

## Towards a quantitative understanding of the $\Delta I = 1/2$ rule

---

**P. Hernández**<sup>\*†</sup>

*IFIC and Universidad de Valencia, Burjassot, Valencia, Spain*

*E-mail: pilar.hernandez@ific.uv.es*

A new strategy to quantify the role of the charm quark mass in the  $\Delta I = 1/2$  rule is summarized. Results for the low-energy couplings of the  $\Delta S = 1$  chiral effective Hamiltonian in a theory with a light charm quark (GIM limit) are obtained through a matching of three-point correlation functions computed in Chiral Perturbation Theory and in quenched lattice QCD. We observe a large  $\Delta I = 1/2$  enhancement, which is not large enough to explain the experimental result, but suggests significant long-distance contributions to the physical  $\Delta I = 1/2$  which are unrelated to penguin contractions or operators.

*XXIV International Symposium on Lattice Field Theory*

*July 23-28 2006*

*Tucson Arizona, US*

---

<sup>\*</sup>Speaker.

<sup>†</sup>The work reviewed here has been done in collaboration with L. Giusti, M. Laine, M. Lüscher, C. Pena, P. Weisz, J. Wennekers and H. Wittig.

## 1. The $\Delta I = 1/2$ rule

One of the most striking hierarchies in hadronic physics is the famous  $\Delta I = 1/2$  rule, which refers to the experimental observation that the kaon decay amplitude in two pions with total isospin,  $I = 0$ , is twenty times larger than that into an  $I = 2$  state:

$$T(K^0 \rightarrow \pi\pi|_{I=\alpha}) = A_\alpha e^{i\delta_\alpha} \quad \frac{A_0}{A_2} = 22.1. \quad (1.1)$$

The explanation of this large number is one of the notorious failures of large  $N_C$  which predicts [1] the ratio to be  $\sqrt{2}$ .

It was soon realized that there are many scales relevant in the dynamics of these decays,  $M_W, m_c, M_K, \dots$  and maybe subleading orders in  $N_C$  could get enhanced by large renormalization group logarithms [2].

Below the scale of the  $W$  mass, this boson can be integrated out of the theory. The resulting effective Hamiltonian for the  $\Delta S = 1$  transitions can be obtained through the Operator Product Expansion (OPE) of two left-handed currents. The use of CPS symmetry and the flavour symmetry restricts the possible dimension six four-quark operators to just two,  $Q_1^\pm$  and  $Q_2^\pm$ , which are singlets under the  $SU(4)_R$  and transform in the **84** and **20** representations of  $SU(4)_L$  [3]:

$$\mathcal{H}_W^{QCD} = 2\sqrt{2}G_F(V_{us})^*V_{ud} \sum_{\sigma=\pm} k_1^\sigma Q_1^\sigma + k_2^\sigma Q_2^\sigma, \quad (1.2)$$

where

$$\begin{aligned} Q_1^\pm &= \left\{ (\bar{s}\gamma_\mu P_- u)(\bar{u}\gamma_\mu P_- d) \pm (\bar{s}\gamma_\mu P_- d)(\bar{u}\gamma_\mu P_- u) \right\} - (u \rightarrow c), \\ Q_2^\pm &= (m_u^2 - m_c^2) \left\{ m_d (\bar{s}P_+ d) + m_s (\bar{s}P_- d) \right\}. \end{aligned} \quad (1.3)$$

The operators  $Q_2^\pm$  do not contribute to the physical amplitudes and vanish identically if  $m_u = m_c$ . The **84** operator,  $Q_1^+$ , contributes both to the  $\Delta I = 1/2, 3/2$  transitions, while the **20**,  $Q_1^-$ , only contributes to  $\Delta I = 1/2$ . This Hamiltonian can be run down to lower scales resumming leading logarithms through the renormalization group. A  $\Delta I = 1/2$  enhancement of the Wilson coefficients is then observed at low scales in commonly used schemes [2]. For example in the renormalization group invariant scheme (RGI) at 2-loops:

$$\left[ \frac{k_1^-(\mu)}{k_1^+(\mu)} \right]_{\mu=m_c} \simeq 2. \quad (1.4)$$

Although this goes in the right direction, a much bigger hierarchy must come from the hadronic matrix elements to match the experiment. Besides at  $\mu = m_c$  higher-order corrections to eq. (1.4) are  $\mathcal{O}(100\%)$ .

One can nevertheless try to push this perturbative analysis below the charm quark mass to obtain some qualitative understanding. When the charm is integrated out of the theory, one moves from a four-flavour theory to a three-flavour one. The OPE then allows any four-quark operator that transforms as the **27** or **8** dimensional representations of  $SU(3)_L$ . Among the latter there are operators that are no longer the product of two left currents, but involve the product of left and right

currents or densities, such as the famous *penguin* operators. Arguments were put forward to argue that the hadronic matrix elements of the penguin operators are larger than those of the left current operators and this could be the origin of the rule [4].

The computation of these matrix elements requires however a non-perturbative method. Many approaches to estimate the matrix elements using large  $N_C$  arguments or models have been pursued in the past. For a review and further references see e.g. Ref. [5]. Although plausible arguments seem to indicate that the rule could be the result of the accumulation of several instances of the octet enhancement, combined with large penguin matrix elements [6], the approach relies on the use of perturbation theory down to dangerously low scales.

## 2. The $\Delta I = 1/2$ rule on the lattice

It was realized in the early days of Lattice QCD [7] that the  $\Delta I = 1/2$  rule would be a very well suited problem for the lattice approach, since it is such a large effect! Even if there are approximations like quenching, or in the presence of large systematic uncertainties, such a large enhancement would be hard to miss.

