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Higher N-point Amplitudes in Open Superstring Theory

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In this work we report on recent progress in the calculation of open superstring scattering amplitudes, at tree level, with more than four external massless states. We also report on the corresponding terms in the low energy effective lagrangian.

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1. Introduction and review of known results

1.1 The Born-Infeld lagrangian in the context of String Theory

As a starting point we consider the low energy interaction of *abelian* open (bosonic) strings in Minkowski spacetime, which is given by the Born-Infeld lagrangian [1]:

$$\mathscr{L}_{\rm BI} = -\frac{1}{(2\pi\alpha')^{D/2}}\sqrt{\det(\eta_{\mu\nu}+2\pi\alpha' F_{\mu\nu})}\ , \tag{1.1}$$

as long as $F_{\mu\nu}$ is kept constant. D is the spacetime dimension.
The α' expansion of $\mathscr{L}_{\rm BI}$ has the following form:

$$\mathscr{L}_{\rm BI} = (\text{constant}) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\pi^2}{2}\alpha'^2\left(F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) + \mathscr{O}(\alpha'^4)\ . \tag{1.2}$$

The first non constant term clearly is identified as the Maxwell lagrangian and the $\mathscr{O}(\alpha'^2)$ term is the first low energy correction coming from String Theory. Only *even* powers of α' show up in this expansion.
The general situation for the low energy effective lagrangian includes as well derivatives of the F's:

$$\mathscr{L}_{\rm eff} = \mathscr{L}_{\rm BI} + (\text{derivative terms})\ . \tag{1.3}$$

The situation for the low energy effective lagrangian in Open Superstring Theory is similar to the one in (1.3), the only difference being the fact that fermionic degrees of freedom are also present in the lagrangian [2]. Anyway, the bosonic part of this lagrangian has a similar structure as the one in (1.3).
A *nonabelian* generalization of (1.3) is of interest for Type I theory. A first guess would be to consider the trace of the lagrangian in (1.1) and (1.2), leading to

$$\begin{aligned}\mathscr{L}_{\rm BI}^{non-ab} = {} & (\text{constant}) - \frac{1}{4}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \frac{\pi^2}{2}\alpha'^2\mathrm{tr}\left(F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) + \\ & + \mathscr{O}(\alpha'^4)\ ,\end{aligned} \tag{1.4}$$

where, clearly, the first non constant term is the Yang-Mills lagrangian. From now on we will desconsider the constant term in (1.4).
The problem arises as soon as we consider the $\mathscr{O}(\alpha'^2)$ contribution in (1.4) since it is ambiguously defined, for example, terms like $F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\nu}$ and $F_{\mu\nu}F^{\nu\rho}F^{\sigma\nu}F_{\rho\sigma}$ which are equivalent from the *abelian* point of view are not so from the *nonabelian* point of view. The reason lies in the fact that the commutator of two field strengths is not zero in the nonabelian case. So a nonabelian generalization of the Born-Infeld lagrangian, in the context of Superstring Theory is not an immediate task to acheive.
A nonabelian calculation by means of a 4-point amplitude calculation leads to the following expression [2, 3]:

$$\begin{aligned}\mathscr{L}_{\rm BI}^{non-ab} = {} & -\frac{1}{4}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \frac{\pi^2}{2}\alpha'^2\mathrm{tr}\left(\frac{1}{3}F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\nu} + \frac{2}{3}F_{\mu\nu}F^{\nu\rho}F^{\sigma\mu}F_{\rho\sigma} - \right. \\ & \left. -\frac{1}{6}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - \frac{1}{12}F_{\mu\nu}F_{\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\right) + \mathscr{O}(\alpha'^3)\ .\end{aligned} \tag{1.5}$$

It is easy to see that the abelian limit of the $\mathscr{O}(\alpha'^2)$ terms in (1.5) agrees exactly with the corresponding terms in (1.2).

1.2 Introducing the symmetrized trace

In [4] it was seen that the abelian and the nonabelian expressions of the Born-Infeld lagrangian could be related, at least up to $\mathscr{O}(\alpha'^2)$ terms, by introducing a symmetrized trace in the abelian expression (1.2),

$$\mathscr{L}_{\rm BI}^{non-ab} = -\frac{1}{4}\mathrm{str}\Big(F_{\mu\nu}F^{\mu\nu}\Big) + \frac{\pi^2}{2}\alpha'^2\mathrm{str}\Big(F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big) + \\ +\mathscr{O}(\alpha'^3)\ . \tag{1.6}$$

The symmetrized trace, denoted by 'str', is defined as an average of the trace of all possible permutations of matrices. The result in (1.6) is very nice in the sense that it looks like a democratic way of constructng the nonabelian lagrangian from the abelian one.
The complete proposal of [4] for the nonabelian Born-Infeld lagrangian is simply

$$\mathscr{L}_{\rm BI}^{non-ab} = (\text{constant})\ \mathrm{str}\Big(\sqrt{\det(\eta_{\mu\nu}+2\pi\alpha' F_{\mu\nu})}\Big)\ . \tag{1.7}$$

