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Strong to weak coupling transitions of SU(N) gauge theories in 2+1 dimensions

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We find a strong-to-weak coupling cross-over in $D = 2+1$ $SU(N)$ lattice gauge theories that appears to become a third-order phase transition at $N = \infty$, in a similar way to the Gross-Witten transition in the $D = 1+1$ $SU(N \to \infty)$ lattice gauge theory. There is, in addition, a peak in the specific heat at approximately the same coupling that increases with N, which is connected to Z_N monopoles (instantons), reminiscent of the first order bulk transition that occurs in $D = 3+1$ for $N \geq 5$. Our calculations are not precise enough to determine whether this peak is due to a second-order phase transition at $N = \infty$ or to a third-order phase transition with different critical behaviour to that of the Gross-Witten transition. We investigate whether the trace of the Wilson loop has a non-analyticity in the coupling at some critical area, but find no evidence for this. However we do find that, just as one can prove occurs in $D = 1+1$, the eigenvalue density of a Wilson loop forms a gap at $N = \infty$ at a critical value of its trace. We show that this gap formation is in fact a corollary of a remarkable similarity between the eigenvalue spectra of Wilson loops in $D = 1+1$ and $D = 2+1$ (and indeed $D = 3+1$): for the same value of the trace, the eigenvalue spectra are nearly identical. This holds for finite as well as infinite N; irrespective of the Wilson loop size in lattice units; and for Polyakov as well as Wilson loops.

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1. Introduction

A phase transition requires an infinite number of degrees of freedom. One can have phase transitions on finite volumes at $N = \infty$ because in this case we have an infinite number of degrees of freedom at each point in space. The classic example in the context of gauge field theories is the Gross-Witten transition [1] that occurs in the $D = 1+1$ $SU(\infty)$ lattice gauge theory.

In $D = 3+1$ $SU(N)$ gauge theories numerical studies reveal the existence for $N \geq 5$ of a first order 'bulk' transition separating the weak and strong coupling regions [2, 3].

These are both in some sense strong to weak coupling transitions which has led to the conjecture [4, 5] that Wilson loops in general will show such $N = \infty$ transitions when the physical size of the loop passes some critical value. Such a transition in $D = 3+1$ could have interesting implications for dual string approaches to large-N gauge theories and provide a natural explanation for the observed rapid onset of non-perturbative physics in the strong interactions [1, 6, 5].

2. Background

2.1 The 'Gross-Witten' transition

The $D = 1+1$ $SU(N)$ lattice gauge theory can be explicitly solved [1]. One finds a cross-over between weak and strong coupling that sharpens with increasing N into a third order phase transition at $N = \infty$. In terms of the plaquette u_p this shows up in a change of functional behaviour

$$\langle u_p \rangle \stackrel{N\to\infty}{=} \begin{cases} \frac{1}{\lambda} & \lambda \geq 2, \\ 1 - \frac{\lambda}{4} & \lambda \leq 2. \end{cases} \tag{2.1}$$

More detailed information about the behaviour of plaquettes and Wilson loops can be gained by considering their eigenvalues, which are are just phases, $\lambda = \exp\{i\alpha\}$. As $\beta \to 0$ the eigenvalue distribution $\rho(\alpha)$ of a Wilson loop becomes uniform while as $\beta \to \infty$ it becomes increasingly peaked around $\alpha = 0$. At the Gross–Witten transition a gap opens in the density of plaquette eigenvalues: in the strongly–coupled phase the eigenvalue density is non–zero for all angles, but in the weakly-coupled phase it is only non–zero in the range $-\alpha_c \leq \alpha \leq \alpha_c$, where $\alpha_c < \pi$ [1].

In $D = 3+1$ it is known that for $N \geq 5$ [2, 3] there is a strong first order transition as β is varied from strong to weak coupling. Calculations in progress [9] suggest that the plaquette eigenvalue distribution shows a gap formation at $N = \infty$ that is similar to the $D = 1+1$ transition.

