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Thermofield Dynamics and Path-Integral Formalism

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Thermofield Dynamics (TFD), a real time formalism for thermal quantum field theory, is formulated in terms of a path-integral approach following the Weinberg procedure. In order to assure precise rules for the development, we use a representation of Lie-algebras such that the TFD algebraic ingredients are derived, including the tilde conjugation rules. The association with the canonical formalism is obtained, and for the case of bosons, we introduce the Feynman diagrams and derive the n-point functions.

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1. Introduction

The main goal of the present paper is to build a path integral formalism to Thermofield Dynamics (TFD) approach, proposed by Takahashi and Umezawa as a real-time thermal quantum field theory fully structured on the notion of linear (Hilbert) spaces. With the emergence of TFD, as an operator version of the Matsubara (imaginary time) formalism, vast applications were available for treating thermal phenomena in the realm of quantum fields[1, 2, 3, 4], in particular exploring algebraic representations[5]. Despite the success and the effort over the last five decades, since the seminal paper by Matsubara in 1955, the construction of a general thermal quantum field theory remains an open problem, demanding improvements to derive, for instance, transport equations for systems described by a quantum field in a curved space-time background[6, 7]; or to deal with the imminent possibility of experimental informations, in high energy physics, about the phase transition from the state of hadrons to a quark-gluon plasma state[8, 9]. Under such a situation, TFD arises as a strong candidate for new developments due to its algebraic nature.

TFD is defined through two expedients: the tilde (dual) conjugation rules and a Bogoliubov transformation. The former defines a doubling in the dynamic variables, whilst the latter introduces thermal effects through a vacuum condensation. Because of these algebraic constituents, TFD has proved to be useful for developing thermal field theories, with particular emphasis on symmetry groups. Support for this can be found in some instances: the generators of Bogoliubov transformations in TFD are the generators of $SU(2)$ in the case of a fermionic system , and $SU(1,1)$ for bosons[4]; elements of the q-groups have been explored in connection with the notion of dual conjugation rules and the Bogoliubov transformations, introducing the effect of temperature through a deformation in the Weyl-Heisenberg algebra[10, 11, 12]; the nature of the doubling has been analyzed via the bialgebras, and also in this context, a connection among elements of Hopf and w^*-algebras has been suggested[13, 14].

Furthermore, the TFD vector space has been taken as a representation space for Lie algebras[15]. Thus, the equations of motion in TFD have been derived from a study of the Galilei and Poincaré symmetries[15, 16]; Liouville-von Neumann equations have been introduced for the Klein-Gordan and Dirac field associated with the kinetic theory[17]; and a classical counterpart of TFD has been identified[18, 19]. A constitutive element of this representation theory for Lie algebras is the doubling structure in the set of mappings defined in the TFD Hilbert space (actually, this doubling is present in any field formalism dealing with thermal phenomena[1]). Such a doubling gives rise to the fact that the generators of symmetries and the dynamic observables, although playing the same role in the algebraic level, are distinct from each other in a dynamic sense. This result has indicated an association between the Lie algebras defined in the TFD Hilbert space and the standard representation of w^*-algebras[20, 21, 22], in particular, as a generalization of the Ojima's work[23] in order to treat non-equilibrium systems[16, 17]. This type of representation has been called *thermo-Lie algebras* and it is useful to set forth a thermal quantum field theory from quite precise mathematical and physical roles. For instance, the tilde conjugation rules in TFD are derived under general basis and can be then applied to interacting systems; this is not the case of the original TFD formulation, which is based under physical reasoning but restrict to free systems giving rise to different definitions sometimes in contradiction to each other. In this paper we use such an algebraic procedure of thermoalgebra to introduce a functional integral formalism for TFD. The presentation follows the derivation of some results which are useful for practical applications. In the literature there are just two attempts to write a functional formalism for TFD (at least to our best knowledge) [24, 25]. Here we improve those results by following the Weinberg approach [26] to derive n-point functions at zero temperature. An advantage in the method is its generality to treat with different fields.