In the pioneering work of Ref. [8], it was proposed to use the lattice to perform the matching of the  $\Delta S = 1$  effective Hamiltonian of eq. (1.2) to an effective Hamiltonian in terms of the hadronic degrees of freedom, that is a chiral Lagrangian. The possibility to include the  $\Delta S = 1$  interactions in the chiral Lagrangian as a perturbation was first proposed by Cronin in Ref. [9]. In addition to the chiral Lagrangian that describes the strong interactions,  $\mathcal{L}_{\chi PT}$ , one would have an effective weak Hamiltonian with the same flavour symmetries as that in eq. (1.2),  $\mathcal{H}_w^\chi$ . The operators  $Q_i^\pm$  can be decomposed into **27** and **8** of  $SU(3)_L$ . Therefore all the operators that can be constructed with the building blocks of the chiral Lagrangian,  $U, \partial_\mu, M$ , with the same transformation properties as  $Q_i^\pm$  should be included. At the leading order (LO) in a momentum or mass expansion there are just three of them, one 27-plet and two octets. The weak Hamiltonian can be written [8]:

$$\mathcal{H}_w^\chi \equiv 2\sqrt{2}G_F V_{ud}V_{us}^* \left\{ \frac{5}{3}g_{27}\mathcal{O}_{27} + 2g_8\mathcal{O}_8 + 2g'_8\mathcal{O}'_8 \right\} + \text{H.c.} , \quad (2.1)$$

where  $g_8, g_{27}$  and  $g'_8$  are low-energy couplings that contain the non-perturbative dynamics that is not fixed by symmetry arguments. The operators read

$$\mathcal{O}_{27} \equiv [\hat{\mathcal{O}}_w]_{suud}^+ = \frac{3}{5} \left( [\mathcal{O}_w]_{sudu} + \frac{2}{3} [\mathcal{O}_w]_{suud} \right) , \quad (2.2)$$

$$[\mathcal{O}_w]_{rsuv} \equiv \frac{F^4}{4} \left( \partial_\mu U U^\dagger \right)_{ur} \left( \partial_\mu U U^\dagger \right)_{vs} , \quad (2.3)$$

$$\mathcal{O}_8 \equiv \frac{1}{2} \sum_{k=u,d,s} [\mathcal{O}_w]_{skkd} , \quad (2.4)$$

$$\mathcal{O}'_8 \equiv \frac{F^2}{2} \Sigma \left( e^{i\theta/N_f} M U + U^\dagger M^\dagger e^{-i\theta/N_f} \right)_{ds} , \quad (2.5)$$

where we have made use of  $\text{Tr}[\partial_\mu U U^\dagger] = 0$  to simplify the expressions.

If these couplings were known, the ratio of the  $\Delta S = 1$  amplitudes at LO in the chiral expansion would be:

$$\frac{A_0}{A_2} = \frac{1}{\sqrt{2}} \left( \frac{1}{5} + \frac{9}{5} \frac{g_8}{g_{27}} \right) . \quad (2.6)$$

LO chiral perturbation theory is not expected to be very precise at the scale of  $M_K$ , but again the enhancement is such a large effect that as long as the expansion is reasonably well-behaved, the effect should already be there at the LO.

It was then proposed in [8] that these low-energy couplings could be determined by matching appropriate (the simplest) correlation functions between the chiral effective theory and lattice QCD, and in particular that this could be done through the computation of three-point functions and two-point functions. Note that to compute directly the amplitude  $K \rightarrow \pi\pi$ , four-point functions would be needed.

This nice proposal turned out to be extremely difficult to implement in practice. Firstly the renormalization of four-fermion operators is extremely challenging when there is explicit breaking of chiral symmetry in the regularization. Not only there are a large number of additional mixings with wrong-chirality operators, but there is even mixing with lower dimensional ones and therefore power-diverging coefficients [10]. With the advent of Ginsparg-Wilson regularizations [11, 12, 13], that preserve an exact chiral symmetry [14], the renormalization of the effective Hamiltonian of eq. (1.2) becomes as simple as it is in the continuum [15]<sup>1</sup>:

$$\begin{aligned} Q_1^\pm &= Z_{11}^\pm Q_1^{\pm,\text{bare}} + Z_{12}^\pm Q_2^{\pm,\text{bare}}, \\ Q_2^\pm &= Z_{22}^\pm Q_2^{\pm,\text{bare}}. \end{aligned} \quad (2.7)$$

Indeed the computations of the  $\Delta I = 1/2$  rule in Refs. [16, 17] have been performed in the quenched approximation using domain-wall fermions [18], that approximately satisfy the Ginsparg-Wilson relation. The lattice spacing used in these computations was however too low to keep the charm active, so they considered the effective Hamiltonian where the charm is integrated out perturbatively. The effect of integrating out the charm brings in important complications. On the one hand, the renormalization involves power-divergent subtractions, which require a very good control over statistical as well as systematic uncertainties. A second difficulty was pointed out in Ref. [19] and is related to the quenched ambiguities, which occur at the level of the OPE. More concretely there are operators in the OPE that are octets under the valence group and contain ghost quarks, in the supersymmetric formulation of the quenched approximation [20]. Now, if these operators are not included in the OPE, by assuming for instance that the quenched approximation is only used to define the matrix elements at some low scale, it is unclear what the meaning of non-perturbative renormalization is, since it requires to compute the matrix elements in the quenched approximation up to very high scales. In other words it is not clear whether the use of the quenched approximation to compute matrix elements combined with the full theory OPE is really a consistent method.

Finally the simulations in Refs. [16, 17] were carried out at relatively large quark masses above  $m_s/2$ . Chiral corrections were shown to be very large and a large systematic error resulted from the long chiral extrapolations. In the end, both collaborations found a large enhancement but there was almost a factor of 2 discrepancy between the two computations.

### 3. New strategy

The approach to the  $\Delta I = 1/2$  rule that was presented in Ref. [21] was designed not to reach the final result directly, but to try to reveal if the large enhancement is coming from one single leading

<sup>1</sup>Only in Ref. [21] the mixing with GIM suppressed operators  $Q_2^\pm$  has been discussed.

effect. The point is that on the lattice, the different physical scales that are involved in these decays can be modified at will in order to understand their relevance. In particular this is quite clearly the case for the charm quark mass.

Most of the ideas that have been put forward to explain the enhancement are related in one way or the other to the charm quark mass. If the large enhancement is due to the large separation between  $m_c$  and  $\Lambda_{QCD}$  or the up quark mass, there should be no enhancement whatsoever in a theory with a light charm quark.