In $D = 2+1$ there has been, as far as we are aware, no systematic search for a Gross-Witten or 'bulk' transition, and this is one of the gaps that the present work intends to fill.

2.2 Wilson loop transitions

The Gross-Witten transition involves the smallest possible Wilson loop, the plaquette. On the weak coupling side the plaquette can be calculated in perturbation theory; but this breaks down abruptly at the Gross-Witten transition [1]. The coupling is the bare coupling and hence a coupling on the length scale of the plaquette. One might interpret this as saying that there is a critical length scale at which perturbation theory in the running coupling will suddenly break down.

One might imagine that this generalises to other Wilson loops: i.e. when we scale up a Wilson loop, at some critical size, in 'physical units', there is a non-analyticity. In fact precisely such a scenario has been conjectured for $SU(N \to \infty)$ gauge theories in $D = 3+1$ [4, 5].

Such a non-analyticity does in fact occur for the $SU(N \to \infty)$ continuum theory in $D = 1+1$ [7, 8]. The transition occurs at a fixed physical area $A_{crit} = \frac{8}{g^2 N}$. Very much larger Wilson loops have a flat eigenvalue spectrum $\rho(\alpha)$ which becomes peaked as $A \to A_{crit}^+$. As A decreases through A_{crit} a gap appears in the spectrum near the extreme phases $\alpha = \pm\pi$. However, unlike the Gross-Witten transition this is not a phase transition: the partition function is analytic. Thus it is unclear what if any is the physical significance of this non-analyticity.

In this talk we investigate whether such a non-analyticity develops in $D = 2+1$.

3. Results

3.1 Preliminaries

At a phase transition appropriate derivatives of $\frac{1}{V}\log Z$, where V is the volume and Z is the partition function, will diverge or be discontinuous as $V \to \infty$. The lowest order of such a singular derivative determines the order of the phase transition.

With the standard plaquette action, a conventional first order transition has a discontinuity at $V = \infty$ in the average plaquette.

A conventional second order transition has a continuous first derivative of Z but a diverging second derivative and a specific heat $C \to \infty$ as $V \to \infty$. Defining $\overline{u_p}$ to be the average value of u_p over the space-time volume for a single lattice field, C can be written as

$$C = N_p(\langle \overline{u_p}^2 \rangle - \langle \overline{u_p} \rangle^2). \tag{3.1}$$

A conventional third order transition has continuous first and second order derivatives but a singular third-order derivative, $C' \equiv N_p^{-1} \partial^3 \log Z / \partial \beta^3$, at $V = \infty$. This may be written as

$$C' = N_p^2(\langle \overline{u_p}^3 \rangle - 3\langle \overline{u_p} \rangle \langle \overline{u_p}^2 \rangle + 2\langle \overline{u_p} \rangle^3). \tag{3.2}$$

Since, in general, fluctuations in the pure gauge theory decrease by powers of N we define the rescaled quantities $C_2 = N^2 \times C$ and $C_3 = N^4 \times C'$ which one expects generically to have finite non-zero limits when $N \to \infty$. If we find a crossover in C_2 or C_3 which does not sharpen with increasing volume at fixed N, but rather becomes a divergence or a discontinuity only in the large–N limit, then this will indicate a second– or third–order $N = \infty$ phase transition respectively.

Since large-N phase transitions can arise from completely local fluctuations we also consider local versions of C_2 and C_3 where we replace $\overline{u_p}$ by u_p, which we call P_2 and P_3 respectively.

To search for non-analyticities in Wilson loops we search for non-analyticities in $\langle u_w \rangle$ and its derivatives. We also calculate 'local' versions of the latter, just as we do for $\langle u_p \rangle$, and various moments of the Wilson loops. Finally, we also calculate and analyse their eigenvalue spectra.

3.2 Bulk transition

In 3+1 dimensions the bulk transition is easily visible as a large discontinuity in the action for $N \geq 5$ and as a (finite) peak in the specific heat for $N \leq 4$. We have searched for an analogous jump or rapid crossover in 2+1 dimensional $SU(6)$, $SU(12)$, $SU(24)$ and $SU(48)$ gauge theories. What we see is that the action appears to be approaching a smooth crossover in the large–N limit.