This can be considered as a prescription for writing the F^n terms of the lagrangian, at any order in α'. Its abelian limit clearly agrees with the usual Born-Infeld lagrangian (1.1).
So, the general structure of the low energy effective lagrangian in the open string sector of Type I theory may be written as

$$\mathscr{L}_{\rm eff} = (\text{constant})\ \mathrm{str}\Big(\sqrt{\det(\eta_{\mu\nu}+2\pi\alpha' F_{\mu\nu})}\Big) + \begin{pmatrix}\text{covariant}\\ \text{derivative}\\ \text{terms}\end{pmatrix} + (\text{fermions})\ . \tag{1.8}$$

Equation (1.8) may seem to be a quite simple and a very strong result and, although it is correct, it has a serious problem. Due to the $[D_\mu, D_\nu]F_{\alpha\beta} = -ig[F_{\mu\nu}, F_{\alpha\beta}]$ identity, the F^n and the $D^{2p}F^{n-p}$ terms can be related, so the covariant derivative terms in (1.8) are as important as the F^n ones. Therefore, the separation between F^n terms (i.e. nonabelian Born-Infeld lagrangian) and covariant derivative ones is a purely artificial fact in the case of the nonabelian theory.
The conclusion is that the complete determination of the $\mathscr{L}_{\rm eff}$ lagrangian in (1.8) can only be obtained by perturbative theory in α'.
The approach that we will follow in this work is the scattering amplitude one.

2. Some details about the interactions

2.1 General formula for the string scattering amplitude

The general formula for the (open string) massless boson amplitude (at tree level) is [5]

$$\mathscr{A}^{(M)} = i\ (2\pi)^{10}\delta(k_1+k_2+\ldots+k_M)\cdot \sum_{j_1,j_2,\ldots,j_M}{}' \mathrm{tr}(\lambda^{a_{j_1}}\lambda^{a_{j_2}}\ldots\lambda^{a_{j_M}})\, A(j_1,j_2,\ldots,j_M)\ , \tag{2.1}$$

where M is the number of bosons and the sum $\sum'$ in the indices $\{j_1, j_2, \ldots, j_M\}$ is done over *non* $-$ *cyclic* equivalent permutations of the group $\{1,2,\ldots,M\}$. The matrices λ^{a_j} are in the adjoint representation of the Lie group. $A(j_1, j_2, \ldots, j_M)$ is the main object of study, called *subamplitude*. It corresponds to the M-point amplitude of open superstrings which do not carry color indices and which are placed in the ordering $\{j_1, j_2, \ldots, j_M\}$ (modulo cyclic permutations). Using vertex operators, the RNS formalism leads to the following integral formula for $A(1,2,\ldots,M)$ [6], for $M \geq 3$:

$$
\begin{aligned}
A(1,2,\ldots,M) \;=\; & 2\frac{g^{M-2}}{(2\alpha')^{7M/4+2}}(x_{M-1}-x_1)(x_M-x_1)\times \\
& \times \int dx_2 \ldots dx_{M-2} \int d\theta_1 \ldots d\theta_{M-2} \prod_{i>j}^{M} |x_i - x_j - \theta_i\theta_j|^{2\alpha' k_i \cdot k_j} \times \\
& \times \int d\phi_1 \ldots d\phi_M \; e^{f_M(\zeta,k,\theta,\phi)}\,,
\end{aligned}
\tag{2.2}
$$

where

$$
f_M(\zeta,k,\theta,\phi) = \sum_{i\neq j}^{M} \frac{(\theta_i-\theta_j)\phi_i(\zeta_i\cdot k_j)(2\alpha')^{11/4} - 1/2\phi_i\phi_j(\zeta_i\cdot\zeta_j)(2\alpha')^{9/2}}{x_i - x_j - \theta_i\theta_j}\,. \tag{2.3}
$$

The θ_i's and the ϕ_i's in (2.2) and (2.3) are Grassmann variables, while the x_i's are real variables such that $x_1 < x_2 < x_3 < \ldots < x_M$. The k_i's and the ζ_i's are the i-th string momentum and polarization, respectively.

Although not manifest, the subamplitude $A(1,2,\ldots,M)$ in (2.2) has the following symmetries[5]:

1. Cyclicity:

$$
A(1,2,\ldots,M-1,M) = A(2,3,\ldots,M,1) = \ldots = A(M,1,\ldots,M-2,M-1)\,.
$$

2. On-shell gauge invariance:

$$
A(1,2,\ldots,M)|_{\zeta_i=k_i} = 0, \;\text{ for }\; i=1,\ldots,M\,.
$$

3. World-sheet parity:

$$
A(1,2,\ldots,M-1,M) = (-1)^M A(M-1,M-2,\ldots,1,M)\,.
$$

Formula (2.2), together with (2.1), contains *all* the information to construct the low energy efective lagrangian, which has the form

$$
\mathscr{L}_{\text{eff}} = \text{tr}\left\{F^2 + \alpha'^2 F^4 + \alpha'^3(F^5 + D^2F^4) + \alpha'^4(F^6 + D^2F^5 + D^4F^4) + \ldots\right\}. \tag{2.4}
$$

Formula (2.4) already considers the fact that the string 3-point amplitude, $A(1,2,3)$, agrees completely with the corresponding Yang-Mills 3-point amplitude (i.e., it has no α' corrections [5]).