More concretely the strategy that we proposed to quantify the role of the charm quark mass is the following:

- Step 1: light charm quark

A theory with four degenerate quarks,  $m_c = m_u = m_d = m_s$  (GIM limit) is matched to a  $SU(4)$  chiral effective theory to extract the low-energy couplings that would mediate  $\Delta S = 1$  transitions

- Step 2:  $\Lambda_{\chi PT} \gg m_c \gg m_u = m_d = m_s$

If the charm gets significantly heavier than the up quark, but remains in the realm of the chiral theory, that is charmed mesons are well below the chiral theory cutoff  $4\pi F$ , the charm quark can be integrated out from the  $SU(4)$  chiral theory to obtain the  $SU(3)$  chiral effective theory. If the charm quark is not too large this can be done analytically in chiral perturbation theory [22].

- Step 3:  $m_c \geq \Lambda_{\chi PT} \gg m_u = m_d = m_s$

If the charm gets too heavy to be describable in terms of an effective chiral theory, the matching to the  $SU(3)$  chiral theory has to be done non-perturbatively. The couplings  $g_{27}(m_c)$  and  $g_8(m_c)$  can then be monitored as a function of the charm quark mass.

#### 4. $K \rightarrow \pi\pi$ amplitudes in the GIM limit

We will now describe the formulation of the problem in the GIM limit, that is for  $m_u = m_d = m_s = m_c$ .

##### 4.1 Lattice formulation

In this limit there are only two operators in the OPE at first order in  $G_F$ ,  $Q_1^\pm$  in eq. (1.2), which renormalize therefore multiplicatively

$$Q_1^\pm = Z_{11}^\pm Q_1^{\pm, \text{bare}}. \quad (4.1)$$

In the quenched approximation no spurious operator can appear [21].

It can be shown [21] that using overlap fermions [13, 14, 23] this simple renormalization pattern is preserved. In Ref. [24], the renormalization constants have been computed non-perturbatively through an intermediate matching to twisted-mass Wilson fermions at some large reference quark mass. The corresponding renormalization constants for twisted-mass Wilson fermions have been previously computed using the Schrödinger functional approach [25].

	bare P.T.	MFI P.T.	N.P.
$Z_{11}^-/Z_{11}^+$	0.525	0.582	0.584(62)
$Z_{11}^+/Z_A^2$	1.242	1.193	1.15(12)
$Z_{11}^-/Z_A^2$	0.657	0.705	0.561(61)

**Table 1:** Comparison of the perturbative, mean-field improved and non-perturbative renormalization constants in the RGI scheme.

The result for the renormalization constants in the RGI scheme (for details see [21, 24]) is summarized in Table 1, where the perturbative (one-loop) and mean-field estimates are also shown for comparison.

## 4.2 $\chi PT$ formulation

In the GIM limit, the chiral Lagrangian has a  $SU(4)_L \times SU(4)_R$  symmetry group. It can be shown that at leading order in the momentum expansion, and in contrast with the  $SU(3)$  case, only two operators appear one **84** and a **20**:

$$\mathcal{H}_w^{\chi PT} = 2\sqrt{2}G_F (V_{us})^* V_{ud} \sum_{\sigma=\pm} g^\sigma [\mathcal{O}^\sigma]$$

where

$$\mathcal{O}^\pm = \frac{F^4}{4} [(U \partial_\mu U^\dagger)_{us} (U \partial_\mu U^\dagger)_{du} \pm (U \partial_\mu U^\dagger)_{uu} (U \partial_\mu U^\dagger)_{ds} - (u \rightarrow c)].$$

The normalization is such that in the large  $N_C$  limit

$$[g^+]_{N_c} = [g^-]_{N_c} = 1. \quad (4.2)$$

The ratio of the  $K \rightarrow \pi\pi$  amplitudes in the GIM limit would be given by

$$\frac{A_0}{A_2} = \frac{1}{\sqrt{2}} \left( \frac{1}{2} + \frac{3}{2} \frac{g^-}{g^+} \right), \quad (4.3)$$

and therefore at LO in the chiral expansion the hierarchy of the amplitudes is directly related to a hierarchy in the couplings  $g_+$  and  $g_-$ . Our primary goal is therefore to determine these couplings.

We also note that no further operators appear in the quenched or partially quenched approximation at this order, so there are no Golterman-Pallante ambiguities in the effective theory [21].

## 4.3 The matching

In order to determine the couplings  $g^\pm$ , we will match suitable correlation functions computed in lattice QCD to those computed to next-to-leading (NLO) order in the chiral expansion. Even though we will do the matching at very small quark masses, we will be in a finite volume, entering the  $\varepsilon$ -regime of chiral perturbation theory [26].

We will consider three-point bare correlation functions of the four-fermion operators and two left-handed currents [27]:

$$C^\sigma(x_0, y_0) \equiv \sum_{\mathbf{x}, \mathbf{y}} \langle [J_{L0}(x)]_{du} Q^\pm(0) [J_{L0}(y)]_{us} \rangle. \quad (4.4)$$

More concretely we will match the ratio of these three-point functions to left-current correlators:

$$R^\sigma(x_0, y_0) \equiv \frac{C^\sigma(x_0, y_0)}{C(x_0)C(y_0)}, \quad (4.5)$$

where

$$C(x_0) \equiv \sum_{\mathbf{x}} \langle [J_{L0}(x)]_{\alpha\beta} [J_{L0}(0)]_{\beta\alpha} \rangle. \quad (4.6)$$

Factorizing out the unknown couplings, the same ratio can be computed in the effective theory [28]:

$$\mathcal{R}^\sigma(x_0, y_0) \equiv \frac{\mathcal{C}^\sigma(x_0, y_0)}{\mathcal{C}(x_0)\mathcal{C}(y_0)}, \quad (4.7)$$

where

$$\mathcal{C}^\pm(x_0, y_0) \equiv \int d^3x \int d^3y \langle [\mathcal{J}_{L0}(x)]_{du} \mathcal{O}^\pm(0) [\mathcal{J}_{L0}(y)]_{us} \rangle \quad (4.8)$$

$$\mathcal{C}^\pm(x_0) \equiv \int d^3x \langle [\mathcal{J}_{L0}(x)]_{\alpha\beta} [\mathcal{J}_{L0}(0)]_{\beta\alpha} \rangle. \quad (4.9)$$

The couplings can be extracted from the matching:

$$g^\pm = k^\sigma(M_W) \frac{Z_{11}^\sigma(g_0)}{Z_A^2} \frac{R^\sigma(x_0, y_0)}{\mathcal{R}^\sigma(x_0, y_0)}, \quad (4.10)$$

where the Wilson coefficients are obtained in the same renormalization scheme as the operators, i.e. the RGI one. For the explicit expressions see [21].

