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Symmetries in Non Commutative Configuration Space

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Extending earlier work [7], we examine the deformation of the canonical symplectic structure in a cotangent bundle $T^\star(\mathscr{Q})$ by additional terms implying the Poisson non-commutativity of both configuration and momentum variables. In this short note, we claim this can be done consistently when $\mathscr{Q}$ is a Lie group.

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1. Introduction

When a symplectic manifold is a cotangent bundle with projection, $\kappa : T^*(\mathscr{Q}) \to \mathscr{Q}$, and canonical symplectic structure $\omega_0 = dq^i \wedge dp_i$, the action of a diffeomorphism ϕ on $\mathscr{Q}$ induces a diffeomorphism Φ on $T^*(\mathscr{Q})$ conserving ω_0 :

$$\Phi : T^\star(\mathscr{Q}) \to T^\star(\mathscr{Q}) : \{q^i, p_k\} \to \left\{q'^i = \phi^i(q), p'_k\right\} \; ; \; p_l = p'_k \, \frac{\partial \phi^k(q)}{\partial q^l} \tag{1.1}$$

In particular a group action being a homomorphism $G \to \mathbf{Diff}(\mathscr{Q})$, induces a strictly Hamiltonian action on $T^*(\mathscr{Q})$:

$$\Phi_g : T^\star(\mathscr{Q}) \to T^\star(\mathscr{Q}) : (q^i, p_k) \to \left(q'^i = \phi^i(g,q), p'_k\right) \; ; \; p_l = p'_k \, \frac{\partial \phi^k(g,q)}{\partial q^l} \tag{1.2}$$

Let $\mathbf{F}$ be a closed two-form on configuration space, then it is well known [1] that a change in the symplectic structure, $\omega_0 \to \omega_1 = \omega_0 + \kappa^\star \mathbf{F}$, induces a "magnetic" interaction without changing the "free" Hamiltonian. With this new symplectic structure, the momenta variables cease to Poisson commute and one needs to introduce a potential to switch to Darboux variables.
It is then tempting to introduce also a closed two-form in the p-variables in such a way that Poisson non commuting q-variables will emerge[1]. In this way, we obtain a (pre-)symplectic structure :

$$\omega = \omega_0 - \frac{1}{2} F_{ij}(q)\, dq^i \wedge dq^j + \frac{1}{2} G^{kl}(p)\, dp_k \wedge dp_l \; ; \; d\omega = 0 \tag{1.3}$$

Obviously the structure of such a two-form is not maintained by general diffeomorphisms of type **(1.1)**. But for an affine configuration space, there is the privileged group of affine transformations, $q^i \to q'^i = A^i{}_j\, q^j + b^i$, which conserve such a structure. When an origin is fixed, this configuration space is identified with the translation group $\mathscr{Q} = G \equiv \mathbf{R}^N$ with commutative Lie algebra $\mathscr{G} \equiv \mathbf{R}^N$ and dual $\mathscr{G}^\star \equiv \mathbf{R}^{*N}$. Furthermore, if $\mathbf{F}$ and $\mathbf{G}$ are constant, ω is invariant under translations. Such a situation was examined for the N-dimensional case in our previous work [7]. From the work of Souriau and others [1, 2, 4, 5] it is clear how to generalize the first term of this extension of the canonical symplectic two-form when configuration space is a Lie group G such that phase space is trivialised $T^*G \approx G \times \mathscr{G}^\star$. This is done introducing a symplectic one-cocycle, defined below.

2. The symplectic one-cocycle

A 1-chain θ on $\mathscr{G}$ with values in $\mathscr{G}^\star$, on which $\mathscr{G}$ acts with the coadjoint representation $\mathbf{k}$, $\theta \in C^1(\mathscr{G}, \mathscr{G}^\star, \mathbf{k})$, is a linear map $\theta : \mathscr{G} \to \mathscr{G}^\star : \mathbf{u} \to \theta(\mathbf{u})$.
Let $\{\mathbf{e}_\alpha\}$ be a basis of the Lie algebra $\mathscr{G}$ with dual basis $\{\varepsilon^\beta\}$ of $\mathscr{G}^\star$ and structure constants $[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\mu\, {}^\mu\mathbf{f}_{\alpha\beta}$. The 1-cochain is given by $\theta(\mathbf{u}) = \theta_{\alpha,\mu}\, u^\mu\, \varepsilon^\alpha$, where $\theta_{\alpha,\mu} \doteq \langle \theta(\mathbf{e}_\mu) | \mathbf{e}_\alpha \rangle$.
It is a 1-cocycle, $\theta \in Z^1(\mathscr{G}, \mathscr{G}^\star, \mathbf{k})$, if it has a vanishing coboundary:

$$\begin{aligned}
(\delta_1 \theta)(\mathbf{u}, \mathbf{v}) &\doteq \mathbf{k}(\mathbf{u})\theta(\mathbf{v}) - \mathbf{k}(\mathbf{v})\theta(\mathbf{u}) - \theta([\mathbf{u}, \mathbf{v}]) = 0 \\
\langle (\delta_1 \theta)(\mathbf{u}, \mathbf{v}) | \mathbf{w} \rangle &\doteq -\langle \theta(\mathbf{v}) | [\mathbf{u}, \mathbf{w}] \rangle + \langle \theta(\mathbf{u}) | [\mathbf{v}, \mathbf{w}] \rangle - \langle \theta([\mathbf{u}, \mathbf{v}]) | \mathbf{w} \rangle = 0 \\
(\delta_1 \theta)_{\alpha,\mu\nu} &\doteq \langle (\delta_1 \theta)(\mathbf{e}_\mu, \mathbf{e}_\nu) | \mathbf{e}_\alpha \rangle \\
&\doteq -\theta_{\kappa,\nu}\, {}^\kappa\mathbf{f}_{\mu\alpha} + \theta_{\kappa,\mu}\, {}^\kappa\mathbf{f}_{\nu\alpha} - \theta_{\kappa,\alpha}\, {}^\kappa\mathbf{f}_{\mu\nu} = 0
\end{aligned}$$

[1] Such an approach towards non commutative coordinates was originally proposed in [6] in the two-dimensional case with posible application to anyon physics.

The 1-cocycle is called symplectic if $\Theta(\mathbf{u},\mathbf{v}) \doteq \langle\theta(\mathbf{u})|\mathbf{v}\rangle$ is antisymmetric :

$$\Theta(\mathbf{u},\mathbf{v}) = -\Theta(\mathbf{v},\mathbf{u}) \; ; \; \Theta_{\alpha\mu} \doteq \theta_{\alpha,\mu}$$

Any antisymmetric Θ defined in terms of $\theta \in C^1(\mathscr{G},\mathscr{G}^\star,\mathbf{k})$ is actually a 2-cochain on $\mathscr{G}$ with values in $\mathbf{R}$ and trivial representation : $\Theta \in C^2(\mathscr{G},\mathbf{R},\mathbf{0})$. Furthermore, when $\theta \in Z^1(\mathscr{G},\mathscr{G}^\star,\mathbf{k})$, Θ is a 2-cocycle of $Z^2(\mathscr{G},\mathbf{R},\mathbf{0})$:

$$(\delta_2\Theta)(\mathbf{u},\mathbf{v},\mathbf{w}) \doteq -\Theta([\mathbf{u},\mathbf{v}],\mathbf{w}) + \Theta([\mathbf{u},\mathbf{w}],\mathbf{v}) - \Theta([\mathbf{v},\mathbf{w}],\mathbf{u}) = 0$$

$$(\delta_2\Theta)(\mathbf{e}_\alpha,\mathbf{e}_\beta,\mathbf{e}_\gamma) \doteq -\Theta_{\kappa\gamma}\,{}^\kappa\mathbf{f}_{\alpha\beta} + \Theta_{\kappa\beta}\,{}^\kappa\mathbf{f}_{\alpha\gamma} - \Theta_{\kappa\alpha}\,{}^\kappa\mathbf{f}_{\beta\gamma} = 0 \tag{2.1}$$