2. Emergence of TFD

We can think of the operation $\langle A\rangle = Tr(A\rho)$, as an average of an operator A in some Hilbert state, say $|0(\beta)\rangle$, such that $\langle A\rangle = \langle 0(\beta)|A|0(\beta)\rangle$, where $\beta = \frac{1}{T}$, T being the temperature. In this way, the system would be described by a pure like state, but still compatible with the description by a mixed state. Actually, the equation above provides an alternative formula for $\langle A\rangle$. The ensemble average of an operator A in thermal equilibrium, is given by $\langle A\rangle = Z^{-1}(\beta)Tr(e^{-\beta H}A)$. Assuming that $H|n\rangle = E_n|n\rangle$, with $\langle n|m\rangle = \delta_{nm}$, we write $\langle A\rangle = Z^{-1}(\beta)\sum_n e^{-\beta E_n}\langle n|A|n\rangle$.

We are looking for a state $|0(\beta)\rangle$ such that

$$\langle A\rangle = \langle 0(\beta)|A|0(\beta)\rangle = Z^{-1}(\beta)\sum_n e^{-\beta E_n}\langle n|A|n\rangle. \tag{2.1}$$

Let us write $|0(\beta)\rangle = \sum_n |n\rangle\langle n|0(\beta)\rangle = \sum_n f_n(\beta)|n\rangle$, which results in

$$\langle 0(\beta)|A|0(\beta)\rangle = \sum_{nm} f_n^*(\beta)f_m(\beta)\langle n|A|m\rangle.$$

This gives us the required expression in Eq.(2.1) if $f_n^*(\beta)f_m(\beta) = Z^{-1}(\beta)e^{-\beta E_n}\delta_{nm}$. But we know that $f_n(\beta)$ are c-numbers and, then, it is impossible to satisfy this relation so long as we restrict ourselves to the original Hilbert space. The above condition is like an orthogonality condition. Therefore if we introduce a doubling in the Hilbert space, we may be able to satisfy this condition. Let us do this, resulting in a tensor product of spaces, with $|n,\tilde{m}\rangle = |n\rangle\otimes|\tilde{m}\rangle$. We can write $|0(\beta)\rangle = \sum_n f_n(\beta)|n,\tilde{n}\rangle$, such that

$$\langle 0(\beta)|A|0(\beta)\rangle = \sum_n f_n^*(\beta)f_n(\beta)\langle n|A|n\rangle,$$

where we have assumed that $\langle n,\tilde{n}|A|m,\tilde{m}\rangle = \langle n|A|m\rangle\langle\tilde{n}|\tilde{m}\rangle = A_{nm}\delta_{nm}$. Now we have $f_n^*(\beta)f_n(\beta) = Z^{-1}(\beta)e^{-\beta E_n}$, which has the solution $f_n(\beta) = Z^{-1/2}(\beta)e^{-\beta E_n/2}$. Therefore we obtain,

$$|0(\beta)\rangle = \frac{1}{Z(\beta)^{1/2}}\sum_n e^{-\beta E_n/2}|n,\widetilde{n}\rangle.$$

The thermal state $|0(\beta)\rangle$ is then defined in this doubled Hilbert space. Doubling is not a characteristic of TFD, but rather an ingredient present in all thermal theory. In terms of density matrix, the doubling is present when we write the Liouville-von Neumann equation, $i\partial\rho = L\rho$, with $L = [H,]$ being the Liouvillian. In this case we have L the generator of time evolution of $\rho(t)$ and H the Hamiltonian of the system. They are in the correspondence, but are different to each other in mathematical and physical ground. In the sequence we explore this result in a more general context.

3. The Meaning of the Doubling in Thermal Theories

3.1 Generator of Symmetry and Observable

In order to introduce a formalism based on states $|0(\beta)\rangle$ from general assumptions, we assume that the set of kinematical variables, say $\mathscr{V}$, is a vector space of mappings in a Hilbert space denoted by $\mathscr{H}_T$. The set $\mathscr{V}$ is composed of two subspaces and is written as $\mathscr{V} = \mathscr{V}_{ob} \oplus \mathscr{V}_{gen}$, where $\mathscr{V}_{ob}$ stands for the set of kinematical observables while $\mathscr{V}_{gen}$ is the set of kinematical generators of symmetries.