#### 4.4 $\mathcal{R}^\sigma$ to NLO in $\chi PT$

In order to obtain the LO couplings, the matching between lattice QCD and  $\chi PT$  should be done as close as possible to the chiral limit. The use of Ginsparg-Wilson regularization makes it possible to do simulations with extremely small quark masses: below a few MeV with volumes larger than 2 fm. However it becomes very costly to increase the volume much further than 2 fm or so. While the usual way to take the chiral limit is to first take  $V$  to infinity and only then take  $m$  to zero, there are advantages in taking  $m$  as small as possible at finite  $V$ , entering the so-called  $\varepsilon$ -regime defined by the condition:

$$m\Sigma V \leq 1. \quad (4.11)$$

For  $FL \gg 1$ , this limit is equivalent to  $ML \ll 1$ , so the Compton wavelength of the pion is larger than the box size. Finite volume effects are large in this regime, however they are calculable in chiral perturbation theory [26]. The counting rules of the chiral expansion that ensure the condition of eq. (4.11) are:

$$m\Sigma \sim \varepsilon^4 \quad L^{-1}, T^{-1} \sim \varepsilon \quad p \sim \varepsilon \quad (4.12)$$

which are different to the usual ones. As a result of the new power-counting:

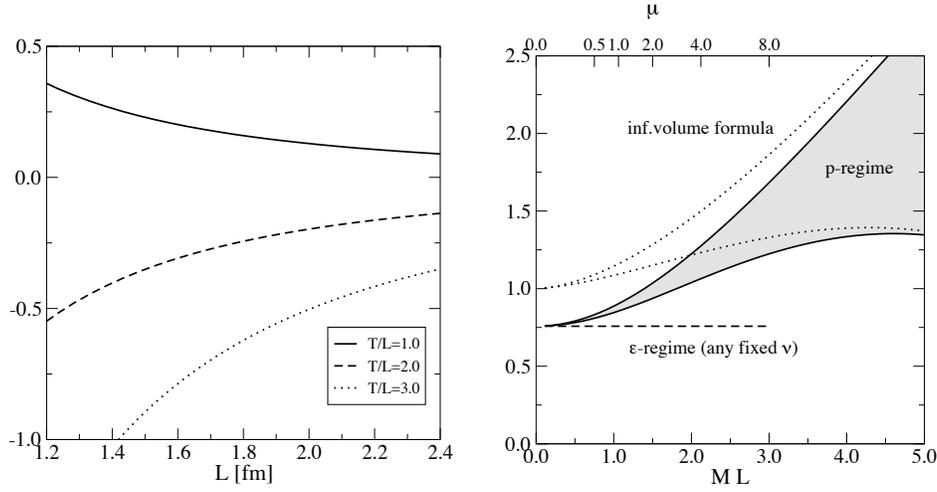
	$p$ -regime	$\varepsilon$ -regime
$\mathcal{L}_{QCD}$	$L_4 \langle D_\mu U^\dagger D^\mu U \rangle \langle \mathcal{S} \rangle$	×
	$L_5 \langle D_\mu U^\dagger D^\mu U \mathcal{S} \rangle$	×
	$L_6 \langle \mathcal{S} \rangle^2$	×
	$L_8 \langle \mathcal{S}^2 \rangle$	×
$\mathcal{H}_w^{\chi PT}$	$D_2^\pm t_{ij,kl}^\pm \mathcal{P}_{ij} \mathcal{P}_{kl}$	×
	$D_4^\pm t_{ij,kl}^\pm (\mathcal{L}_\mu)_{ij} (\{\mathcal{L}^\mu, \mathcal{S}\})_{kl}$	×
	$D_7^\pm t_{ij,kl}^\pm (\mathcal{L}_\mu)_{ij} (\mathcal{L}_\mu)_{kl} \langle \mathcal{S} \rangle$	×
	$D_{20}^\pm t_{ij,kl}^\pm (\mathcal{L}_\mu)_{ij} (\partial_\nu \mathcal{W}_{\mu\nu})_{kl}$	×
	$D_{24}^\pm t_{ij,kl}^\pm (\mathcal{W}_{\mu\nu})_{ij} (\mathcal{W}_{\mu\nu})_{kl}$	×

**Table 2:** Strong and weak operators that would contribute to the observables considered in eqs.(4.9) at NLO in infinite-volume chiral perturbation theory.  $\mathcal{S} \equiv U\chi^\dagger + \chi U^\dagger$ ,  $\mathcal{P} \equiv i(U\chi^\dagger - \chi U^\dagger)$ ,  $\mathcal{L}_\mu \equiv U\partial_\mu U^\dagger$  and  $\mathcal{W}_{\mu\nu} \equiv 2(\partial_\mu \mathcal{L}_\nu - \partial_\nu \mathcal{L}_\mu)$ , where  $\chi \equiv 2m\Sigma/F^2$ . The tensors  $t_{ij,kl}^\pm$  project onto the appropriate representations **84** or **20**.

- The zero-momentum modes of the pions become non-perturbative and have to be resummed to all orders. This is achieved by factorizing out the constant field configurations and treating them as collective variables in the partition functional.
- There is a reordering of the chiral expansion and at any given order less relevant couplings appear. In particular for the processes at hand the strong or weak higher-order couplings [30, 31] shown in Table 2 that would appear at NLO in infinite (or large enough) volume are all suppressed at the same order in the  $\varepsilon$ -regime.