Our results for the specific heat for $SU(6)$ and $SU(12)$ show a clear peak around $\gamma \simeq 0.42$ which appears to grow stronger with increasing N. We have calculated P_2, the 'local' version of C_2, for $SU(6)$, $SU(12)$, $SU(24)$ and $SU(48)$. We see no significant evidence for a peak, which indicates that if there is a second order transition at $N = \infty$ it will primarily involve correlations between different plaquettes rather than arising from the fluctuations of individual plaquettes. However, what we do see in P_2 is definite evidence for a cusp developing at $\gamma \simeq 0.43$.

To investigate this further, we show in Fig. 1 our results for P_3 (the 'local' version of C_3) for $SU(6)$, $SU(12)$, $SU(24)$ and $SU(48)$. There is clearly an increasingly sharp transition as N increases around $\gamma \simeq 0.43$. This behaviour is remarkably similar to what happens in $D = 1+1$.

Finally, if we compare plaquette eigenvalue densities across the $D = 2+1$ and $D = 1+1$ transitions directly, we find that they are very similar both below and above the transition.

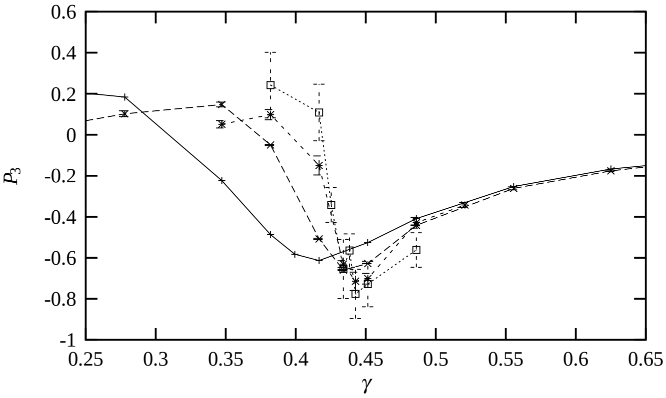

Figure 1: P_3 as a function of $\gamma = \frac{\beta}{2N^2}$ for SU(6) (+), SU(12) (×), SU(24) (∗) and SU(48) (□).

Despite these striking similarities, when we look in more detail we observe significant differences between the bulk transition in 2+1 dimensions and the Gross–Witten transition. In particular there is a peak in the specific heat in $D = 2+1$ which is not present in $D = 1+1$. To investigate this we calculate the contribution to the specific heat C_2 from correlations between a plaquette and its neighbours which share an edge but are not in the same plane, C_o. We find a clear peak, growing with N, in our results, plotted in Fig. 2. We see a similar peak, but almost exactly a factor of four lower, in C_f, the contribution from correlations between a plaquette and the plaquettes facing it across an elementary cube, as expected if the correlations are due to a flux emerging from the cube symmetrically through every face, i.e. due to the presence of monopole–instantons.

There are several scenarios for what happens at $N = \infty$ that are consistent with our results. One possibility is a third–order phase transition with critical exponent α different from -1. Alternatively there could be a second–order phase transition driven either by local fluctuations or by the correlation length diverging (or both). Our results cannot distinguish between these scenarios.

To search for the possibility of a diverging correlation length, we measured the mass of the lightest particle that couples to the plaquette, in both $SU(6)$ and $SU(12)$. Our results show a modest dip in the masses near the transition, which becomes more significant as we increase N. However,

