(\eta)\xi'^{R}_N(\eta)\Big) = 1 \tag{7.5}$$

For later purposes it is convenient to choose the basis in such a way that

$$\xi'^{L}_0(-k) = \xi'^{R}_0(k) = 0, \qquad \xi'^{R}_n(\eta) = \xi'^{L}_n(-\eta) = 0, \quad k>0, \quad n = 1,2,\dots \tag{7.6}$$

This will allow us to separate the continuous from the discrete spectrum–dependent objects.

Now, in analogy with the previous section, we define the coefficients

$$b'_{NM} = \langle\xi'^{L}_N|V_M\rangle, \qquad \tilde{b}'_{NM} = \langle\xi'^{R}_N|V_M\rangle \tag{7.7}$$

and the operators

$$\beta_N^\mu = \sum_{M=0}^{\infty} b'_{NM} {s'_M}^\mu, \qquad \tilde{\beta}_N^\mu = -\sum_{M=0}^{\infty} \tilde{b}'_{NM} {s'_M}^\mu \tag{7.8}$$

Needless to say they satisfy the algebra

$$[\beta_M^\mu, \beta_N^{\nu\dagger}] = \eta^{\mu\nu}\delta_{MN}, \qquad [\tilde{\beta}_M^\mu, \tilde{\beta}_N^{\nu\dagger}] = \eta^{\mu\nu}\delta_{MN}, \tag{7.9}$$

while the other commutators vanish. Here μ, ν are any two transverse directions. We remark that we have dropped the prime from the β's, in order to use a uniform notation for the closed string operators. However it should be kept in mind that the $\beta_n, \tilde{\beta}_n$ operators are different from those defined in the previous section.

We are now ready to discuss the momentum eigenstates. To start with let us define the state

$$|p,q\rangle = \frac{1}{K}\sqrt{\frac{b}{2\pi}}\, e^{-\frac{b}{4}(p^2+q^2)+\sqrt{b}(q\beta_0^\dagger + p\tilde{\beta}_0^\dagger)-\frac{1}{2}(\beta_0^{\dagger 2}+\tilde{\beta}_0^{\dagger 2})}|0'_c\rangle \tag{7.10}$$

where p and q are real numbers, K is the constant that appear in eq.(3.10), and $|0'_c\rangle$ stands for the lump $|\Xi'\rangle$. For notational simplicity we drop Lorentz indexes. We remark the β_0 and $\tilde{\beta}_0$ are not self-adjoint, therefore they cannot be interpreted as momenta, not even as half–momenta. We define the self–adjoint half–momenta operators as

$$\hat{q} = \frac{1}{2\sqrt{b}}(\beta_0 + \beta_0^\dagger), \qquad \hat{p} = \frac{1}{2\sqrt{b}}(\tilde{\beta}_0 + \tilde{\beta}_0^\dagger) \tag{7.11}$$

It is easy to verify that the states (7.10) satisfy

$$\hat{p}|p,q\rangle = \frac{p}{2}|p,q\rangle, \qquad \hat{q}|p,q\rangle = \frac{q}{2}|p,q\rangle$$

The star product of two $|p,q\rangle$ states can be easily computed. The details are given in [14]. The only caution is to introduce a regulator since a naive calculation would bring about infinite factors. This is easily accomplished by multiplying the term $(\beta_0^{\dagger 2} + \tilde{\beta}_0^{\dagger 2})$ in the exponent of (7.10) by a parameter ε and eventually taking the limit $\varepsilon \to 1$. The result is as follows

$$|p_1,q_1\rangle * |p_2,q_2\rangle = \lim_{\varepsilon\to 1} C(\varepsilon, q_1, p_2)\, |p_1,q_2\rangle$$

where

$$C(\varepsilon, q_1, p_2) = \frac{1}{2}\sqrt{\frac{b}{\pi(1-\varepsilon)}}\, e^{-\frac{b(q_1-p_2)^2}{4(1-\varepsilon)}}$$

The limit for $\varepsilon \to 1$ of this expression is $\delta(q_1 - p_2)$. Therefore

$$|p_1,q_1\rangle * |p_2,q_2\rangle = \delta(q_1 - p_2)|p_1,q_2\rangle \tag{7.12}$$

This equation is clearly the natural generalization of equations like (6.2) when continuous parameters are involved (instead of discrete indexes). For this reason we say that $|p,p\rangle$ is a star algebra projector (by slightly extending this notion). We remark that this happens when the left half–momentum is equal to the right half–momentum.

We can therefore improve our description of the closed string states, by giving them a nonzero momentum in the transverse directions: we tensor the states discussed in the previous sections (constructed as in the previous sections, but out of $\beta_n^{\mu\dagger}$ and $\tilde{\beta}_n^{\mu\dagger}$ given by eq.(7.8)) with momentum eigenstates $|p,p\rangle$. The resulting state will have transverse momentum p^μ, which is the eigenvalue of $\frac{1}{2\sqrt{b}}(\beta_0^\mu+\beta_0^{\mu\dagger}+\tilde{\beta}_0^\mu+\tilde{\beta}_0^{\mu\dagger})$.