This classification of $\mathscr{V}$ is usual in quantum (as in classical) theory, but in that case $\mathscr{V} = \mathscr{V}_{ob} = \mathscr{V}_{gen}$. This is so because to each generator of symmetry there exists a corresponding observable and both are described by the same algebraic element.

It has to be emphasized that, although the $1-1$ correspondence among observables and generators of symmetry is based on physical ground, there exists no *a priori* mechanical (or kinematical) imposition to consider a generator of symmetry and the corresponding observable as described by the same mathematical quantity. Actually we are free to assume a more general situation. Here we consider then the same $1-1$ correspondence among generators and observables, but $\mathscr{V}_{ob}$ and $\mathscr{V}_{gen}$ will be took as different from each other. That is, $\mathscr{V}_{ob}$ and $\mathscr{V}_{gen}$ are described by different mappings in $\mathscr{H}_T$. To emphasize these aspect, we denote an arbitrary elements of $\mathscr{V}_{ob}$ by A and by $\widehat{A}$ the corresponding element in $\mathscr{V}_{gen}$. Now we analyze the consequence of such *separability condition* in a general situation.

3.2 Doubled Lie Algebra

Taking $\mathscr{H}_T$ as the carrier space for representation of a Lie algebra l we can write

$$[\widehat{A}_i, \widehat{A}_j] = ic^k_{ij}\widehat{A}_k, \tag{3.1}$$

where $\widehat{A}_i \in \mathscr{V}_{gen}$. The imaginary i is to emphasize that we treat a unitary representation. Since we have a hat-representation for l, the non-hat operators have to be taken into consideration in a representation in the full space $\mathscr{H}_T$ (otherwise the representation will be restricted to a subspace of $\mathscr{H}_T$ in which the set of hat operators, $\mathscr{V}_{gen}$, is defined; resulting in the usual representations). Therefore we have additional commutation relations among A and $\widehat{A}$ operators, and among the operators A. Let us then write

$$[\widehat{A}_i, A_j] = i f^k_{ij} A_k, \tag{3.2}$$

$$[A_i, A_j] = i g^k_{ij} A_k, \tag{3.3}$$

where f^k_{ij} and g^k_{ij} are constants to be fixed with the following reasoning. Observe that this comutation relations describe a Lie algebra (to be denoted by $\mathscr{L}_T$) which is called a semidirect product of two subalgebras, $\mathscr{V}_{gen}$ and $\mathscr{V}_{obs}$, with $\mathscr{V}_{obs}$ the invariant subalgebra. The motive for that is a physical imposition. Since non-hat operators describe kinematical observables, Eq.(3.2) interpreted as the infinitesimal action of a symmetry generated by $\widehat{A}_i$ on the observable A_i, resulting in another observable given by $i f^k_{ij} A_k$. Here we take $f^k_{ij} = g^k_{ij} \equiv c^k_{ij}$ [18].

3.3 Tilde Conjugation Rules

Some properties of such an algebra l_T, that will be useful in the study of representations, can be immediately derived. Defining the variable $\widetilde{A} = A - \widehat{A}$, the commutation relations given by Eqs.(3.1)-(3.3) are now rewritten as

$$\begin{aligned} [A_i, A_j] &= ic_{ij}^k A_k, \\ [\widetilde{A}_i, \widetilde{A}_j] &= -ic_{ij}^k \widetilde{A}_k, \\ [A_i, \widetilde{A}_j] &= 0. \end{aligned}$$

This result shows that a doubling of the degrees of freedom has been introduced, compared with the standard (irreductible) group theory representation. This is a direct consequence of the algebraic separability between mappings in $\mathscr{H}_T$ describing the generators of symmetry and those describing the observables.