An additional complication is the quenched approximation, where we shall be working. The quenched chiral-perturbation-theory version of the  $\varepsilon$ -regime was studied in detail in Ref. [32]. Analytical treatment is possible if averages are considered in fixed topological sectors. Ginsparg-Wilson fermions satisfy an exact index theorem [12] and therefore these averages can be computed on the lattice as well. The different role of topology in the  $\varepsilon$  and  $p$ -regimes was first discussed in [29]. The fixing of topology can be seen as a type of boundary condition, which should not affect the local properties of the theory, and therefore the dependence on the topological charge is in principle predictable in terms of the low-energy couplings of the effective theory, just as the finite-volume dependence is.

The result of  $\mathcal{R}_V^\sigma$  at NLO in the  $\varepsilon$ -regime, in a fixed-topological sector of charge  $\nu$ , was



**Figure 1:** Left: NLO correction  $K$  as defined in eq. (4.13) as a function of the spatial extent of the box  $L$  in fm for three different ratios  $T/L$ . Right:  $2R^+(-T/3, T/3)$  for a lattice of 2 fm as a function of  $ML$  for  $T/L = 2$ . The band corresponds to varying the NLO couplings within a large reasonable range. The dotted and dashed lines are the  $\infty$ -volume and the  $\varepsilon$ -regime results respectively.

computed in [28, 21] and the result is:

$$2 \mathcal{R}_v^\pm(x_0, y_0) = 1 \pm \frac{2}{(FL)^2} \left[ \rho^{-1/2} \beta_1 - \rho k_{00} \right] = 1 \pm K, \quad (4.13)$$

with  $\rho \equiv T/L$  and  $\beta_1, k_{00}$  are shape coefficients of the box that depend only on  $\rho$ . The ratio turns out to be the same in the full and in the (partially-)quenched theories. It is quite remarkable that the result does not depend on the insertion of the currents  $x_0, y_0$ , it is the same for all topological sectors  $v$ , and as expected no higher order coupling enters at this order. In the left plot of Figure 1 we show the numerical result for the NLO correction,  $K$ , as a function of the spatial extent of the box, for three values of the aspect ratio. If  $\rho$  is not too large<sup>2</sup>, the corrections are reasonably small for lattice sizes above 2 fm or so.

Since we will simulate a number of quark masses, including those for which  $m\Sigma V \gg 1$ , which is the usual regime of all previous calculations, we would need the results for the ratio  $\mathcal{R}^\sigma$  in the  $p$ -regime, where the counting rules are the same as in infinite volume and therefore at NLO the couplings of Table 2 become relevant. The result for  $\mathcal{R}^\sigma$  in the  $p$ -regime has been presented in [31] and is shown in the right plot of Figure 1 as a function of  $ML$  for a lattice of extent 2 fm. The band corresponds to changing the value of the unknown NLO couplings within a reasonable range. For comparison the  $\varepsilon$ -regime and  $\infty$ -volume results are also shown. Surprisingly we find that deviations from the  $\infty$ -volume expectations are significant up to  $ML \leq 5$  for these observables.

#### 4.5 $R^\sigma$ in lattice QCD

The computation of  $R^\sigma$  for small quark masses in a finite volume is non-trivial. It was observed that large fluctuations in observables containing quark propagators occur in the kinematical

<sup>2</sup>The appropriate regime for large values of  $\rho$  is the so-called  $\delta$ -regime, so actually the  $\varepsilon$ -regime expansion breaks down in the  $\rho \rightarrow \infty$  limit.

	$\beta$	$L/a$	$T/a$	$n_{\text{low}}$	$L[\text{fm}]$	$m$	# cfs
$\varepsilon$ -regime	5.8485	16	32	20	2	$m_s/40, m_s/60$	$O(800)$
$p$ -regime	5.8485	16	32	20	2	$m_2/2 - m_s/6$	$O(200)$

**Table 3:** Simulation parameters

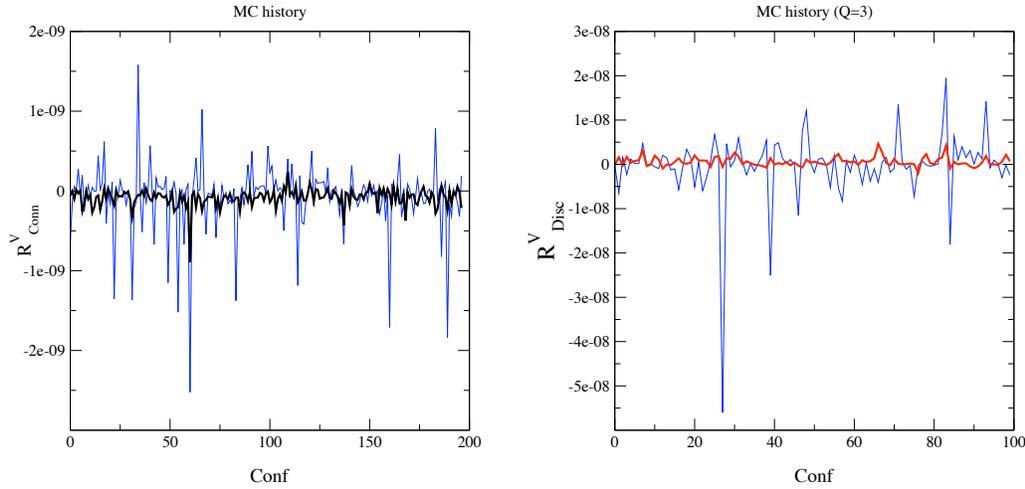
conditions of the  $\varepsilon$ -regime [33]. These fluctuations result from the fact that the low-lying spectrum of the Dirac operator is discrete in this regime. Both the low-lying eigenvalues of the massless Dirac operator and the splittings between them are controlled by the quantity  $(\Sigma V)^{-1}$ :

$$\langle \lambda_i \rangle \simeq \frac{\mathcal{O}(1)}{\Sigma V} \quad \Delta\lambda = \lambda_{i+1} - \lambda_i \simeq \frac{\mathcal{O}(1)}{\Sigma V}, \quad (4.14)$$