8. The boundary state in the transverse direction

It is very instructive to redo the computation we did at the beginning of section 6 for transverse directions. Let ij denote transverse directions and let us consider the identity

$$\begin{aligned}\sum_n \beta_n^{i\dagger}\tilde{\beta}_n^{j\dagger}\eta_{ij} &= -\sum_n \langle s'^{i\dagger}|\xi_n'^L\rangle\langle\xi_n'^R|s'^{j\dagger}\rangle\eta_{ij}\\ &= -\sum_n \langle s'^{i\dagger}|\xi_n'^L\rangle\langle\xi_n'^L|Cs'^{j\dagger}\rangle\eta_{ij} = -\frac{1}{2}\sum_{k=1}^{\infty} s_k'^{i\dagger}C_{kl}s_l'^{j\dagger}\eta_{ij}\end{aligned}$$

The only difference with section 6 is the – sign, which comes from the definition (7.8). This is not compensated anymore now by the twist properties of the basis since

$$\xi'^R = C\xi'^L, \tag{8.1}$$

which in turn descends from the sign change implied by passage from the 'sliver basis' to the 'lump basis', see [14].

For the transverse directions we have therefore the following identity

$$e^{\sum_n \beta_n^{i\dagger}\tilde{\beta}_n^{j\dagger}\eta_{ij}}|0_c\rangle = e^{-\frac{1}{2}\sum_{k=1}^{\infty} s_k'^{i\dagger}C_{kl}s_l'^{j\dagger}\eta_{ij}}|\Xi\rangle \sim e^{-\frac{1}{2}\sum_{k=1}^{\infty} a_k^{i\dagger}C_{kl}a_l^{j\dagger}\eta_{ij}}|0\rangle \tag{8.2}$$

where $|0\rangle$ is the original open string vacuum.

Suppose we have Dk–brane in closed string theory, i.e. we have $25-k$ transverse directions and $k+1$ parallel ones (including time). Then the oscillator part of the corresponding boundary state in closed string theory is the tensor product of a factor like the LHS of eq.(6.5) and a factor given by the LHS of the above eq.(8.2). As one can see the RHS of the two equations takes the same form. This miracle has to be traced back to the twist properties of the 'sliver basis' and the 'lump basis'.

The identification (8.2) generalizes the corresponding result in section 6. But we are now in a position to offer an interpretation for it. The LHS is proportional to the boundary state in closed string theory, the right hand side is the identity state in open string field theory. The boundary state represents a Dk–brane in the closed string language. The identity state represents absence of interaction in the open string field theory language. We can interpret the above equality in the following way: closed strings are reflected by the Dk–brane (they feel it). Open string live on the Dk–brane, therefore they perceive the corresponding state as an identity state (they don't feel it).

At this stage it is also clear that one cannot speak about closed string states in absolute generality but only with respect to a given background. The closed string states we have introduced are the ones that interact with the open string excitations of a given D–brane, which is manifest in the structure of the vacuum they act upon.

The elements brought forth in this section are evidence in favor of our identification of closed string modes with open string star algebra projectors. In particular the above mentioned automatic change in boundary conditions can hardly be a mere accident.

9. Conclusions and discussion

In this paper we have first discussed a correspondence between 1/2 BPS supergravity solutions (LLM geometries) and SFT projectors. This example suggests that supergravity solutions can be constructed out of open string theory bricks. Led by this example we have proposed a translation dictionary between open and closed string theory in the framework of open string field theory. We can summarize our proposal with the slogan: closed string modes are star algebra projectors, where the star algebra is the one that appear in open string field theory. Our starting point has been the identification of the left and right sectors of the open string theory with the holomorphic and antiholomorphic sectors of the closed string via a Bogoliubov transform. The latter, in particular, maps the open string vacuum into the sliver string field, which is identified with the closed string vacuum. We have shown that zero momentum level–matched (off–shell) closed string states are associated under our dictionary with star algebra projectors (or families thereof) in the open string side. To associate a momentum to a given state we had to shift to the lump vacuum and to tensor the previous states by a momentum eigenstate which is itself a star algebra projector. So, altogether, we can claim that according to our dictionary, off-shell closed string states (i.e. momentum and level-matched closed string states) correspond to star algebra projectors in the open string side.

We have presented one important consistency check of our proposal, by showing that the boundary state that represents a Dk–brane in the closed string language is translated into the identity state in the open string side, which is precisely the result one expects if our identification is correct. In [14] we have tested this result by explicitly showing how one can compute the closed string exchange between two boundary states by using elementary star algebra operations.

The string states that in section 4 were set in correspondence with the 1/2 BPS LLM geometries, [7], turn out to be, in the light of the present paper, infinite superpositions of closed string states of the type (6.3) with $n = m$ and $k = l$. This is another element that fits the general scheme presented here.

References

[1] A. Sen, *Open-closed duality: Lessons from matrix model,* Mod. Phys. Lett. A **19** (2004) 841 [arXiv:hep-th/0308068].

[2] D. Berenstein, *Large N BPS states and emergent quantum gravity,* [arXiv:hep-th/0507203].