Such a doubling can be described as a mapping in $\mathscr{V} = \mathscr{V}_{ob} \oplus \mathscr{V}_{gen}$, say $\tau : \mathscr{V} \to \mathscr{V}$, denoted by $\tau A \tau = \widetilde{A}$, fulfilling the following conditions:

$$\begin{aligned} (A_i A_j)\widetilde{\ } &= \widetilde{A}_i \widetilde{A}_j, \\ (cA_i + A_j)\widetilde{\ } &= c^* \widetilde{A}_i + \widetilde{A}_j, \\ (A_i^\dagger)\widetilde{\ } &= (\widetilde{A}_i)^\dagger, \\ (\widetilde{A}_i)\widetilde{\ } &= A_i. \end{aligned}$$

These properties are called *tilde conjugation rules*.

4. Thermal Generating Functional and Feynman Rules

Lets us consider the example of a real scalar field. The tilde conjugation rules can be applied to any relation among the dynamical variables, which enable to add for the scalar field and conjugated momentum the tilde components. In this case the canonical commutation relations can be written as

$$[\Phi_m(\mathbf{x}), \Pi_n(\mathbf{y})] = i\delta_{mn}\delta(\mathbf{x}-\mathbf{y}), \quad [\Phi_m(\mathbf{x}), \Phi_n(\mathbf{y})] = [\Pi_m(\mathbf{x}), \Pi_n(\mathbf{y})] = 0,$$

where $(\Phi_1(\mathbf{x}), \Phi_2(\mathbf{x})) = (\Phi(\mathbf{x}), -\tilde{\Phi}(\mathbf{x}))$ and $(\Pi_1(\mathbf{x}), \Pi_2(\mathbf{x})) = (\Pi(\mathbf{x}), \tilde{\Pi}(\mathbf{x}))$. We can apply the tilde conjugation to the equations of motion in the Heisenberg picture, by writing the generator of time translation as $\hat{H} = H - \tilde{H}$ and in this way we derive the associated Lagrangian as $\hat{L} = L - \tilde{L}$ such that the free Lagrangian of the scalar theory is given by

$$\hat{L}_0 = -\frac{1}{2}\sigma_{ml}(\partial_\mu \phi^m(\mathbf{x})\partial^\mu \phi^l(\mathbf{x}) + m^2 \phi^m(\mathbf{x})\phi^l(\mathbf{x})),$$

where σ_{ml} are the components of the Pauli matrix σ_3. The free action for this theory is given by

$$S_0[\phi] = \int d^4x \hat{L}_0(\phi(x), \partial\phi(x)) + \frac{i}{2}\varepsilon \int d^4x d^4y E_{lm}(\vec{x}, \vec{y})\delta(x^0 - y^0)\phi^m(x)\phi^l(x),$$

where $E_{lm}(\mathbf{x}, \mathbf{y}) = \delta_{ml} \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} E(\mathbf{p})$ and $E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$. Since both terms in S_0 are quadratic in the fields, we may write S_0 in the quadratic form

$$S_0[\phi] = -\frac{1}{2}\int d^4x d^4y \mathscr{D}_{lm}(x,y)\phi^m(x)\phi^l(y),$$

where

$$\mathscr{D}_{lm}(x,y) = \sigma_{lm}(\partial_{y,\mu}\partial_x^{\mu}\delta^4(x-y) + m^2\delta^4(x-y)) - i\varepsilon E_{lm}(\mathbf{x},\mathbf{y})\delta(x^0-y^0).$$

The propagator is given by the inverse of $\mathscr{D}$ [26], that is $\Delta_{ml}(x,y) = (\mathscr{D}^{-1})_{ml}(x,y)$; and since

$$\mathscr{D}_{ml}(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{ip.(x-y)}[\sigma_{ml}(p^2+m^2) - i\delta_{ml}\varepsilon E(\mathbf{p})],$$

we have

$$\Delta(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{ip.(x-y)}\mathscr{D}^{-1}(p),$$

such that

$$\Delta(p) = \mathscr{D}^{-1}(p) = \begin{pmatrix} \frac{1}{p^2+m^2-i\varepsilon E(\mathbf{p})} & 0 \\ 0 & \frac{-1}{p^2+m^2+i\varepsilon E(\mathbf{p})} \end{pmatrix}.$$

