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Gauge Symmetries in Fokker-Planck Equations

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In this work we review some aspects of a metric approach to Galilean invariance to construct equations of Fokker-Planck type with the content of a gauge theory. The metric formalism is based on the projection from an extended $(4+1)$ Minkowski manifold to the usual $(3+1)$ Newtonian space-time. Whilst the symmetry $U(1)$ gives rise to the usual Fokker-Planck equation, the extension for a nonabelian gauge theory provides us with a Fokker-Planck dynamics including color indices. As an application a non-stationary color-like Ornstein-Uhlenbeck process is studied. In order to derive explicit expression for the drift and diffusion terms, we use an approach based on the theory of Lie algebras for differential equations, working in $(1+1)$ space-time. The formalism is presented in detail with three-dimensional Lie algebras.

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1. Introduction

Recently we have derived the Fokker-Planck equation via a variational principle with gauge-invariant Lagrangians[1]. The starting point is the notion of Galilean-covariance for non-relativistic theories, working in a $(4+1)$ Minkowski space-time $\mathcal{G}$ with light-cone coordinates [2]-[4]. In the context of $U(1)$ gauge group, the drift and the diffusion terms are then derived with the physical content of a metric tensor on a pseudo-Riemannian manifold defined in such way that $\mathcal{G}$ is the local manifold. In the generalization for non-abelian symmetries, we studied an $SU(2)$ gauge invariant Fokker-Planck equation, considering a stationary Ornstein-Uhlenbeck process. Here we review these results but we analyze, as one of our proposal, a non-stationary color-like Ornstein-Uhlenbeck process.

One of the most outstanding interest and motivation for such a study is the possibility to construct an effective color Fokker-Planck dynamics. This follows a recent trend to develop, using methods of field theories, phenomenological Yang-Mills-like equations describing fluids as the deconfined quark-gluon plasm [5, 6, 7]. However,a difficult aspect in these formalisms is to obtain the transport coefficients. Another goal of the present paper is, using Lie algebras, to point a systematic way to derive such coefficients for the Fokker-Planck dynamics; that is, the drift and diffusion tensors.

Lie symmetries have often been invoked to solve and generalize the Fokker-Planck equation with non-trivial drift and diffusion terms [8]-[18]. The central ingredient of these approaches is a starting basic symmetry which is usually considered as the symmetry of a more restrictive set of equations. For instance, the well-known symmetry group of the diffusion equation has been used to derive Fokker-Planck equations [13]; but the problem of how to choose the starting symmetry and its realizations remains open. To find a way to treat this problem is of interest since it can be useful to understand and solve several Fokker-Planck equations which are available in the literature. That is the case of some equations with log-terms in the transport coefficients, associated with quantum chaocity and experimental results describing nucleation in metals [11, 19, 20, 22, 23]. In this paper, we exploit a method by considering different realizations of three-dimensional Lie algebras in the $(1+1)$ dimensions of space-time. Let us emphasize that our central aim is to show that a Fokker-Planck dynamics can be thought of as a field theory fully defined in terms of symmetries.

The plan of this article is as follows. In Section 2, we briefly review the derivation of classes of Fokker-Planck equations with $U(1)$. In Section 3 the $SU(2)$ gauge-invariant Lagrangians are studied. In Section 4 we develop the procedure to derive drift and diffusion terms from Lie algebras, with applications presented in Section 5. Section 6 is dedicated to some concluding remarks.