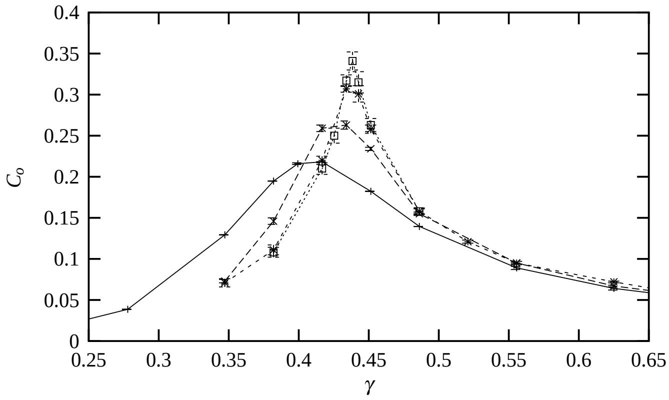

Figure 2: The plaquette correlator, C_o, for SU(6) (+), SU(12) (×), SU(24) (∗) and SU(48) (□).

the masses are large, and if the correlation length is going to show any sign of diverging it will be at much larger values of N than are accessible to our calculations.

3.3 Wilson loops

When we calculate how $\langle u_w \rangle$ varies with λ we see no sign of any singularity developing in this quantity, or in our simulataneous calculations of $\partial \langle u_w \rangle / \partial \lambda$. The more accurately calculated local version of the correlator that is equivalent to the derivative, also shows no evidence of developing the sort of cusp that might suggest an $N = \infty$ singularity in the second derivative. We have also looked at quantities analogous to P_2 and P_3 for the plaquette. The variation of these quantities with γ also does not become sharper with N. All our results are in fact essentially identical to those we obtain in similar calculations in $D = 1+1$.

It is possible that there are more subtle non-analyticities associated with a gap forming in the eigenvalue spectrum, of the kind that exist in $D = 1+1$. To search for such behaviour we directly compare Wilson loop eigenvalue spectra in 1+1 and 2+1 dimensions. We first evaluate the spectrum in 1+1 dimensions at the critical coupling at which the gap forms. A true gap only forms at $N = \infty$; for finite N we use the same value of the 't Hooft coupling, $\lambda_c = \frac{1}{\gamma_c} = 4(1 - e^{\frac{-2a^2}{A}})$, where A is the area of the Wilson loop in physical units. Having obtained the spectrum (numerically) in $D = 1+1$ we then calculate the spectrum in $D = 2+1$ for the same size loop and for the same N, varying the coupling to a value where the two spectra match.

We find that it is always possible to achieve such a match, for any N and for any size of Wilson loop. We show an example in Fig. 3, where we compare the eigenvalue densities of the 3×3 Wilson loops in SU(12) in 1+1 and 2+1 dimensions. The spectra are clearly very similar and indeed indistinguishable on this plot. We also find that the spectra can be matched when they are away from the critical coupling.

The fact that at finite but large N we can match so precisely the $D = 1+1$ and $D = 2+1$ eigenvalue spectra provides convincing evidence that the Wilson loops in the $D = 2+1$ $N = \infty$ theory also undergo a transition involving the formation of a gap in the eigenvalue spectrum.

All the above is an immediate corollary of a much stronger and rather surprising result concerning the matching of Wilson loop eigenvalue spectra in 1+1 and 2+1 (and indeed 3+1) dimensions.

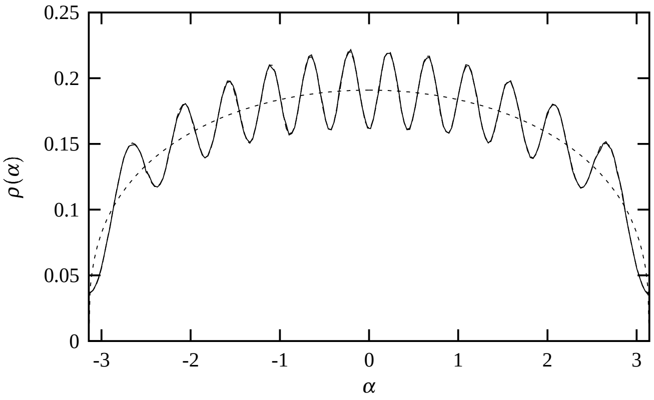

Figure 3: 3×3 Wilson loop eigenvalue density, $e^{i\alpha}$, for SU(12) in 1+1 dimensions at $\gamma = \frac{\beta}{2N^2} = 1.255$ (solid line) and in 2+1 dimensions at $\gamma = 0.722$ (long dashes), and the continuum large–N distribution in 1+1 dimensions at $A = A_{crit}$ (short dashes).