We proceed further by introducing the Bogoliubov transformation

$$U(\beta) = e^{-iG[\theta(\beta)]},$$

where $G[\theta(\beta)] = -i\int d^3\mathbf{p}\theta(E(\mathbf{p});\beta)(a(\mathbf{p})\tilde{a}(\mathbf{p}) - a^{\dagger}(\mathbf{p})\tilde{a}^{\dagger}(\mathbf{p}))$ is the generator of the Bogoliubov transformation and since it is hermitian $G[\theta(\beta)] = G^{\dagger}[\theta(\beta)]$, we have $U^{-1}(\beta) = U^{\dagger}(\beta)$. The fuctional parameter $\theta(E(\mathbf{p});\beta)$ is such that

$$u(E(\mathbf{p});\beta) = \cosh\theta(E(\mathbf{p});\beta) = \frac{1}{\sqrt{1-e^{-\beta E(\mathbf{p})}}},$$

$$v(E(\mathbf{p});\beta) = \sinh\theta(E(\mathbf{p});\beta) = \frac{e^{\frac{-\beta E(\mathbf{p})}{2}}}{\sqrt{1-e^{-\beta E(\mathbf{p})}}}.$$

In order to add the temperature in the duplicated system we proceed, just like it is done with the time, by defining

$$\Phi^l(x,\beta) \equiv U(\beta)\Phi^l(x)U^{-1}(\beta),$$

$$\Pi^m(x,\beta) \equiv U(\beta)\Pi^m(x)U^{-1}(\beta).$$

The Bogoliubov transformation is a canonical transformation, such that

$$[\Phi_m(\mathbf{x},\beta),\Pi_n(\mathbf{y},\beta)] = i\delta_{mn}\delta(\mathbf{x}-\mathbf{y}), \quad [\Phi_m(\mathbf{x},\beta),\Phi_n(\mathbf{y},\beta)] = [\Pi_m(\mathbf{x},\beta),\Pi_n(\mathbf{y},\beta)] = 0,$$

where $(\Phi_1(\mathbf{x},\beta),\Phi_2(\mathbf{x},\beta)) = (\Phi(\mathbf{x},\beta),-\tilde{\Phi}(\mathbf{x},\beta))$ and $(\Pi_1(\mathbf{x},\beta),\Pi_2(\mathbf{x},\beta)) = (\Pi(\mathbf{x},\beta),\tilde{\Pi}(\mathbf{x},\beta))$. These operators have the eigenstates $|\phi;\tau,\beta\rangle \equiv U(\beta)|\phi;\tau\rangle$ and $|\pi;\tau,\beta\rangle \equiv U(\beta)|\pi;\tau\rangle$, such that

$$\Phi^l(x,\beta)|\phi;\tau,\beta\rangle \equiv \phi^l(\mathbf{x})|\phi;\tau,\beta\rangle, \quad \Pi^l(x,\beta)|\pi;\tau,\beta\rangle \equiv \pi^l(\mathbf{x})|\pi;\tau,\beta\rangle.$$

These eigenstates satisfy the completeness and ortonormality conditions for each time τ and temperature β^{-1}.

In terms of creation and annihilation operators we have

$$a^l(\mathbf{p};\beta) \equiv U(\beta)a^l(\mathbf{p})U^{-1}(\beta) = B^l_m(p^0;\beta)a^m(\mathbf{p}),$$

and $[a^l(\mathbf{p};\beta), a^{m\dagger}(\mathbf{q};\beta)] = \delta^{lm}\delta(\mathbf{p}-\mathbf{q})$, where $(a^1(\mathbf{p}), a^2(\mathbf{p})) = (a(\mathbf{p}), -\tilde{a}^\dagger(\mathbf{p}))$ with

$$B(p^0;\beta) = [B^l_m(p^0;\beta)] = \begin{pmatrix} u(p^0;\beta) & v(p^0;\beta) \\ v(p^0;\beta) & u(p^0;\beta) \end{pmatrix}.$$