2. $U(1)$-Gauge Fokker-Planck Lagrangian

In this section, we review the derivation of Fokker-Planck equations via a gauge-invariant Lagrangian and the notion of Galilean covariance [1]. Let us consider the five-vector $p^\mu = (\mathbf{p}, p^4, p^5)$, where $\mathbf{p}$ is the Euclidean momentum vector, $p^4 = H/v$ (H is the energy and v has units of velocity), and $p^5 = mv$ (m is the mass). Then, by using the metric

$$\eta = \delta_{ij} dx^j \otimes dx^i - dx^4 \otimes dx^5 - dx^5 \otimes dx^4, \tag{2.1}$$

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we find the following dispersion relation:

$$p_\mu p^\mu = p_\mu p_\nu g^{\mu\nu} = \mathbf{p}^2 - 2p^4 p^5 = k^2, \tag{2.2}$$

where k is a constant. This relation is consistent with the fact that a free particle with mass m has total energy equal to $H = \mathbf{p}^2/2m$, so that

$$p_\mu p^\mu = \mathbf{p}^2 - 2mH = 0.$$

Henceforth, we will take k equal to zero, or absorb it within H. The metric in Eq. (2.1) defines a $(4+1)$ Minkowski space with a 15-dimensional Poincaré algebra (corresponding to the inhomogeneous group of linear transformations in the extended configuration space) which contains both the usual 10-dimensional Poincaré algebra and the Galilei algebra. Therefore, this extended manifold provides a unifying scheme for treating both relativistic and non-relativistic physics in $(3+1)$ dimensions.

We can associate with the five-vector p^μ a set of canonical conjugate coordinates $q^\mu = (\mathbf{q}, q^4, q^5)$ in a configuration space $\mathscr{G}$ with metric η. They can be interpreted as follows: $\mathbf{q}$ are the canonical conjugate coordinates of $\mathbf{p}$, q^4 is the conjugate coordinate of p^4 (the energy, H/v), so that q^4 is a time coordinate, and q^5 is conjugate of p^5 (the mass, mv). Thus, we can write q^5 as a function of $\mathbf{q}$ and q^4 which obeys the following expression for the analogue of an interval in $\mathscr{G}$:

$$q_\mu q^\mu = q^\mu q^\nu g_{\mu\nu} = \mathbf{q}^2 - 2q^4 q^5 = s^2.$$

Since such an expression is the canonical coordinates counterpart of the dispersion relation, Eq. (2.2), we choose $s = 0$, which corresponds to $k = 0$, so that we find

$$q^5 = \mathbf{q}^2/2q^4.$$

With $q^4 = vt$, it follows that $q^5 = \mathbf{q}^2/2vt$. In short, we have defined an embedding of the Euclidian space into $\mathscr{G}$:

$$(\mathbf{q}, t) \rightarrow q^\mu = (\mathbf{q}, q^4, q^5),$$

according to the aforementioned prescription. As noted earlier, the isometries of the extended space-time manifold $\mathscr{G}$ contain the symmetries of both Poincaré (relativistic) and Galilean (non-relativistic) physics so that the embedding defines the kinematics.

Let us consider U(1) gauge-invariant Lagrangian written in terms of the 2-form tensor field F, with components $F^{\mu\nu}$:

$$\mathscr{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}. \tag{2.3}$$

The tensor $F^{\mu\nu}$ is written in terms of the Abelian gauge fields J as

$$F_{\mu\nu} = \partial_\mu J_\nu - \partial_\nu J_\mu,$$

where J remains to be specified. This leads to the usual Euler-Lagrange equations

$$\partial^\mu \partial_\mu J^\nu - \partial^\nu \partial_\mu J^\mu = 0. \tag{2.4}$$

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The Lagrangian $\mathscr{L}$ is invariant under the gauge transformation: $J^\mu \to \bar{J}^\mu = J^\mu + \partial^\mu h(x)$. We take the gauge condition as being $\partial^\mu \partial_\mu J^\nu = 0$, such that $h(x)$ satisfies the constraint equation: $\partial^\mu \partial_\mu h(x) = \beta$, where β is an arbitrary constant. Therefore, we find $\partial_\mu J^\mu = \alpha$, where α is another arbitrary constant, which we can take equal to zero. Then the Euler-Lagrange equations can be written as follows,

$$\partial_\mu J^\mu = 0. \tag{2.5}$$

In order to specify the five-dimensional vector field theory, we have analyzed this gauge theory in a pseudo-Riemannian manifold $\mathscr{R}(\mathscr{G})$, with metric $g^{\mu\nu}(x)$ given by the tensor