When $\mathscr{G}$ is semisimple, Θ is exact. Indeed, the Whitehead lemmata state that $H^1(\mathscr{G},\mathbf{R},\mathbf{0}) = 0$ and $H^2(\mathscr{G},\mathbf{R},\mathbf{0}) = 0$. So, Θ is a coboundary of $B^2(\mathscr{G},\mathbf{R},\mathbf{0})$ and there exists an element ξ of $C^1(\mathscr{G},\mathbf{R},\mathbf{0}) \equiv \mathscr{G}^\star$ such that $\Theta(\mathbf{u},\mathbf{v}) = (\delta_1(\xi))(\mathbf{u},\mathbf{v}) = -\xi([\mathbf{u},\mathbf{v}])$ or $\Theta_{\alpha\beta} = -\xi_\mu\,{}^\mu\mathbf{f}_{\alpha\beta}$.
In general, $\Theta = \frac{1}{2}\Theta_{\alpha\beta}\,\varepsilon^\alpha \wedge \varepsilon^\beta$, with Θ obeying the cocycle condition **(2.1)**. Acting with $L^\star{}_{g^{-1}|g}$: $T_e^\star(G) \to T_g^\star(G)$, yields the left-invariant forms :

$$\varepsilon_L^\alpha(g) \doteq L^\star{}_{g^{-1}|g}\,\varepsilon^\alpha = L^\alpha{}_\beta(g^{-1};g)\,\mathbf{d}g^\beta$$
$$\Theta_L(g) \doteq L^\star{}_{g^{-1}|g}\,\Theta = (1/2)\,\Theta_{\alpha\beta}\,\varepsilon_L^\alpha(g) \wedge \varepsilon_L^\beta(g)$$

where $L^\alpha{}_\beta(g;h) \doteq \partial(gh)^\alpha/\partial h^\beta$. Using the cocycle relation **(2.1)** and the Maurer-Cartan structure equations,

$$\mathbf{d}\varepsilon_L^\alpha(g) = -\frac{1}{2}\,{}^\alpha\mathbf{f}_{\mu\nu}\,\varepsilon_L^\mu(g) \wedge \varepsilon_L^\nu(g)$$

it is seen that $\Theta_L(g)$ is a closed left-invariant two-form on G.

3. G Actions on $T^\star(G)$

Natural coordinates of points $x = (g,\mathbf{p}) \in T^\star(G)$ are given by (g^α, p_β), where $\mathbf{p} = p_\beta\,\mathbf{d}g^\beta$. There are two canonical trivialisations of the cotangent bundle.

- The left trivialisation :

  $$\lambda : T^\star(G) \to G \times \mathscr{G}^\star : (g, p_g) \to \left(g, \pi^L = L^\star{}_{g|e}\,p_g = \pi^L{}_\mu\,\varepsilon^\mu\right)_{\mathbf{B}}$$

  which yields "body" coordinates, given by $(g^\alpha, \pi^L{}_\mu)_{\mathbf{B}}$.

- The right trivialisation :

  $$\rho : T^\star(G) \to G \times \mathscr{G}^\star : (g, p_g) \to \left(g, \pi^R = R^\star{}_{g|e}\,p_g = \pi^R{}_\mu\,\varepsilon^\mu\right)_{\mathbf{S}}$$

  which yields "space" coordinates, given by $(g^\alpha, \pi^R{}_\mu)_{\mathbf{B}}$.

They are related by : $\pi^R = R^{\star}{}_{g^{-1}|g} \circ L^{\star}{}_{g|e}\, \pi^L = \mathbf{K}(g)\, \pi^L$, where $\mathbf{K}(g)$ is the coadjoint representation of G in $\mathscr{G}^{\star}$.
Lifting the left multiplication of G by G to the cotangent bundle yields

$$\Phi^L_a : T^{\star}(G) \rightarrow T^{\star}(G) : x = (g, p_g) \rightarrow y = (ag, p'_{ag} = L^{\star}_{a^{-1}|ag}\, p_g)$$