Using these definitions, the thermal field $\phi(x,\beta)$ can be written in terms of non-thermal fields $\phi(x)$ and a thermal Bogoliubov operator $B(i\partial_0;\beta)$, that is

$$\begin{aligned} \phi^l(x,\beta) &= \int \frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}\sqrt{2p^0}}(e^{ipx}a^l(\mathbf{p};\beta) + e^{-ipx}a^{\dagger l}(\mathbf{p};\beta)) \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}\sqrt{2p^0}} B^l_m(p^0;\beta)(e^{ipx}a^m(\mathbf{p}) + e^{-ipx}a^{\dagger m}(\mathbf{p})) \\ &= B^l_m(i\partial_0;\beta)\int \frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}\sqrt{2p^0}}(e^{ipx}a^m(\mathbf{p}) + e^{-ipx}a^{\dagger m}(\mathbf{p})) \\ &= B^l_m(i\partial_0;\beta)\phi^m(x), \end{aligned}$$

where the action of $B(i\partial_0;\beta)$ is defined by

$$B^l_m(i\partial_0;\beta)e^{\pm ip^0x^0} = B^l_m(p^0;\beta)e^{\pm ip^0x^0}. \tag{4.1}$$

Once we introduce the temperature in the fields, we may show that the themalized action is given by

$$\begin{aligned} S_0[\phi;\beta] &= -\frac{1}{2}\int d^4x d^4y \mathscr{D}_{lm}(x,y)\phi^m(x;\beta)\phi^l(y;\beta) \\ &= -\frac{1}{2}\int d^4x d^4y \mathscr{D}_{lm}(x,y;\beta)\phi^m(x)\phi^l(y), \end{aligned}$$

where

$$\mathscr{D}_{lm}(x,y;\beta) = B^j_l(i\partial_{y_0};\beta)B^k_m(i\partial_{x_0};\beta)\mathscr{D}_{jk}(x,y).$$

Since $\mathscr{D}_{jk}(x,y) = \int \frac{d^4p}{(2\pi)^4}e^{ip.(x-y)}\mathscr{D}_{jk}(p)$, we have

$$\mathscr{D}_{lm}(p;\beta) = B^j_l(p^0;\beta)\mathscr{D}_{jk}(p)B^k_m(p^0;\beta),$$

whose inverse is $\Delta(p;\beta) = \mathscr{D}^{-1}(p;\beta) = B^{-1}(p^0;\beta)\mathscr{D}^{-1}(p)B^{-1}(p^0;\beta)$; or explicitly:

$$\Delta(p;\beta) = \begin{pmatrix} \frac{1}{p^2+m^2-i\varepsilon} + 2\pi i\delta(p^2+m^2)n(p^0;\beta) & -2\pi i\delta(p^2+m^2)e^{\frac{\beta p^0}{2}}n(p^0;\beta) \\ -2\pi i\delta(p^2+m^2)e^{\frac{\beta p^0}{2}}n(p^0;\beta) & \frac{-1}{p^2+m^2+i\varepsilon} + 2\pi i\delta(p^2+m^2)n(p^0;\beta) \end{pmatrix},$$

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where $n(p^0;\beta) = v^2(p^0;\beta)$ is boson ocupation number. The physical component of the thermal propagator is

$$\Delta_{1,1}(p;\beta) = \frac{1}{p^2+m^2-i\varepsilon} + 2\pi i\delta(p^2+m^2)n(p^0;\beta),$$

in agreement with the usual procedures.

As a start point to derive the path integral formalism in TFD for scalar theories, we can show the following result,

$$\langle 0(\beta);out|T\{\Phi(\mathbf{x}_1,t_1)\ldots\Phi(\mathbf{x}_n,t_n)\}|0(\beta);in\rangle$$

$$= |N|^2\int\prod_{\mathbf{x},\tau,l} d\phi^l(\mathbf{x},\tau)\phi(\mathbf{x}_1,t_1)\ldots\phi(\mathbf{x}_n,t_n)\exp[-iS[\phi;\beta]], \tag{4.2}$$

where: we have assumed a quadratic Hamiltonian in the momenta,

$$|0(\beta);in\rangle = U(t\to-\infty)|0(\beta)\rangle,\ |0(\beta);out\rangle = U(t\to+\infty)|0(\beta)\rangle,$$