$$g = P(x)B_{ij}(x)dx^j \otimes dx^i - dx^4 \otimes dx^5 - dx^5 \otimes dx^4, \tag{2.6}$$

such that, at each point of $\mathscr{R}(\mathscr{G})$, there is a flat space $\mathscr{G}$. The result is to write the components of J^μ as

$$\begin{aligned} J^i &= A^i(x)P(x) + \partial_j P(x)B^{ij}(x), \\ J^4 &= P(x), \\ J^5 &= 0. \end{aligned}$$

We obtain from Eq. (2.5) that

$$\partial_t P(\mathbf{x},t) = \frac{\partial}{\partial x^i}\left[-A^i(\mathbf{x},t)P(\mathbf{x},t) + \frac{\partial}{\partial x^j}B^{ij}(\mathbf{x},t)P(\mathbf{x},t)\right]. \tag{2.7}$$

This is the Fokker-Planck equation with the *drift term* $A^i(\mathbf{x},t)$, and the *diffusion tensor* $B^{ij}(\mathbf{x},t)$. We can take $P(\mathbf{x},t)$ to be a real positive and normalized function, so that it can be interpreted as a (covariant) probability density. In the following section this procedure is generalized to non-abelian gauge-symmetries.

3. $SU(2)$-Gauge Fokker-Planck Lagrangian

Consider a gauge-invariant non-Abelian Lagrangian defined on the manifold $\mathscr{G}$,

$$\mathscr{L} = -\frac{1}{4}F^{a\mu\nu}F_{a\mu\nu}, \tag{3.1}$$

where the Latin index a stands for the gauge group, with generators t^a, $a = 1,...,n$, satisfying the Lie algebra $[t^a,t^b] = C_c^{ab}t^c$, where C_c^{ab} are structure constants of the gauge group (summation convention over Latin indices is assumed). The field strength tensor $F^a_{\mu\nu}$ is given by

$$F_{\mu\nu a} = \partial_\mu J_{\nu a} - \partial_\nu J_{\mu a} - \lambda C_a^{bc} J_{\mu b} J_{\nu c},$$

for which the equation of motion is $D_a^{\mu b} F_{\mu\nu b} = 0$, where $D_a^{b\mu}$ is the covariant derivative $D_a^{\mu b} = \partial^\mu \delta_a^b + \lambda C_a^{bc} J_c^\mu$. Using the gauge condition $\partial_\mu \partial^\mu J_{\nu a} = 0$, the equations of motion for each component of J are written as

$$\begin{aligned} \partial_\nu \partial_\mu J_a^\mu =& \lambda C_a^{bc} \partial_\mu (J_c^\mu J_{\nu b}) + \lambda C_a^{bc} J_c^\mu \partial_\mu J_{\nu b} \\ &+ \lambda C_a^{cb} J_c^\mu \partial_\nu J_{\mu b} + \lambda^2 C_a^{cb} C_b^{de} J_c^\mu J_{\mu d} J_{\nu e}. \end{aligned} \tag{3.2}$$

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Despite the non-linear structure of these equations, a Fokker-Planck system can be recognized if J is defined as in the Abelian case, and if we discard the non-linear terms in Eq. (3.2) [1], such that

$$\partial_\mu \partial_\nu J_a^\mu = 0.$$

As a consequence

$$\partial_\mu J_a^\mu = \alpha, \tag{3.3}$$

where α is a constant. If we choose $\alpha = 0$, we obtain Eq. (2.5), which leads to a Fokker-Planck equation for each gauge index a.