From $\lambda \circ L^{\star}_{a^{-1}|ag} : p_g \rightarrow L^{\star}_{ag|e} \circ L^{\star}_{a^{-1}|ag}\, p_g = L^{\star}_{g|e}\, p_g = \pi$, it is seen that, in body coordinates, $\left(\Phi^L_a\right)_{\mathbf{B}} \doteq \lambda \circ \Phi^L_a \circ \lambda^{-1} : (g, \pi^L)_{\mathbf{B}} \rightarrow (ag, \pi^L)_{\mathbf{B}}$.
The pull-back of the cotangent projection $\kappa : T^{\star}(G) \rightarrow G : x \doteq (g, \mathbf{p}) \rightarrow g$, yields differential forms on the cotangent bundle :

$$\begin{aligned} \langle \varepsilon^{\alpha}_L(x)| &= \kappa^{\star}_x\, \varepsilon^{\alpha}_L(\kappa(x)) \\ \widetilde{\Theta}_L(x) &= \kappa^{\star}_x\, \Theta_L(\kappa(x)) = -\frac{1}{2}\, \Theta_{\alpha\beta}\, \langle \varepsilon^{\alpha}_L(x)| \wedge \langle \varepsilon^{\beta}_L(x)| \end{aligned} \tag{3.1}$$

Since $\Theta(g)$ is closed on G, its pull-back, $\widetilde{\Theta}_L(x)$, is a closed 2-form on $T^{\star}(G)$.
Furthermore, the left-invariance of $\varepsilon^{\alpha}_L(g) : L^{\star}_{a^{-1}|ag}\, \varepsilon^{\alpha}(g) = \varepsilon^{\alpha}(ag)$ implies the Φ^L_a-invariance of its pull-back : $(\Phi^L_a)^{\star}_{|x}\, \langle \varepsilon^{\alpha}_L(\Phi^L_a(x))| = \langle \varepsilon^{\alpha}_L(x)|$ and so is $\widetilde{\Theta}_L(x)$. A Φ^L_a-invariant basis of one-forms on $T^{\star}(T^{\star}(G))$ is

$$\{\langle \varepsilon^{\alpha}_L|; \langle \mathbf{d}\pi^L{}_{\mu}|\} \tag{3.2}$$

The right multiplication by a^{-1} induces another *left* action by :

$$\Phi^R_a : T^{\star}_g(G) \rightarrow T^{\star}_{ga^{-1}}(G) : (g, p_g) \rightarrow (ga^{-1}, p'_{ga^{-1}} = R^{\star}_{a|ga^{-1}}\, p_g)\ ,$$

Computing : $L^{\star}_{ga^{-1}|e} \circ R^{\star}_{a|ga^{-1}} \circ L^{\star}_{g|e}\, \pi^L = L^{\star}_{a^{-1}|e} \circ R^{\star}_{a|a^{-1}}\, \pi^L$, it follows that, in body coordinates, Φ^R_a acts as : $\Phi^R_a : (g, \pi^L)_{\mathbf{B}} \rightarrow (g' = ga^{-1}, \pi'^L = \mathbf{K}(a)\pi^L)_{\mathbf{B}}$. Under Φ^R_a, the Φ^L_a-invariant basis **(3.2)** transforms as

$$\begin{aligned} (\Phi^R_a)^{\star}_{|x}\, \langle \varepsilon^{\alpha}_L(\Phi^R_a(x))| &= \mathbf{Ad}^{\alpha}{}_{\beta}(a)\, \langle \varepsilon^{\beta}_L(x)| \\ (\Phi^R_a)^{\star}_{|x}\, \langle \mathbf{d}\pi'^L{}_{\mu}| &= \langle \mathbf{d}\pi^L{}_{\nu}|\, \mathbf{Ad}^{\nu}{}_{\mu}(a^{-1}) \end{aligned} \tag{3.3}$$