$S[\phi;\beta] = S_0[\phi;\beta] + S_{int}[\phi]$ and $|N|^2$ is a normalization factor. With this result we are ready to use the path-integral formalism to derive the thermal Feynman rules in a scalar theory with interaction. We will concentrate on the themal vacuum expectation values of the time-ordered products of the non-tilde components, which is defined by

$$M(x_1\ldots x_n;\beta) = \frac{\langle 0(\beta);out|T\{\Phi(\mathbf{x}_1,t_1)\ldots\Phi(\mathbf{x}_n,t_n)\}|0(\beta);in\rangle}{\langle 0(\beta);out|0(\beta);in\rangle}. \tag{4.3}$$

From this definition, the physical $S(\beta)$-matrix elements may be obtained following in parallel with the Weinberg procedure [26]. Using Eq.(4.2) in Eq.(4.3), we obtain

$$M(x_1\ldots x_n;\beta) = \frac{\int\prod_{l,x}d\phi^l(x)\phi(x_1)\ldots\phi(x_n)e^{iS[\phi;\beta]}}{\int\prod_{l,x}d\phi^l(x)e^{iS[\phi;\beta]}}. \tag{4.4}$$

To deal with interactions we expand the exponential in powers of $S_{int}[\phi]$, that is

$$\exp(iS[\phi;\beta]) = \exp(iS_0[\phi;\beta])\sum_{k=0}^{\infty}\frac{i^k}{k!}(S_{int}[\phi])^k, \tag{4.5}$$

The general integral that we find in the numerator and denominator of the Eq.(4.3) is in the form

$$\mathscr{S}_{m_1,\ldots,m_k}(y_1,\ldots,y_k;\beta) = \int\prod_{l,x}d\phi^l(x)\phi_{m_1}(y_1)\ldots\phi_{m_k}(y_k)\phi(x_1)\ldots\phi(x_n)e^{iS_0[\phi;\beta]}, \tag{4.6}$$

where the fields $\phi_{m_1}(y_1),\ldots,\phi m_k(y_k)$ come from $(S_{int}[\phi])^k$ in Eq.(4.5). Since $S_0[\phi;\beta]$ is quadratic in the fields ϕ, we can evaluate the functional gaussian integrals (4.6) obtaining the following resuts

$$\mathscr{S}_{m_1,\ldots,m_k}(y_1,\ldots,y_k;x_1\ldots x_n,\beta) = [Det(\frac{i\mathscr{D}(\beta)}{2\pi})]^{-\frac{1}{2}}\sum_{pairings\,of\,fields}\ \prod_{pairs}[-i\mathscr{D}^{-1}(\beta)].$$

This expression gives rise to the coordinate-space Feynman rules for calculating the numerator of Eq(4.3): we proceed by expanding in the interaction term S_{int}, and sum over the ways of pairings the fields in the S_{int}'s with each other and with the fields $\phi(x_1),\ldots,\phi(x_n)$, with the contribution of each pairing being given by space-time integral ($\int dy_1 \ldots dy_k$) of the product of the coefficient of the fields in S_{int} and the product of propagators $-i\mathscr{D}^{-1}(\beta)$ [26].

This approach can be extended for gauge and fermion fields following the basic structure of TFD, with the thermal Bogoliubov operator introduced in Eq. (4.1). One interesting aspect is that the Bogoliubov transformation can be generalized to describe a quantum field in a confined region in space [5]. This aspect can be straightforward brought to the form of the thermal Bogoliubov operator.

5. Summary

In this work we use the Thermofield Dynamics formalism, based on the tilde conjugation rules and a thermal Bogoliubov transformation, to obtain a thermal n-point functions for the scalar field theory. In order to achieve our results we have analyzed the structure of TFD on the basis of Lie algebra; so deriving general algebraic properties that can be used in the context of interacting fields. Our approach to derive the thermal n-point functions follows the Weinberg scheme for a path-integral (without the use of sources), and is based on a definition of the thermal Bogoliubov transformation written as a differential operator acting on the fields. This formalism can be generalized, along the same lines, for spinors and gauge fields. Such a subject matter will be presented in detail elsewhere.

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