On the other hand, by considering $\alpha << 1$, then Eq. (3.2) reduces, up to second order terms in $\lambda\alpha$, to

$$\partial_\nu(\partial_\mu J_a^\mu + \lambda C_a^{bc} J_c^\mu J_{\mu b}) = 2\lambda C_a^{bc} J_c^\mu \partial_\mu J_{\nu b} + \lambda C_a^{bc}(\partial_\nu J_c^\mu) J_{\mu b}. \tag{3.4}$$

The left-hand side of this equation can be integrated for each $\nu = 1,...,5$, such that the right-hand side results in a non-local term along each direction. In a heuristic construction, if we discard as a first approximation this non-local terms we obtain the following nonlinear [1] equation

$$\partial_\mu J_a^\mu + \lambda C_a^{bc} J_c^\mu J_{\mu b} = 0. \tag{3.5}$$

Let us consider as an example the $su(2)$ symmetry with J_a^μ defined by:

$$\begin{aligned} J_a^i &= \varepsilon_{aij}[A_j^k P_k + \partial_k(B_j^{nk} P_n)] \ , \\ J_a^4 &= P_a \ , \\ J_a^5 &= 0 \ , \end{aligned}$$

where both gauge and tensor indices are of the same nature (that is, i, j, k and a, b, c are all equal to $1,2,3$), $A_j^k = A_j^k(x)$ describes the drift term (which is now a rank-two tensor, taking into account the vector and the gauge index), whilst $B_j^{nk}(x)$ stands for the diffusion term. Notice that this definition can be developed with the reasoning used in the abelian case. From Eq. (3.5), it follows that $\varepsilon_{abc} J_c^\mu J_{\mu b} = \varepsilon_{abc} J_{ic} J_{ib} = 0$. Hence

$$\partial_t P_a = \varepsilon_{aji}[\partial_i(A_j^b P_b) + B_j^{cb} \partial_i \partial_b P_c]. \tag{3.6}$$

Let us analyze the content of this Fokker-Planck-like equation in some particular situations. First, define

$$\begin{aligned} P_2 &= P_3 = P, \\ A_2^1 &= f(z), \ \ A_3^1 = g(y), \\ B_2^{13} &= a(y), \ \ B_3^{12} = b(z), \end{aligned}$$

where P is a constant and the other components of A_j^b and B_j^{cb} are zero. With the above expressions for the drift terms A_2^1 and A_3^1, and the diffusion tensor components B_2^{13} and B_3^{12}, we are assured that we have an arbitrary process for this theory with color as the gauge index, and yet, with the characteristics of a Fokker-Planck like dynamics. Indeed, if we write

$$P_1(y,z,t) = \varphi(y)\phi(z)e^{wt},$$

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we get

$$w = \frac{1}{\phi(z)}\left[\frac{d^2}{dz^2}[b(z)\phi(z)] + \frac{d}{dz}[f(z)\phi(z)]\right] - \frac{1}{\varphi(y)}\left[\frac{d^2}{dy^2}[a(y)\varphi(y)] + \frac{d}{dy}[g(y)\varphi(y)]\right].$$

Therefore, with

$$\frac{1}{\phi(z)}\left[\frac{d^2}{dz^2}[b(z)\phi(z)] + \frac{d}{dz}[f(z)\phi(z)]\right] = F_1, \tag{3.7}$$

$$\frac{1}{\varphi(y)}\left[\frac{d^2}{dy^2}[a(y)\varphi(y)] + \frac{d}{dy}[g(y)\varphi(y)]\right] = F_2, \tag{3.8}$$

we find $F_1 - F_2 = w$. By multiplying Eq. (3.7) by $\exp(F_1 t)$, we use

$$F_1\phi(z,t) = \frac{\partial}{\partial t}\phi(z,t),$$

together with

$$\phi(z,t) = \phi(z)\exp(F_1 t),$$

to find the equation

$$\frac{\partial}{\partial t}\phi(z,t) = \frac{\partial^2}{\partial z^2}[b(z)\phi(z,t)] + \frac{\partial}{\partial z}[f(z)\phi(z,t)]. \tag{3.9}$$