4. The modified symplectic structure on $T^{\star}(G)$

The canonical Liouville one-form on $T^{\star}(G)$ and its associated symplectic two-form are $\langle \theta_0| = p_{\alpha}\, \langle dg^{\alpha}| = \pi_{\mu}\, \langle \varepsilon^{\mu}_L|$, and

$$\begin{aligned} \omega_0 &= -\mathbf{d}\langle \theta_0| = -\pi_{\mu}\, \mathbf{d}\langle \varepsilon^{\mu}_L| + \langle \varepsilon^{\mu}| \wedge \langle \mathbf{d}\pi_{\mu}| \\ &= \frac{1}{2}\, \pi_{\mu}\, {}^{\mu}\mathbf{f}_{\alpha\beta}\, \langle \varepsilon^{\alpha}| \wedge \langle \varepsilon^{\beta}| + \langle \varepsilon^{\mu}| \wedge \langle \mathbf{d}\pi_{\mu}| \end{aligned} \tag{4.1}$$

A modified symplectic two-form is obtained adding the closed two-form **(3.1)**, constructed from the symplectic cocycle:

$$\omega = \omega_0 + \widetilde{\Theta}_L = \frac{1}{2}\, \left(\pi_{\mu}\, {}^{\mu}\mathbf{f}_{\alpha\beta} + \Theta_{\alpha\beta}\right)\, \langle \varepsilon^{\alpha}| \wedge \langle \varepsilon^{\beta}| + \langle \varepsilon^{\mu}| \wedge \langle \mathbf{d}\pi_{\mu}| \tag{4.2}$$

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For semisimple $\mathscr{G}$, this reduces to :

$$\omega = \frac{1}{2}\,(\pi_\mu - \xi_\mu)\,{}^\mu\mathbf{f}_{\alpha\beta}\,\langle\varepsilon^\alpha|\wedge\langle\varepsilon^\beta| + \langle\varepsilon^\mu|\wedge\langle\mathbf{d}\pi_\mu| = -\,\mathbf{d}\,\big((\pi_\mu - \xi_\mu)\,\langle\varepsilon_L^\mu|\big) \tag{4.3}$$

This means that the Liouville form is modified $\langle\theta_L| = \big((\pi_\mu - \xi_\mu)\,\langle\varepsilon_L^\mu|\big)$ such that $\omega = -\mathbf{d}\langle\theta_L|$ and that $\{g, p'_g \doteq L^\star{}_{g^{-1}|g}(\pi - \xi)\}$ and there are global Darboux coordinates : $\{g^\alpha, p'{}_\mu = p_\mu - \xi_\beta\, L^\beta{}_\mu(g^{-1};g)\}$.
Finally we may add another left-invariant and closed two-form in the π variables $\widetilde{\Upsilon}_L = (1/2)\,\Upsilon^{\mu\nu}\,\langle\mathbf{d}\pi_\mu|\wedge\langle\mathbf{d}\pi_\nu|$ such that

$$\omega_L = \omega_0 + \widetilde{\Theta}_L + \widetilde{\Upsilon}_L \tag{4.4}$$

defines a Φ_a^L-invariant (pre)-symplectic two form on $T^\star(G)$.
Under Φ_a^R, this (pre-)symplectic two-form **(4.5)** is invariant if a belongs to the intersection of the isotropy groups of $\widetilde{\Theta}_L$ and $\widetilde{\Upsilon}_L$:

$$\Theta_{\alpha\beta}\,\mathbf{Ad}^\alpha{}_\mu(a)\,\mathbf{Ad}^\beta{}_\nu(a) = \Theta_{\mu\nu}\,;\;\mathbf{Ad}^\alpha{}_\mu(a^{-1})\,\mathbf{Ad}^\beta{}_\nu(a^{-1})\,\Upsilon^{\mu\nu} = \Upsilon^{\alpha\beta} \tag{4.5}$$

5. Conclusions

The degeneracy of the two-form **(4.4)** will be examined in further work, as was done in [7] for the abelian group. If ω_L is not degenerate, Poisson Brackets can be defined and, in the degenerate case, the constrained formalism of [3] is applicable. Finally, if the isotropy group of **(4.5)** is not empty, the remaining Φ_a^R-invariance will provide momentum mappings. Equations of motion of the Euler type will follow from a Hamiltonian of the form

$$H \doteq \frac{1}{2}\,\mathscr{I}^{\mu\nu}\,\pi^L{}_\mu\,\pi^L{}_\nu$$

The momenta mentionned above will be conserved if the isotropy group above also conserves the *inertia tensor* $\mathscr{I}$.

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