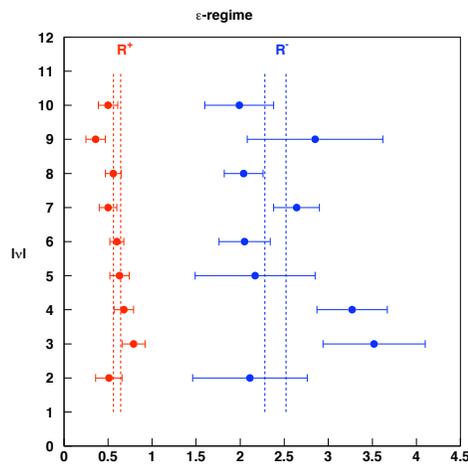
therefore the eigenvalues of the massive Dirac operator in the  $\varepsilon$ -regime where  $m \leq (\Sigma V)^{-1}$  are of the same order as their splittings. In this situation, observables with point-to-all quark propagators can get large contributions from a few eigenfunctions. Space-time fluctuations in these eigenfunctions can lead to large and rare fluctuations in observables whenever peaks in the wavefunction happen to be near the fixed-point of the point-to-all propagator [34]. It has been shown [34] that these fluctuations could be cured provided  $m\Sigma V \simeq 1$  through the technique of low-mode averaging (LMA) [34, 35]. The idea is to rewrite the quark propagator as a sum over its spectral decomposition onto the subspace spanned by a few low-lying eigenvalues (low part) and the rest (heavy part). Once this representation is included in the correlation functions, those contributions that contain the low-parts only, can be averaged over all possible spatial insertions of the sources, because the wavefunctions of the low-modes are known in all points. The mixed contributions often can also be averaged provided a few additional inversions of the Dirac operator on the low-modes are performed. This is indeed the case for the two-point functions [34] and three-point functions [21] that we need in the GIM limit. The contributions from just the heavy parts remain unaveraged, but since the low-modes are no longer affecting these contributions, they should be much better behaved.

The LMA typically requires the computation of the low-lying eigenvalues and eigenfunctions and the inversion of the Dirac operator on them, which is a non-negligible overhead, but still pays off as Figure 2 clearly shows. The two Montecarlo histories with and without LMA for one of the contractions of the three-point correlators of eq.(4.4) show that the improvement from LMA is very significant.

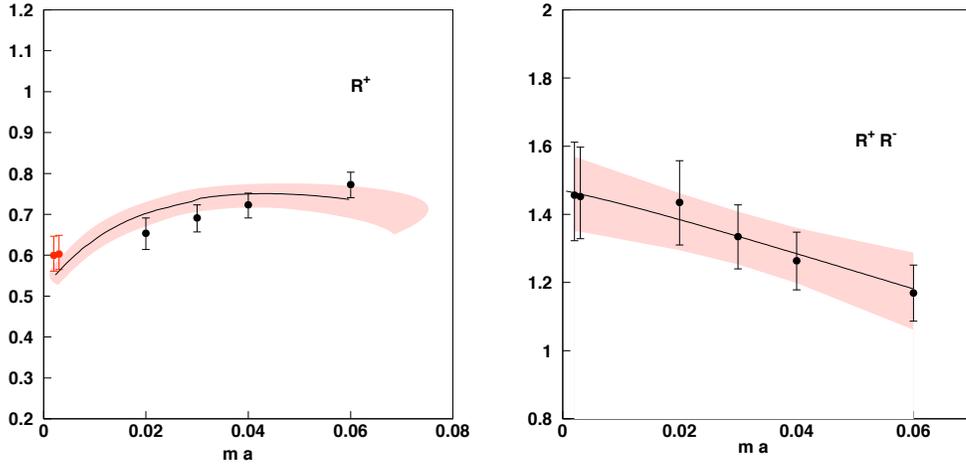
In [36] we have presented the first results of a lattice determination of  $R^\sigma$  in the quenched approximation using overlap fermions [13]. The simulation parameters are summarized in Table 3. The expected features of  $R_V^\sigma$  in the  $\varepsilon$ -regime as predicted from chiral perturbation theory in eq. (4.13) are well reproduced by the data. At large time separations, the ratio shows a flat behaviour in  $x_0$  and  $y_0$ . There is no signal of  $v$  dependence in  $R_V^\pm$  as shown in Figure 3. A weighted averaged is used to combine the result for all topologies. Finally there is no visible dependence with the quark mass for the small values considered. On the other hand the quark mass dependence of the ratios  $R^\pm$  in the  $p$ -regime is quite significant. Note that in the  $p$ -regime all topological sectors are averaged. In Figure 4 we show the results of the fits to NLO chiral perturbation theory expressions of the two combinations with smaller mass dependence:  $R^+$  and the product  $R^+R^-$ . The latter combination has the nice property that the NLO chiral corrections vanish at zero quark



**Figure 2:** Left: Montecarlo history with and without LMA of the color-connected contraction in  $R^\sigma$  for the lightest mass in the  $p$ -regime. Right: Montecarlo history with and without LMA of the color-disconnected contraction of  $R^\sigma$  in the  $\varepsilon$ -regime for  $|\nu| = 3$ .



**Figure 3:** Ratios  $R^\pm$  as a function of the topological charge  $2 \leq \nu \leq 10$ .



**Figure 4:** Left: NLO chiral perturbation theory fits of the bare  $R^+$  as a function of the quark mass. Right: NLO chiral perturbation theory fits of the bare  $R^+R^-$  as a function of the quark mass. The bands represent the statistical errors on the fitting function.

mass. Each of the fits has two free parameters  $((g^+)_{bare}, \Lambda_+)$  and  $((g^+g^-)_{bare}, \Lambda_\pm)$ , where  $\Lambda_+, \Lambda_\pm$  is a combination of the NLO couplings that enter in the  $p$ -regime. The bands contain the statistical (bootstrap) errors. Combining the results obtained from the fits of the bare ratios with the Wilson coefficients and the non-perturbative renormalization constants we obtain our main result:

$$g^+ = 0.51(3)(5)(6), \quad g^- = 2.6(1)(3)(3), \quad (4.15)$$

where the first error is statistical, the second comes from the renormalization factor and the third is a systematic error estimated from the dispersion of the values of  $g^\pm$  obtained from different fitting strategies (for example fitting only the  $\varepsilon$ -regime or only the  $p$ -regime points). Although the results have been obtained for just one lattice spacing, scaling studies of several observables with overlap fermions have shown that scaling violations tend to be very small [37].