2.2 Case of the 4-point amplitude

An interesting (and very well known) application of formula (2.2) is the case of the 4-point subamplitude. It leads to

$$A(1,2,3,4) = 8\, g^2\, \alpha'^2 \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)} K(\zeta_1,k_1;\zeta_2,k_2;\zeta_3,k_3;\zeta_4,k_4)\ , \tag{2.5}$$

where

$$K(\zeta_1,k_1;\zeta_2,k_2;\zeta_3,k_3;\zeta_4,k_4) = t_{(8)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} \zeta^1_{\mu_1} k^1_{\nu_1} \zeta^2_{\mu_2} k^2_{\nu_2} \zeta^3_{\mu_3} k^3_{\nu_3} \zeta^4_{\mu_4} k^4_{\nu_4} \tag{2.6}$$

is a kinematic factor, $t_{(8)}$ being a known tensor [5]. The s and t variables in (2.5) are part of the three Mandelstam variables. They may be written as [7]

$$s = -k_1\cdot k_2 - k_3\cdot k_4,\ \ t = -k_1\cdot k_4 - k_2\cdot k_3 \quad . \tag{2.7}$$

The Gamma factor in (2.5) has a completely known α' expansion, which begins like

$$\alpha'^2 \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)} = \frac{1}{st} - \frac{\pi^2}{6}\alpha'^2 + \mathscr{O}(\alpha'^3)\,. \tag{2.8}$$

Using (2.7) and the known symmetries of the $t_{(8)}$ tensor [5], the expression for $A(1,2,3,4)$ in (2.5) has all three symmetries (cyclic invariance, on-shell gauge invariance and world-sheet parity) manifest.

The general method of finding the string corrections to the Yang-Mills lagrangian, at a given α' order, consists in writing all the possible terms with unknown coefficients. Consider, for example, the $\mathscr{O}(\alpha'^2)$ corrections. Up to that α' order the effective lagrangian looks like [2]

$$\begin{aligned} \mathscr{L}_{\text{eff}} = & -\frac{1}{4}\text{tr}\Big(F_{\mu\nu}F^{\mu\nu}\Big) + \frac{\pi^2}{2}\alpha'^2 \text{tr}\Big(c_1 F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\nu} + c_2 F_{\mu\nu}F^{\nu\rho}F^{\sigma\mu}F_{\rho\sigma} + \\ & + c_3 F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + c_4 F_{\mu\nu}F_{\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\Big)\ . \end{aligned} \tag{2.9}$$

This expression is the final result, after having considered also derivative terms and used the Bianchi identity, integration by parts and the $[D,D]F = [F,F]$ relations. In (2.9), we have also omitted terms which vanish on-shell (i.e., terms which contain $D_\mu F^{\mu\nu}$).

Then, the tree level 4-point amplitude is calculated using (2.9) and compared with the corresponding expression in (2.5) up to $\mathscr{O}(\alpha'^2)$ order. This comparison determines the unknowns c_1, c_2, c_3 and c_4, leading to the expression in (1.5) [2]).

Although this procedure can be applied to compute the correction terms (sensible to a 4-point amplitude) up to any α' order, it gets longer and harder as the α' order grows.

Based on the idea of [7], in [8] was done the explicit construction of all $D^{2n}F^4$ ($n = 0,1,2,\ldots$) in the effective lagrangian, arriving to

$$\begin{aligned} \mathscr{L}_{D^{2n}F^4} = & -\frac{1}{8} g^2 \alpha'^2 \int\!\!\int\!\!\int\!\!\int \left\{ \prod_{j=1}^{4} d^{10}x_j\ \delta^{(10)}(x-x_j) \right\} \times \\ & \times f_{\text{sym}}\left(\frac{(D_1+D_2)^2+(D_3+D_4)^2}{2}, \frac{(D_1+D_4)^2+(D_2+D_3)^2}{2} \right) \times \\ & \times t_{(8)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} \text{tr}\Big(F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)F_{\mu_3\nu_3}(x_3)F_{\mu_4\nu_4}(x_4)\Big)\ , \end{aligned} \tag{2.10}$$

where the function f is given by

$$f(s,t) = \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)} - \frac{1}{\alpha'^2 st}\,. \tag{2.11}$$

(See [8] for more details about the relation between functions $f_{\rm sym}(s,t)$ and $f(s,t)$.)