Similarly, Eq. (3.7) becomes

$$\frac{\partial}{\partial t}\varphi(y,t) = \frac{\partial^2}{\partial y^2}[a(y)\varphi(y,t)] + \frac{\partial}{\partial y}[g(y)\varphi(y,t)]. \tag{3.10}$$

Note that Eqs. (3.9) and (3.10) are as general as Eq. (2.7); hence we study, as an example, a non-stationary Ornstein-Uhlenbeck process, providing expressions for $a(y), g(y), b(z)$ and $f(z)$; that is, we assume that

$$a(y) = D \quad g(y) = -\gamma y,$$
$$b(z) = D \quad f(z) = -\gamma z.$$

Then we have two known equations that are written as

$$\frac{\partial}{\partial t}\Phi(x,t) = \frac{\partial^2}{\partial x^2}[D\Phi(x,t)] + \frac{\partial}{\partial x}[-\gamma x\Phi(x,t)]. \tag{3.11}$$

where $\Phi(x,t)$ stands for $\varphi(y)$ and $\phi(z)$. We take for the initial condition $\Phi(x,0) = \delta(x)$. Then the solution of Eq. (3.11) is given by [24]

$$\Phi(x,t) = \left[\frac{\gamma}{2\pi D(1-e^{-2\gamma t})}\right]^{1/2} \exp\left[\frac{\gamma x^2(1-e^{-\gamma t})^2}{2D(1-e^{-2\gamma t})}\right],$$

which leads to

$$P_1(y,z,t) = \varphi(y)\phi(z)e^{wt} = \left[\frac{\gamma}{2\pi D(1-e^{-2\gamma t})}\right] \exp\left[\frac{\gamma(y^2+z^2)(1-e^{-\gamma t})^2}{2D(1-e^{-2\gamma t})} + w\right].$$

In the following section we address the problem of deriving explicitly, in $(1+1)$ space-time, the diffusion and the drift terms by considering general arguments of symmetry.

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4. Symmetries of drift terms and diffusion tensors

In order to derive the drift vector and diffusion tensor terms in Eq. (2.7) or (3.6), we proceed with arguments based on Lie groups [17, 18]. First, we discuss the symmetries of the differential equations. In order to do so, we take a generic element G of the symmetry Lie group. If G is connected to the identity, we can write it as

$$G = \exp\left(\sum_{k=1}^{m} \alpha_k T_k\right), \tag{4.1}$$

where T_k denotes the generators of symmetries, and the coordinates α_k are finite numbers. A partial differential equation can be cast into the following general form

$$\Delta(x)\theta(x) = 0, \tag{4.2}$$

where $\Delta(x)$ is a partial differential (field) operator defined in $\mathbf{R}^m$ with coordinates $x = (x_1, x_2, ..., x_m)$, and $\theta(x)$ is a function of $\mathbf{R}^m$. As explained in Refs. [8, 9, 17, 18, 25], to say that G is a symmetry group of Eq. (4.2) means that for a symmetry transformation generator $L(x)$ which belongs to the Lie algebra of G, we have

$$L(x)\Delta(x)\theta(x) = 0.$$

Since we can write this generator in terms of the generators T_k as $L(x) = a_k T_k(x)$, then we can rewrite the invariance condition above as follows:

$$[T_k(x), \Delta(x)] \;= r_k(x)\Delta(x), \quad k = 1, ..., \dim(G), \tag{4.3}$$

where $r_k(x)$'s are functions in $\mathbf{R}^m$.

Our purpose is to utilize Eq. (4.3) with Δ, a generic Fokker-Planck type differential operator, and T's the generators of of a given symmetry, in order to determine explicitly the form of drift and diffusion terms. The problem at this point is that there is no prescription to provide us with a specific symmetry. Then we have started a systematic study using the classification of Lie algebras.