[3] V. Balasubramanian, J. de Boer, V. Jejjala and J. Simon, *The library of Babel: On the origin of gravitational thermodynamics,* [arXiv:hep-th/0508023].

[4] P. J. Silva, *Rational foundation of GR in terms of statistical mechanic in the AdS/CFT framework,* [arXiv:hep-th/0508081].

[5] L. Bonora, C. Maccaferri, R. J. Scherer Santos and D. D. Tolla, *Bubbling AdS and vacuum string field theory*, Nucl. Phys. B **749** (2006) 338

[6] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the giant gravitons from anti-de Sitter space*, JHEP **0006** (2000) 008 [arXiv:hep-th/0003075].

[7] H. Lin, O. Lunin and J. Maldacena, *Bubbling AdS space and 1/2 BPS geometries*, JHEP **0410** (2004) 025 [arXiv:hep-th/0409174].

[8] S. Corley, A. Jevicki and S. Ramgoolam, *Exact correlators of giant gravitons from dual N = 4 SYM theory*, Adv. Theor. Math. Phys. **5** (2002) 809 [arXiv:hep-th/0111222].

[9] R. C. Myers and O. Tafjord, *Superstars and giant gravitons*, JHEP **0111** (2001) 009 [arXiv:hep-th/0109127].

[10] M. Hillery, R. F. O'Connell, M. O. Scully, E. P. Wigner *Distribution functions in physics: fundamentals*, Phys.Rep. **106** (1984) 121.

[11] M. V. Berry, *Semi–classical Mechanics in Phase Space: A Study of Wigner's Function*, Phil. Trans. Royal Soc. London. Series A, Mathematical and Physical Sciences, vol. 287 (1977), 237–271.

[12] G. Milanesi and M. O'Loughlin, *Singularities and closed time-like curves in type IIB 1/2 BPS geometries*, JHEP **0509** (2005) 008 [arXiv:hep-th/0507056].

[13] M. Schnabl, *Analytic solution for tachyon condensation in open string field theory*, [arXiv:hep-th/0511286].

[14] L. Bonora, N. Bouatta and C. Maccaferri, *Towards open-closed string duality: Closed Strings as Open String Fields,*, arXiv:hep-th/0609182.

[15] L. Bonora, C. Maccaferri, D. Mamone and M. Salizzoni, *Topics in string field theory*, arXiv:hep-th/0304270.

[16] L.Rastelli, A.Sen and B.Zwiebach, *Classical solutions in string field theory around the tachyon vacuum*, Adv. Theor. Math. Phys. **5** (2002) 393 [hep-th/0102112].

[17] L.Rastelli, A.Sen and B.Zwiebach, *Half-strings, Projectors, and Multiple D-branes in Vacuum String Field Theory*, JHEP **0111** (2001) 035 [hep-th/0105058].

[18] K. Furuuchi and K. Okuyama, *Comma vertex and string field algebra*, JHEP **0109** (2001) 035 [arXiv:hep-th/0107101].

[19] L. Rastelli, A. Sen and B. Zwiebach, *Star algebra spectroscopy*, JHEP **0203** (2002) 029 [arXiv:hep-th/0111281].

[20] I.Ya.Aref'eva, D.M.Belov, A.A.Giryavets, A.S.Koshelev, P.B.Medvedev, *Noncommutative field theories and (super)string field theories*, [hep-th/0111208].

[21] K.Ohmori, *A review of tachyon condensation in open string field theories*, [hep-th/0102085].

K. Ohmori, *On ghost structure of vacuum superstring field theory*, Nucl. Phys. B **648** (2003) 94 [arXiv:hep-th/0208009].

[22] L. Bonora, D. Mamone and M. Salizzoni, *B field and squeezed states in Vacuum String Field Theory*, Nucl. Phys. **B630** (2002) 163 [hep-th/0201060].

L. Bonora, D. Mamone and M. Salizzoni, *Vacuum String Field Theory with B field"*, JHEP **0204** (2002) 020 [hep-th/0203188].

L. Bonora, D. Mamone and M. Salizzoni, *Vacuum String Field Theory ancestors of the GMS solitons*, JHEP **0301** (2003) 013 [hep-th/0207044].

[23] C. Maccaferri, *Chan-Paton factors and higgsing from vacuum string field theory,* JHEP **0509** (2005) 022 [arXiv:hep-th/0506213].

[24] L. Bonora, C. Maccaferri and P. Prester, *Dressed sliver solutions in vacuum string field theory,* JHEP **0401** (2004) 038 [arXiv:hep-th/0311198.]

[25] D.M.Belov, *Diagonal Representation of Open String Star and Moyal Product*, [hep-th/0204164].

B. Feng, Y. H. He and N. Moeller, *The spectrum of the Neumann matrix with zero modes*, JHEP **0204** (2002) 038 [arXiv:hep-th/0202176].

[26] E. Witten, *Noncommutative Tachyons and String Field Theory*, [hep-th/0006071].

[27] M. Schnabl, *String field theory at large B-field and noncommutative geometry*, JHEP **0011**, (2000) 031 [hep-th/0010034].