2.3 Case of the 5-point amplitude

At this point is where the search for *higher* N-point amplitudes begins. In the case of the 5-point subamplitude formula (2.2) becomes

$$\begin{aligned} A(1,2,3,4,5) &= 2\frac{g^3}{(2\alpha')^{43/4}}(x_4-x_1)(x_5-x_1)\times \\ &\times \int_{x_1}^{x_4} dx_3 \int_{x_1}^{x_3} dx_2 \int d\theta_1 d\theta_2 d\theta_3 \prod_{i>j}^{5} |x_i-x_j-\theta_i\theta_j|^{2\alpha' k_i\cdot k_j}\times \\ &\times \int d\phi_1 d\phi_2 d\phi_3 d\phi_4 d\phi_5 e^{f_5(\zeta,k,\theta,\phi)}\,, \end{aligned} \tag{2.12}$$

where $\theta_4 = \theta_5 = 0$.

Once the Grassmann integration has been done, and after some lengthly algebra, it can be written as [9]

$$\begin{aligned} A(1,2,3,4,5) = 2\,g^3\,(2\,\alpha')^2 \Big\{ & L_3(\zeta_1\cdot\zeta_2)(\zeta_3\cdot\zeta_4)(\zeta_5\cdot k_2)(k_1\cdot k_3) + \Big(44\ (\zeta\cdot\zeta)^2(\zeta\cdot k)(k\cdot k)\ \text{terms}\Big) \\ & + K_2(\zeta_1\cdot\zeta_4)(\zeta_5\cdot k_2)(\zeta_2\cdot k_1)(\zeta_3\cdot k_4) + \Big(99\ (\zeta\cdot\zeta)(\zeta\cdot k)^3\ \text{terms}\Big)\Big\}. \end{aligned} \tag{2.13}$$

(See eq. (5.29) of [9] for the complete detailed formula.) L_3 and K_2 are momentum dependent factors (which also depend on α') given by double integrals:

$$\left\{ \begin{array}{c} L_3 \\ K_2 \end{array} \right\} = \int_0^1 dx_3 \int_0^{x_3} dx_2\; x_2^{2\alpha'\alpha_{12}}(1-x_2)^{2\alpha'\alpha_{24}} x_3^{2\alpha'\alpha_{13}}(1-x_3)^{2\alpha'\alpha_{34}}(x_3-x_2)^{2\alpha'\alpha_{23}} \left\{ \begin{array}{c} \frac{1}{x_2 x_3(1-x_3)} \\ \frac{1}{x_2(1-x_3)} \end{array} \right\}, \tag{2.14}$$

where $\alpha_{ij} = k_i\cdot k_j$. They can be calculated in terms of Beta and hypergeometric functions. Independently of the method used to calculate them, the first terms of their α' expansion can be obtained. For example,

$$\begin{aligned} K_2 = & \frac{1}{(2\alpha')^2}\left\{\frac{1}{\alpha_{12}\ \alpha_{34}}\right\} - \frac{\pi^2}{6}\left\{\frac{\alpha_{51}\ \alpha_{12}-\alpha_{12}\ \alpha_{34}+\alpha_{34}\ \alpha_{45}}{\alpha_{12}\ \alpha_{34}}\right\} + \\ & + \zeta(3)\ (2\alpha')\left\{\frac{\alpha_{12}^2\ \alpha_{51}-\alpha_{34}^2\ \alpha_{12}+\alpha_{45}^2\ \alpha_{34}+\alpha_{51}^2\ \alpha_{12}-\alpha_{12}^2\ \alpha_{34}+\alpha_{34}^2\ \alpha_{45}-2\alpha_{12}\ \alpha_{23}\ \alpha_{34}}{\alpha_{12}\ \alpha_{34}}\right\} + \\ & + \mathcal{O}((2\alpha')^2)\,. \end{aligned} \tag{2.15}$$

Formula (2.13) was used in [9] to distinguish from three non equivalent versions of the $\alpha'^3 F^5$ terms of (2.4) [10, 11, 12], obtaining complete agreement with the one in [12].

In [9], formula (2.13) was written in terms of 8 K_i's and 8 L_i's (plus cyclic permutations of the terms in the amplitude). It was seen that some linear relations, coming from integration by parts technique, existed between the K_i's and between the L_i's.
Afterwards, in [13] it was seen that further linear relations existed between those α' dependent factors. The new relations, independent from the ones obtained in [9], were due to the partial fraction technique. So at the end, for the 16 α' dependent factors there were found 7 independent integration by parts relations and 7 independent partial fraction relations. Solving this linear system it allowed to write *all* K_i's and L_i's in terms of only 2 of them. This was summarized and exploited in [14], where the final expression for the 5-point subamplitude was written as

$$A(1,2,3,4,5) = T \cdot A_{YM}(1,2,3,4,5) + (2\alpha')^2 K_3 \cdot A_{F^4}(1,2,3,4,5)\ . \tag{2.16}$$

In this formula $A_{YM}(1,2,3,4,5)$ and $A_{F^4}(1,2,3,4,5)$ are the 5-point subamplitudes coming from the Yang-Mills and the known F^4 terms in (1.4), while T and K_3 are α' dependent factors which have a known α' expansion which go like

$$T = 1 + \mathscr{O}(\alpha'^3)\ , \quad (2\alpha')^2 K_3 = \frac{\pi^2}{6}(2\alpha')^2 + \mathscr{O}(\alpha'^3)\ . \tag{2.17}$$

Formula (2.16), the same as formula (2.5), has the nice property that the cyclic, the on-shell gauge invariance and the world-sheet parity symmetries are <u>manifest</u> (see [14] for further details). We have also tested the factorization properties of the poles [14].