5. Derivation of drift and difusion terms

Let us show this procedure for some 3-dimensional Lie algebras which are given by the following commutation relations [26]

$$[T_1, T_2] \;=\; [T_1, T_3] = [T_2, T_3] = 0, \tag{5.1}$$
$$[T_2, T_3] \;=\; T_1, \tag{5.2}$$
$$[T_1, T_3] \;=\; T_1, [T_2, T_3] = T_1 + T_2, \tag{5.3}$$
$$[T_1, T_3] \;=\; T_1, [T_2, T_3] = T_2, \tag{5.4}$$
$$[T_1, T_3] \;=\; T_1, [T_2, T_3] = -T_2, \tag{5.5}$$
$$[T_1, T_3] \;=\; T_1, [T_2, T_3] = aT_2, 0 <| a |< 1 \tag{5.6}$$
$$[T_1, T_3] \;=\; -T_2, [T_2, T_3] = -T_1, \tag{5.7}$$
$$[T_1, T_3] \;=\; aT_1 - T_2, [T_2, T_3] = T_1 + aT_2, a > 0 \tag{5.8}$$
$$[T_1, T_2] \;=\; 2T_2, [T_1, T_3] = -2T_3, [T_2, T_3] = T_1 \tag{5.9}$$
$$[T_1, T_2] \;=\; T_3, [T_2, T_3] = T_1, [T_3, T_1] = T_2. \tag{5.10}$$

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Therefore, we can consider each one of these algebras, and apply Eq. (4.1) with three generators T_1, T_2 and T_3. Before that, however, we have to study the realizations of each algebra. This procedure demands long calculations, and the full study will be presented elsewhere. Here we limit ourselves to some of these algebras. Let us deal with

$$[T_1,T_2]=2T_2,\ [T_3,T_1]=2T_3,\ [T_2,T_3]=T_1. \tag{5.11}$$

Clearly, it is possible to define many realizations of this Lie algebra in terms of vector fields, even for a specific number of manifold dimensions. We will work with realizations of this algebra in $(1+1)$ space-time of the form

$$\begin{aligned} T_1 &= \partial_t, \\ T_2 &= k_1(x,t)\partial_t + k_2(x,t)\partial_x, \\ T_3 &= k_3(x,t)\partial_t + k_4(x,t)\partial_x, \end{aligned} \tag{5.12}$$

where $k_1(x,t), k_2(x,t), k_3(x,t)$ and $k_4(x,t)$ are functions of x and t constrained by the commutation relations in Eq. (5.11). If we substitute the expressions for T_1 and T_2 from Eq. (5.12) into the commutator $[T_1,T_2]$ of (5.11), then we find

$$[\partial_t k_1(x,t) - 2k_1(x,t)]\partial_t + [\partial_t k_2(x,t) - 2k_2(x,t)]\partial_x = 0,$$

which leads to $\partial_t k_1(x,t) = 2k_1(x,t)$ and $\partial_t k_2(x,t) = 2k_2(x,t)$, the solutions of which are

$$k_1(x,t) = f_1(x)\exp(2t), \quad k_2(x,t) = f_2(x)\exp(2t). \tag{5.13}$$

Similarly, with the commutator of T_1 and T_3 in Eq. (5.11), the realization of Eq. (5.12) gives us

$$[\partial_t k_3(x,t) + 2k_3(x,t)]\partial_t + [\partial_t k_4(x,t) + 2k_4(x,t)]\partial_x = 0,$$

such that

$$k_3(x,t) = f_3(x)\exp(-2t), \quad k_4(x,t) = f_4(x)\exp(-2t). \tag{5.14}$$

Finally, by using the third commutator, $[T_2,T_3]$, of Eq. (5.11), together with Eq. (5.12), we have

$$\begin{aligned} 1 = {} & k_1(x,t)\partial_t k_3(x,t) - k_3(x,t)\partial_t k_1(x,t) \\ & - k_4(x,t)\partial_x k_1(x,t) + k_2(x,t)\partial_x k_3(x,t), \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} 0 = {} & k_1(x,t)\partial_t k_4(x,t) - k_3(x,t)\partial_t k_2(x,t) \\ & + k_2(x,t)\partial_x k_4(x,t) - k_4(x,t)\partial_x k_2(x,t). \end{aligned} \tag{5.16}$$