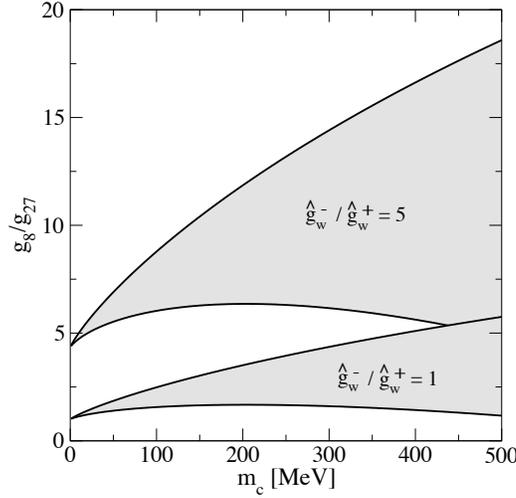
These numbers can be compared with those that would be obtained from experiment if the  $\Delta I = 3/2$  and  $\Delta I = 1/2$  amplitudes would be matched to the corresponding ones in LO chiral perturbation theory in the GIM limit:

$$g^+ \simeq 0.5, \quad g^- \simeq 10.4. \quad (4.16)$$

Therefore the value of  $g_+$  is strikingly close to experiment already in the GIM limit, but the value of  $g_-$  is a factor  $\sim 4$  smaller. A significant enhancement is therefore observed in this limit

$$\frac{A_0}{A_2} \simeq 6, \quad (4.17)$$

which is not large enough to explain the experimental ratio but is already significant and cannot be ascribed to penguin operators nor penguin contractions.



**Figure 5:** Ratio of the low-energy couplings  $g_8/g_{27}$  as a function of the charm quark mass for two different values of the GIM limit ratio  $g^-/g^+ = 1$  and  $5$ . The bands correspond to changing  $\Lambda_\chi = 1 - 4$  GeV.

## 5. Towards a heavier charm

Once the LECS in the GIM limit are known, the charm quark can be increased. If the charm is still light so that charmed mesons can be treated as chiral degrees of freedom, it is possible to match the two chiral theories with and without the charmed mesons analytically, or in other words the charmed mesons can be integrated out of the  $SU(4)$  chiral theory. In this way one recovers an  $SU(3)$  chiral Lagrangian where the couplings  $g_{27}$  and  $g_8$  can be computed as functions of  $g_\pm$  and the charm quark mass. Only when the charm quark mass satisfies

$$\frac{m_u \Sigma}{F^2} \ll \frac{m_c \Sigma}{F^2} \ll (4\pi F)^2. \quad (5.1)$$

is this perturbative matching reliable.

This exercise was carried out in Refs. [22, 21] with the result:

$$g_8(m_c) = \frac{1}{2} \left[ \frac{1}{5} g^+ \left( 1 + 15 \frac{M_c^2}{(4\pi F)^2} \ln \frac{\Lambda_\chi}{M_c} \right) + g^- \left( 1 + 3 \frac{M_c^2}{(4\pi F)^2} \ln \frac{\Lambda_\chi}{M_c} \right) \right] \quad (5.2)$$

$$g_{27}(m_c) = \frac{3}{5} g^+, \quad (5.3)$$

where  $\Lambda_\chi$  contains the unknown information on the NLO couplings that are relevant as  $m_c$  increases. In Figure 5 the dependence on  $m_c$  of the ratio  $g_8/g_{27}$  is shown for two values of the ratio of couplings in the GIM limit  $g^-/g^+ = 1, 5$ . The surprising observation is that only the octet coupling has a logarithmic enhancement. Unfortunately NLO couplings are needed to have predictability. The effect of these is represented by the band where they have been varied within reasonable values (the associated scale  $\Lambda_\chi = 1 - 4$  GeV). For a ratio of  $g^-/g^+ \sim 5$ , close to the value we have obtained in the GIM limit, there could be a large effect coming from the integration of the charm. Unfortunately there is no much predictability unless the NLO couplings are known.

In order to go beyond chiral perturbation theory it is necessary to do a non-perturbative matching this time with a heavy charm quark, which is the next step of our project.

It is well known that the case of a heavier charm will bring additional challenges. On the numerical side, the computation of three-point functions with a heavy charm requires the evaluation of the penguin contractions, which involve a point-point propagator. We are confident that LMA will also help in this case, but this has yet to be demonstrated. On the theoretical side, the quenched ambiguities of Golterman and Pallante will be present in the  $SU(3)$  chiral effective theory. It has been shown in [31] that these quenching ambiguities are rather mild in the  $\varepsilon$ -regime, and it is in principle possible to disentangle the “physical” couplings from the spurious ones.

## 6. Conclusions

The  $\Delta I = 1/2$  rule remains a big challenge for lattice QCD. We have presented a well-defined strategy to quantify the role of the charm quark mass and in particular of the penguin contractions in the enhancement [21]. The idea is to compute the low-energy couplings mediating the  $\Delta S = 1$  transitions in a theory where the charm is light and degenerate with the remaining three quarks (GIM limit) and compare them with those in a theory with just three light flavours, where the corresponding couplings can be computed as a function of the charm quark mass.

The GIM limit is easier to treat in many respects and the first results for the couplings have recently been presented in Ref. [36] in the quenched approximation. The low-energy couplings in this limit already show a significant enhancement of the  $\Delta I = 1/2$  type. Even though the enhancement is not large enough to match the experiment, it already indicates that penguin operator/contractions cannot be the whole story.