<u>Benefits from having a closed formula for A(1,2,3,4,5):</u>

2.3.1 5-point amplitudes involving fermions are immediate

By this we mean that there is no need to compute the 3-boson/2-fermion and the 1-boson/4-fermion amplitudes right from the beginning, in the RNS formalism. The explanation is the following. Up to now, the supersymmetric low energy effective lagrangian is *completely* known up to $\mathscr{O}(\alpha'^2)$ terms. It has the form

$$\mathscr{L}_{\text{eff}} = \mathscr{L}_{\text{SYM}} + \alpha'^2 \mathscr{L}_2 + \mathscr{O}(\alpha'^3)\ , \tag{2.18}$$

where

$$\mathscr{L}_{\text{SYM}} = \text{tr}\left(F^2 + i\bar{\psi}\gamma D\psi \right) \tag{2.19}$$

is the D=10 Super Yang-Mills lagrangian and

$$\mathscr{L}_2 = \text{tr}\left(F^4 + D(\bar{\psi}\gamma\psi)F^2 + D^2(\bar{\psi}\gamma\psi)^2 + F(\bar{\psi}\gamma\psi)^4 \right) \tag{2.20}$$

is the order α'^2 string correction to $\mathscr{L}_{\text{SYM}}$. $\mathscr{L}_2$ has been determined completely in [15].
In this sense, due to the structure of the α' expansion of T and $(2\alpha')^2 K_3$ in (2.17), we could rewrite eq. (2.16) in a slightly different (but equivalent) notation:

$$A^{5b}(1,2,3,4,5) = T \cdot A^{5b}_{SYM}(1,2,3,4,5) + (2\alpha')^2 K_3 \cdot A^{5b}_{\mathscr{L}_2}(1,2,3,4,5)\ , \tag{2.21}$$

where $A^{5b}_{SYM}(1,2,3,4,5)$ and $A^{5b}_{\mathscr{L}_2}(1,2,3,4,5)$ denote the 5-boson subamplitude coming from $\mathscr{L}_{\rm SYM}$ and $\mathscr{L}_2$, respectively.

Now we write down our 'immediate' expression for the 3-boson/2-fermion and the 1-boson/4-fermion subamplitudes. Our ansatz, based on the structure of the α' expansion of T and $(2\alpha')^2 K_3$ in (2.17), is the following, respectively [16]:

$$A^{3b/2f}(1,2,3,4,5) = T \cdot A^{3b/2f}_{SYM}(1,2,3,4,5) + (2\alpha')^2 K_3 \cdot A^{3b/2f}_{\mathscr{L}_2}(1,2,3,4,5) \ , \tag{2.22}$$

$$A^{1b/4f}(1,2,3,4,5) = T \cdot A^{1b/4f}_{SYM}(1,2,3,4,5) + (2\alpha')^2 K_3 \cdot A^{1b/4f}_{\mathscr{L}_2}(1,2,3,4,5) \ . \tag{2.23}$$

As an immediate test of (2.22) and (2.23) we see that, by construction they reproduce the 3-boson/2-fermion and the 1-boson/4-fermion subamplitudes of the low energy effective lagrangian in (2.18). On the other side, and this guarantees that (2.22) and (2.23) are correct to any order in α', these formulas, together with (2.21), satisfy by construction the supersymmetry requirement: the summed variation of $A^{5b}(1,2,3,4,5)$, $A^{3b/2f}(1,2,3,4,5)$ and $A^{1b/4f}(1,2,3,4,5)$ under the supersymmetry transformations [6],

$$\delta A^a_\mu = \frac{i}{2}\bar{\varepsilon}\gamma_\mu \psi^a \ , \tag{2.24}$$

$$\delta \psi^a = -\frac{1}{4} F^a_{\mu\nu}\gamma^{\mu\nu}\varepsilon \ , \tag{2.25}$$

$$\delta \bar{\psi}^a = -\frac{1}{4}\bar{\varepsilon}\gamma^{\mu\nu}F^a_{\mu\nu} \ , \tag{2.26}$$

is zero, after using the on-shell and the physical state conditions, together with momentum conservation.

Formula (2.23) is being used in [16] to determine the $\alpha'^3 D^2 F(\bar{\psi}\gamma\psi)^2$ terms, which are unknown at the present moment.

2.3.2 Determination of the $\alpha'^{n+3}D^{2n}F^5$ terms

In (2.10) it was seen that it was possible to explicitly construct all the effective lagrangian terms which are sensible to the 4-point amplitude.