From Eqs. (5.13) and (5.14), we find that Eqs. (5.15) and (5.16) lead to

$$\begin{aligned} & f_2(x)\partial_x f_3(x) - 4f_1(x)f_3(x) - f_4(x)\partial_x f_1(x) = 1, \\ & f_2(x)\partial_x f_4(x) - 2f_1(x)f_4(x) - 2f_3(x)f_2(x) - f_4(x)\partial_x f_2(x) = 0. \end{aligned} \tag{5.17}$$

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Therefore, we may summarize by rewriting Eq. (5.12) as follows:

$$\begin{aligned} T_1 &= \partial_t, \\ T_2 &= f_1(x)\exp(2t)\,\partial_t + f_2(x)\exp(2t)\,\partial_x, \\ T_3 &= f_3(x)\exp(-2t)\,\partial_t + f_4(x)\exp(-2t)\,\partial_x, \end{aligned} \tag{5.18}$$

where the f's satisfy Eq. (5.17). In the following we explore solutions of such equations to derive drift and diffusions terms for Fokker-Planck equations, by substituting the realizations found above into Eq. (4.3).

The term $\Delta(x)$ in Eq. (4.3) is the Fokker-Planck differential operator (2.7) in $(1+1)$ space-time, given by

$$\Delta(x,t)P(x,t) = \partial_t P(x,t) + \partial_x[A(x,t)P(x,t)] + \partial_{xx}[-B(x,t)P(x,t)] = 0, \tag{5.19}$$

so that the operator Δ reads

$$\Delta = \partial_t + \partial_x A(x,t) - \partial_{xx}B(x,t) + [A(x,t) - 2\partial_x B(x,t)]\,\partial_x - B(x,t)\,\partial_{xx}. \tag{5.20}$$

The drift and the diffusion terms are $A(x,t)$ and $B(x,t)$, respectively.

A rather trivial solution of Eqs. (5.17) is given by

$$f_3(x) = -\frac{1}{4f_1(x)}, \qquad f_2(x) = f_4(x) = 0,$$

so that $f_1(x)$ remains an arbitrary function of x. In other words, we have

$$\begin{aligned} k_1(x,t) &= \exp(2t)\,f_1(x), \\ k_2(x,t) &= 0, \\ k_3(x,t) &= -\exp(-2t)\,\tfrac{1}{4f_1(x)}, \\ k_4(x,t) &= 0, \end{aligned}$$

so that Eq. (5.18) leads to the vector field realization

$$\begin{aligned} T_1 &= \partial_t, \\ T_2 &= \exp(2t)f_1(x)\partial_t, \\ T_3 &= -\exp(-2t)\tfrac{1}{4f_1(x)}\partial_t. \end{aligned} \tag{5.21}$$

Let us see how Eq. (4.3) applied to the realization (5.21) with $\Delta(x)$ given by Eq. (5.20) provides further restrictions of the differential equation. For the sake of illustration, if Eq. (4.3) is applied on a general function $F(x)$ with T_1, then the equation

$$T_1(x,t)\,\Delta(x,t)\,F(x,t) - \Delta(x,t)\,T_1(x,t)\,F(x,t) = r_1(x,t)\,F(x,t),$$

becomes (with the notation $F_t = \partial_t F$, $F_{xt} = \partial_{xt}F$, etc.)

$$\begin{aligned} &[A_{xt} - B_{xxt}]F + [A_t - 2B_{xt}]F_x - B_t F_{xx} = \\ &\qquad r_1[F_t + [A_x - B_{xx}]F + [A - 2B_x]F_x - BF_{xx}]. \end{aligned}$$

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Next we collect the factors of various derivatives of F:

$$\begin{array}{ll} F_t: & 0 = r_1, \\ F_{xx}: & B_t = r_1 B, \\ F_x: & A_t - 2B_{xt} = r_1[A - 2B_x], \\ F: & A_{xt} - B_{xxt} = r_1[A_x - B_{xx}]. \end{array}$$