## Acknowledgements

I would like to thank my collaborators in the work presented here as well as in earlier work related to this subject: P. Damgaard, M.C. Diamantini, L. Giusti, K. Jansen, M. Laine, L. Lellouch, M. Lüscher, C. Pena, P. Weisz, J. Wenekers and H. Wittig for an enjoyable collaboration and many illuminating discussions. I acknowledge partial support from the Spanish CICYT (Project No. FPA2004-00996 and FPA2005-01678) and by the Generalitat Valenciana (Project No. GVA05/164).

## References

- [1] M. Fukugita, T. Inami, N. Sakai and S. Yazaki, *Phys. Lett. B* 72 (1977) 237.
- [2] K. Wilson, *Phys. Rev.* 179 (1969) 1499. M. Gaillard and B.W. Lee, *Phys. Rev. Lett.* 33 (1974) 108. G. Altarelli and L. Maiani, *Phys. Lett. B* 52 (1974) 351.
- [3] H. Georgi, *Weak Interactions and Modern Particle Theory* (Benjamin/Cummings, Menlo Park, California, 1984)
- [4] A.I. Vainshtein, V.I. Zakharov and M.A. Shifman, *JETP* 45 (1977) 670.
- [5] A. J. Buras, *Adv. Ser. Direct. High Energy Phys.* 3 (1989) 575. E. de Rafael, arXiv: hep-ph/9502254.

- [6] W. A. Bardeen, A. J. Buras and J. M. Gerard, Phys. Lett. B 192 (1987) 138.
- [7] N. Cabbibo, G. Martinelli and R. Petronzio, Nucl. Phys. B244 (1984) 381. R.C. Brower *et al*, Phys. Rev. Lett. 53 (1984) 1318.
- [8] C.W. Bernard, T. Draper, A. Soni, H.D. Politzer and M.B. Wise, Phys. Rev. D32 (1985) 2343
- [9] J. A. Cronin, Phys. Rev. 161 (1967) 1483.
- [10] M. Bochicchio, L. Maiani, G. Martinelli, G.C. Rossi and M. Testa, Nucl. Phys. B262 (1985) 331; L. Maiani, G. Martinelli, G.C. Rossi and M. Testa, Nucl. Phys. B289 (1987) 505
- [11] P.H. Ginsparg and K.G. Wilson, Phys. Rev. D25 (1982) 2649
- [12] P. Hasenfratz, Nucl. Phys. B (Proc. Suppl.) 63 (1998) 53; Nucl. Phys. B525 (1998) 401  
P. Hasenfratz, V. Laliena and F. Niedermayer, Phys. Lett. B427 (1998) 125
- [13] H. Neuberger, Phys. Lett. B417 (1998) 141; *ibid.* B427 (1998) 353
- [14] M. Lüscher, Phys. Lett. B428 (1998) 342
- [15] S. Aoki and Y. Kuramashi, Phys. Rev. D63 (2001) 054504 S. Capitani and L. Giusti, Phys. Rev. D62 (2000) 114506
- [16] RBC Collaboration, (T. Blum *et al.*), Phys. Rev. D68 (2003) 114506
- [17] CP-PACS Collaboration (J.I. Noaki *et al.*), Phys. Rev. D68 (2003) 014501
- [18] D.B. Kaplan, Phys. Lett. B288 (1992) 342; Nucl. Phys. B (Proc. Suppl.) 30 (1993) 597 Y. Shamir, Nucl. Phys. B406 (1993) 90 V. Furman and Y. Shamir, Nucl. Phys. B439 (1995) 54
- [19] M. Golterman and E. Pallante, JHEP 0110 (2001) 037; Phys. Rev. D69 (2004) 074503
- [20] C.W. Bernard, M.F.L. Golterman, Phys. Rev. D46 (1992) 853 [arXiv:hep-lat/9204007]. S.R. Sharpe, Phys. Rev. D 46 (1992) 3146 [arXiv:hep-lat/9205020].
- [21] L. Giusti, P. Hernández, M. Laine, P. Weisz and H. Wittig, JHEP 0411 (2004) 016. [arXiv:hep-lat/0407007].
- [22] P. Hernández and M. Laine, JHEP 0409 (2004) 018
- [23] P. Hernández, K. Jansen and M. Lüscher, Nucl. Phys. B552 (1999) 363
- [24] P. Dimopoulos *et al.*, Phys. Lett. B 641 (2006) 118 [arXiv:hep-lat/0607028].
- [25] M. Guagnelli *et al.*, JHEP 0603 (2006) 088 [arXiv:hep-lat/0505002].
- [26] J. Gasser and H. Leutwyler, Phys. Lett. B188 (1987) 477; Nucl. Phys. B307 (1988) 763 F.C. Hansen and H. Leutwyler, Nucl. Phys. B350 (1991) 201 F.C. Hansen, Nucl. Phys. B345 (1990) 685
- [27] L. Giusti, Ch. Hoelbling, M. Lüscher and H. Wittig, Comput. Phys. Commun. 153 (2003) 31
- [28] P. Hernández and M. Laine, JHEP 0301 (2003) 063
- [29] H. Leutwyler and A. Smilga, Phys. Rev. D46 (1992) 5607
- [30] J. Kambor, J. Missimer and D. Wyler, Nucl. Phys. B346 (1990) 17; Phys. Lett. B261 (1991) 496
- [31] P. Hernández and M. Laine, arXiv:hep-ph/0607027.
- [32] P.H. Damgaard, M.C. Diamantini, P. Hernández and K. Jansen, Nucl. Phys. B629 (2002) 445
- [33] W. Bietenholz, T. Chiarappa, K. Jansen, K.I. Nagai and S. Shcheredin, JHEP 0402 (2004) 023

- [34] L. Giusti, P. Hernández, M. Laine, P. Weisz and H. Wittig, JHEP 0404 (2004) 013
- [35] T. DeGrand and S. Schaefer, Comput. Phys. Commun. 159 (2004) 185 R.G. Edwards, Nucl. Phys. B (Proc. Suppl.) 106 (2002) 38, and references therein
- [36] L. Giusti, P. Hernández, M. Laine, C. Pena, J. Wenekers and H. Wittig, arXiv:hep-ph/0607220.
- [37] J. Wenekers and H. Wittig, JHEP 0509 (2005) 059 [arXiv:hep-lat/0507026].

POS(LAT2006)012