We will now see that that using the compact formula (2.16) we have been able to determine all the $\alpha'^{n+3}D^{2n}F^5$ terms in (2.4). For this purpose we compute first the corresponding scattering amplitude as

$$A_{D^{2n}F^5}(1,2,3,4,5) = A(1,2,3,4,5) - A_{YM}(1,2,3,4,5) - A_{D^{2n}F^4}(1,2,3,4,5) \ . \tag{2.27}$$

It is quite remarkable that the resulting expression has no poles, as it should happen (see [14] for further details). With the simplified expression for $A_{D^{2n}F^5}(1,2,3,4,5)$ we find the corresponding lagrangian terms to be [14]:

$$\mathscr{L}_{D^{2n}F^5} = i\, g^3 \int\!\!\int\!\!\int\!\!\int\!\!\int \left\{ \prod_{j=1}^{5} d^{10}x_j\ \delta^{(10)}(x-x_j) \right\} \times$$
$$\times \left[\left\{ \frac{1}{32} H^{(1)}(-D_2\cdot D_3, -D_3\cdot D_4\ , -D_4\cdot D_5)\ t^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5}_{(10)} + \right.\right.$$

$$
\begin{aligned}
&+\frac{1}{16}P^{(1)}(-D_1\cdot D_2,-D_2\cdot D_3,-D_3\cdot D_4\,,-D_4\cdot D_5,-D_5\cdot D_1)\ (\eta\cdot t_{(8)})_1{}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5}\Big\}\times\\
&\qquad\times\mathrm{tr}\Big(F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)F_{\mu_3\nu_3}(x_3)F_{\mu_4\nu_4}(x_4)F_{\mu_5\nu_5}(x_5)\Big)\ -\\
&\qquad-\ U^{(1)}(-D_1\cdot D_2,-D_2\cdot D_3,-D_4\cdot D_5,-D_5\cdot D_1)\times\\
&\times\Big\{\frac{1}{64}t_{(10)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5}\mathrm{tr}\Big(F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)D^{\alpha}F_{\mu_3\nu_3}(x_3)D_{\alpha}F_{\mu_4\nu_4}(x_4)F_{\mu_5\nu_5}(x_5)\Big)+\\
&+\frac{1}{16}t_{(8)}^{\mu_4\nu_4\mu_5\nu_5\mu_1\nu_1\mu_2\nu_2}\mathrm{tr}\Big(D^{\mu_3}F_{\mu_1\nu_1}(x_1)D^{\nu_3}F_{\mu_2\nu_2}(x_2)F_{\mu_3\nu_3}(x_3)F_{\mu_4\nu_4}(x_4)F_{\mu_5\nu_5}(x_5)\Big)+\\
&+\frac{1}{16}t_{(8)}^{\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5\mu_1\nu_1}\mathrm{tr}\Big(D^{\mu_2}F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)D^{\nu_2}F_{\mu_3\nu_3}(x_3)F_{\mu_4\nu_4}(x_4)F_{\mu_5\nu_5}(x_5)\Big)-\\
&-\frac{1}{16}t_{(8)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\mathrm{tr}\Big(D^{\mu_5}F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)F_{\mu_3\nu_3}(x_3)D^{\nu_5}F_{\mu_4\nu_4}(x_4)F_{\mu_5\nu_5}(x_5)\Big)-\\
&-\frac{1}{16}t_{(8)}^{\mu_5\nu_5\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\mathrm{tr}\Big(D^{\mu_4}F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)F_{\mu_3\nu_3}(x_3)F_{\mu_4\nu_4}(x_4)D^{\nu_4}F_{\mu_5\nu_5}(x_5)\Big)\ \Big\}\ -\\
&\qquad-\ \frac{1}{8}W^{(1)}(-D_1\cdot D_2,-D_2\cdot D_3,-D_3\cdot D_4\,,-D_4\cdot D_5,-D_5\cdot D_1)\times\\
&\times t_{(8)}^{\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5}\mathrm{tr}\Big(F_{\mu_1\nu_1}(x_1)D^{\mu_1}F_{\mu_2\nu_2}(x_2)F_{\mu_3\nu_3}(x_3)F_{\mu_4\nu_4}(x_4)D^{\nu_1}F_{\mu_5\nu_5}(x_5)\Big)\ -\\
&\qquad-\ \frac{1}{8}Z^{(1)}(-D_1\cdot D_2,-D_2\cdot D_3,-D_3\cdot D_4\,,-D_4\cdot D_5,-D_5\cdot D_1)\times\\
&\times t_{(8)}^{\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5}\Big\{\mathrm{tr}\Big(F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)D^{\mu_1}F_{\mu_3\nu_3}(x_3)D^{\nu_1}F_{\mu_4\nu_4}(x_4)F_{\mu_5\nu_5}(x_5)\Big)-\\
&\qquad-\mathrm{tr}\Big(F_{\mu_1\nu_1}(x_1)D^{\mu_1}F_{\mu_2\nu_2}(x_2)F_{\mu_3\nu_3}(x_3)F_{\mu_4\nu_4}(x_4)D^{\nu_1}F_{\mu_5\nu_5}(x_5)\Big)\Big\}\ +\\
&\qquad+\ \frac{1}{160}\Delta(-D_1\cdot D_2,-D_2\cdot D_3,-D_3\cdot D_4\,,-D_4\cdot D_5,-D_5\cdot D_1)\times\\
&\times\Big\{t_{(10)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5}\mathrm{tr}\Big(D^{\alpha}D^{\beta}F_{\mu_1\nu_1}(x_1)D_{\alpha}F_{\mu_2\nu_2}(x_2)F_{\mu_3\nu_3}(x_3)F_{\mu_4\nu_4}(x_4)D_{\beta}F_{\mu_5\nu_5}(x_5)\Big)+\\
&\qquad+4(\eta\cdot t_{(8)})_1{}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5}\times\\
&\qquad\times\mathrm{tr}\Big(F_{\mu_1\nu_1}(x_1)D_{\alpha}F_{\mu_2\nu_2}(x_2)D^{\alpha}F_{\mu_3\nu_3}(x_3)D_{\beta}F_{\mu_4\nu_4}(x_4)D^{\beta}F_{\mu_5\nu_5}(x_5)\Big)\ \Big\}\ \Big]\,,
\end{aligned}
\tag{2.28}
$$