From the first two lines, we find that B depends on x only, and from this and line three, we find that A also is independent of t. By proceeding similarly with the generators T_2 and T_3, we find that Eq. (5.20) can be written as

$$\Delta = \partial_t + \frac{dA(x)}{dx} - \frac{d^2(B(x))}{dx^2} + \left(A(x) - 2\frac{dB(x)}{dx}\right)\partial_x - B(x)\partial_{xx}, \tag{5.22}$$

where $A(x)$ and $B(x)$ are arbitrary functions of x.

Following the same scheme, another realization of Eqs. (5.17) for algebra Eqs. (5.9) is (we have used the package in Ref. [27])

$$\begin{aligned} T_1 &= \partial_t, \\ T_2 &= e^{2t}[c\partial_t + f_2(x)\partial_x], \\ T_3 &= e^{-2t}[-\frac{1}{4c}\partial_t + f_3(x)\partial_x], \end{aligned}$$

where

$$\begin{aligned} f_3(x) &= \left(-\frac{1}{2}\int \frac{1}{f_2(x)I(x)}dx + c_1\right) I(x), \\ I(x) &= \exp\left(\int \frac{2c - f_{2x}}{f_2(x)}dx\right). \end{aligned}$$

The final equation reads

$$P_t + \frac{c_2}{\left[\exp\left(\int \frac{c}{f_2(x)}dx\right)\right]^2} + \left[\frac{f_{2x}}{c_3}\frac{1}{I(x)} + \frac{2}{I(x)}\int I(x)dx - \frac{c_4}{c_5}\frac{1}{I(x)}\right] P_x + \frac{f_2(x)}{c_3 I(x)} P_{xx} = 0.$$

Taking $c = 1/2$ and $f_2(x) = x/2$, we get

$$P_t = (-\frac{a}{x^2} + k)P + (kx + \frac{a}{x})P_x - \frac{1}{2}bP_{xx},$$

such that

$$\begin{aligned} A(x,t) &= kx + \frac{a}{x} \\ B(x,t) &= -b. \end{aligned}$$

For $a = 0$ we have a Ornstein-Uhlenbeck process, while for $a \neq 0$ we have a Reyleigh-like process. Interesting results come even from the commutative algebras, Eq.(5.1), including generators in the form $T = k_1(x,t)\partial_t + k_2(x,t)\partial_t + k_3(x,t)$. These aspects of realizations and derivation of the drift and diffusion terms will be presented in a separated publication, considering 2,3 and 4-dimensional Lie algebras.

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6. Concluding remarks

In this paper we have constructed equations of Fokker-Planck type by enforcing various symmetries: (i) Galilean invariance is implemented with an extended Minkowski space to show that the usual Fokker-Planck equation presents a $U(1)$ symmetry; (ii) the theory of Lie symmetries of differential equations is used to obtain explicit Fokker-Planck equations; (iii) non-abelian gauge symmetry is analyzed by using the Galilean covariance, resulting in a Fokker-Planck Lagrangian including color index. Item (i) were discussed to some extend in another publication [1]. However, when rederiving those results here, we have focused our attention on the explicit form for the drift and diffusion coefficients. Observe in addition that the Fokker-Planck equation constructed from the general Δ-operators presents, as it would be expected, symmetries other than the original group. In our case, it is evident that the Fokker-Planck equation is invariant under dilation, for instance; yet this symmetry is not described by the algebra we have studied. This is due to the Lie algebra (not the group) used for our proposal of deriving the transport coefficients.

Although there is no specific criteria for selecting a symmetry to derive Fokker-Planck equations, we proceeded by analyzing the classification of 3-dimensional Lie algebras. This classification is available in the literature of mathematical works, but not their realizations which is of most interest for physicists. It is our contention that by following the scheme presented here, it is possible to carry out such an analysis. This is currently in progress and we will present the results in a separate publication.

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