where $H^{(1)}$, $P^{(1)}$, $U^{(1)}$, $W^{(1)}$, $Z^{(1)}$ and Δ have known expressions (and therefore known α' expansions) in terms of the Gamma factor, T and K_3.
In (2.28) we see that, besides the known $t_{(8)}$ tensor, a new $t_{(10)}$ tensor has arisen [14]. Formula (2.28) has been tested to reproduce the already known α'^3F^5 terms and gives the explicit construction of all covariant derivative terms containing 5 F's (which are sensible to the 5-point amplitude).

3. Towards a closed formula for N-point (tree level) amplitudes in Open Superstring Theory

The method of finding a basis of α' dependent factors that allows to write the scattering amplitude in a short form (see eqs. (2.5) and (2.16)), by using 'integration by parts' and 'partial fractions' techniques, can be used for any N-point amplitude ($N > 3$) and will lead to an expression of the form

$$A(1,2,\ldots,N) = F_1(\alpha_{ij};\alpha')K_1(\zeta,k) + \ldots + F_{m_n}(\alpha_{ij};\alpha')K_{m_n}(\zeta,k) \ . \qquad (3.1)$$

Here, the $F_p(\alpha_{ij};\alpha')$'s are the α' dependent factors and the $K_p(\zeta,k)$'s are the kinematical expressions. By now, the completely known cases are only the $N = 4$ and the $N = 5$ ones.
The ambitious program would consist then in:

1. Finding how many terms are there in formula (3.1): $m_n =$?
2. Finding the specific formulas for:
   - The kinematical expressions $K_p(\zeta,k)$'s, in such a way that the tensors $t_{(8)}$, $t_{(10)}$, ..., $t_{(m_n)}$ can be determined.
   - The α' factors $F_p(\alpha_{ij};\alpha')$'s and its α' expansions.

This is still an open problem. In order to get an insight it would be good to consider the case of the 6-point amplitude, but before that we will make an important comment about how the symmetries of the scattering amplitude restrict its kinematical expression.

3.1 Implementing symmetries in the scattering amplitude

In subsection 2.1 it was seen that the N-point subamplitude $A(1,2,\ldots,N)$ satisfies cyclicity, on-shell gauge invariance and world-sheet parity. We have verified, in the case of $N = 4$ and $N = 5$ that, after doing all Grassmann integrations in (2.2) (like in (2.13), in the case of $N = 5$) and demanding the 3 symmetries to be satisfied, then a set of linear relations between the α' factors is found. This system of relations happens to be linearly equivalent to the one obtained when using 'integration by parts' and 'partial fractions' techniques. We have verified that this works correctly also in the case of bosonic string amplitudes (which have an expression similar to the one in (2.2), but with no Grassmann variables), for $N = 4$ and $N = 5$.
The deeper mean of all this is that, at least up to $N = 5$, the symmetries are enough to fix the kinematics that governs the scattering amplitude, at any order in α'. But, unfortunately, as we will see in the next subsection, this is no longer true already when $N = 6$.

3.2 Case of the 6-point amplitude

After doing the Grassmann integration in (2.2) in the case of $N = 6$ we arrive to an expression of the following type[17]:

$$A(1,2,3,4,5,6) = (2\,\alpha')^2\, g^4 \left\{ \left((2\alpha')I_6(\zeta_4\cdot\zeta_5)(\zeta_6\cdot k_2)(\zeta_1\cdot k_5)(\zeta_2\cdot k_4)(\zeta_3\cdot k_4) + \right.\right.$$
$$\left. + (\text{other } (\zeta\cdot\zeta)(\zeta\cdot k)^4 \text{ terms}) \right) +$$

$$
\begin{aligned}
&+\Big((1-2\alpha' k_3\cdot k_4)I_{45}(\zeta_1\cdot\zeta_2)(\zeta_3\cdot\zeta_4)(\zeta_5\cdot k_1)(\zeta_6\cdot k_2)+\\
&+(\text{other } (\zeta\cdot\zeta)^2(\zeta\cdot k)^2 \text{ terms})\Big)+\\
&+\Big((2\alpha')I_{109}(\zeta_2\cdot\zeta_3)(\zeta_4\cdot\zeta_5)(\zeta_6\cdot\zeta_1)(k_1\cdot k_2)(k_3\cdot k_4)+\\
&+(\text{other } (\zeta\cdot\zeta)^3(k\cdot k)^2 \text{ terms})\Big)\Big\}\,,
\end{aligned}
\tag{3.2}
$$

where I_6, I_{45} and I_{109} are α' dependent factors given by triple integrals:

$$
\left\{\begin{array}{c} I_6 \\ I_{45} \\ I_{109}\end{array}\right\} = \int_0^1 dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2\; x_2^{2\alpha'\alpha_{12}}(1-x_2)^{2\alpha'\alpha_{25}} x_3^{2\alpha'\alpha_{13}}(1-x_3)^{2\alpha'\alpha_{35}} x_4^{2\alpha'\alpha_{14}}(1-x_4)^{2\alpha'\alpha_{45}}
$$
$$
\times(x_3-x_2)^{2\alpha'\alpha_{23}}(x_4-x_2)^{2\alpha'\alpha_{24}}(x_4-x_3)^{2\alpha'\alpha_{34}}\times\left\{\begin{array}{c} \frac{1}{(1-x_4)(x_4-x_2)(x_4-x_3)} \\ \frac{1}{x_2(x_4-x_3)^2} \\ \frac{1}{x_2(1-x_4)(x_3-x_2)(x_4-x_3)}\end{array}\right\}.
\tag{3.3}
$$

Formula (3.2) contains at all 237 different α' dependent factors I_j.
Demanding the symmetries to be satisfied, as explained in subsection 3.1, we obtain a set of linear relations which allows us to write the α' dependent factors in terms of 15 of them.
On the past year an interesting preprint appeared with the calculation of the 6-point amplitude[18]. The authors of it did not find the relations between the α' dependent factors by means of integration by parts neither by partial fractions techniques (although they saw that some of their relations matched with the ones that come from these techniques). They demanded another symmetry to be satisfied by the integral expression of $A(1,\ldots,M)$: the superdiffeomorphism invariance on the string world-sheet. In fact they work with an integral expression for the amplitude which is not exactly the same as we wrote in (2.2), where we have already admitted x_1, x_{M-1} and x_M to be fixed (and not integrated) and also where we had fixed $\theta_{M-1}=\theta_M=0$. Their important result consists in the fact that they find a basis containing 6 α' dependent factors (instead of the 15 dimensional basis that we found). The test that supports their result consists in the fact that the linear system of equations that they find contains a lot more equations than unknowns (and still it is consistent).
Although all the authors of [18] give the first terms of the α' expansions of the 6 factors, they do not make any confirmaton between their expression and the 6-point amplitude that comes from the $D=10$ low energy effective lagrangian (2.4). It would have been nice if they had checked the known terms up to $\mathcal{O}(\alpha'^3)$ order (that have already been checked by S-matrix calculations and other methods) and, moreover, they had confirmed the $\mathcal{O}(\alpha'^4)$ terms obtained in [19].
Anyway, the fact of their basis being 6-dimensional, motivated us to find the linear relations between the α' dependent factors directly by considering the integration by parts and the partial fractions techniques. The result of our computations agreed with their result: the basis is 6-dimensional. So, besides finding agreement with the dimension of the basis of the I_j's space, we conclude that demanding cyclicity, on-shell gauge invariance and world-sheet parity in the scattering amplitude, it does no longer determine the kinematics completely. In the same way, it is

not guaranteed that the method proposed in [18] will be equivalent to the integration by parts and partial fractions technique when $N > 6$.
Up to this moment we have not obtained an explicit closed form for $A(1,2,3,4,5,6)$ since the expressions for the I_j's in terms of the ones in the basis are extremely huge.

4. Final remarks and conclusions

We finish summarizing the main points of this talk:

- There does exist a method to explicitly compute tree level scattering amplitudes in Open Superstring Theory, beyond 4-point calculations. The method is based on the 'integration by parts' and 'partial fraction' techniques of Integral Calculus, for the α' dependent factors that show up in the subamplitude. Any other method, which demands any kind of symmetry present in the scattering amplitudes, should lead to equivalent linear relations for those factors.

- The method has been successfully applied to compute all massless 5-point amplitudes in Open Superstring Theory (5 boson, 3-boson/2-fermion and 1-boson/4-fermion).

- The N-point case is still an open problem.

- In order to look for some generalization, in the N-point case, it would be good to have a closed formula for the 6-point amplitude which had been tested to reproduce the effective lagrangian terms up to $\mathcal{O}(\alpha'^4)$ order. There is some work in progress in this direction.

- The kind of results presented in this talk are of importance in:

  1. Determining the complete low energy effective lagrangian (at least in the open superstring sector) in Type I theory.
  2. Loop amplitudes: it is quite probable that the same kinematic expressions that already appear at tree level also show up in higher loop calculations.
  3. The low energy effective lagrangian of the Type II theories, since closed string amplitudes can be directly obtained from the open ones (by means of the KLT relations [20]).

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