The general statement is that if we take an $n\times n$ Wilson loop $U_w^{n\times n}$ in the SU(N) gauge theory and calculate the eigenvalue spectra in D and D' dimensions, we find that the spectra match at the couplings λ_D and $\lambda_{D'}$ at which the averages of the traces $u_w^{n\times n} = \frac{1}{N}\mathrm{ReTr}\{U_w^{n\times n}\}$ are equal. We have tested this matching for $D = 1+1$ and $D = 2+1$ over groups in the range $N = 2$ to $N = 48$ and for Wilson loops ranging in size from 1×1 (the plaquette) to 8×8. Some sample calculations in $D = 3+1$ [9] strongly suggest that the same is true there.

The fact that such a precise matching is possible implies that the eigenvalue spectrum is completely determined by N, the size of the loop, and its trace. Hence the eigenvalues are not really independent degrees of freedom, which is unexpected. Moreover we find the spectra of Wilson loops that are 2×2 and larger can also be matched with each other, so the size of the Wilson loop is not really an extra variable here. Finally, the N dependence is weak.

We note that our results at this stage rely on a comparison that is visual and impressionistic. We intend to provide a more quantitative and accurate analysis elsewhere [9].

We have also investigated the eigenvalue spectra of Polyakov loops. We found that it is always possible to match the Polyakov loop eigenvalue spectra to those of Wilson loops in 1+1 dimensions (and hence also to Wilson loops in 2+1 dimensions) by choosing couplings at which the trace of the Polyakov loop equals that of the Wilson loop.

The existence of a gap in the eigenvalue spectrum at weak coupling has a rather general origin in terms of Random Matrix Theory. On the other hand we know that in a confining theory $\langle u_w\rangle \stackrel{A\to\infty}{\longrightarrow} 0$ which requires a nearly flat eigenvalue spectrum. Thus as we decrease the lattice spacing, the eigenvalue spectrum of a $L\times L$ Wilson loop must change from being nearly uniform to eventually having a gap. These considerations do not explain the complete matching of eigenvalue spectra across space-time dimension and loop size by merely matching traces.

For the gap formation to be physically significant, it must occur at a fixed physical area in the continuum limit. However, this is not the case in 2+1 dimensions. The reason is the perturbative self-energy of the sources whose propagators are the straight-line sections of the Wilson loop. (Often referred to as the 'perimeter term'.) Due to these the critical area for gap formation will vanish as we approach the continuum limit:

$$A_{crit} \propto \frac{1}{(\log \lambda)^2} \xrightarrow{a \to 0} 0 \tag{3.3}$$

One can imagine regularising the divergent self-energies so that A_{crit} is finite and non-zero in the continuum limit, but then it would appear to depend on the regularisation mass scale μ used.

4. Conclusions

We find a very close match between the behaviour of several observables across the bulk transition in $2+1$ dimensions and the Gross–Witten transition in $1+1$ dimensions. In particular the local contribution to the third derivative of the partition function has a very similar discontinuity in both cases. We find also a very close match in the plaquette eigenvalue spectra.

However, there is clearly more than this going on. We see a peak in the specific heat in $2+1$ dimensions that is mainly due to correlations between nearby plaquettes and that grows with N, which is not present in $1+1$ dimensions. This suggests that there is either a second–order phase transition at $N=\infty$, or a third–order phase transition that has a critical exponent different to -1. The correlations of plaquettes contributing to the specific heat peak behave as if due to monopole condensation, suggesting a connection to the bulk transition in $3+1$ dimensions, which can be understood in terms of the condensation of Z_N monopoles. Thus the bulk transition in $2+1$ dimensions appears to have features in common with both the $D=1+1$ and $D=3+1$ transitions.

We have analysed the behaviour of the eigenvalue spectra of Wilson loops in $2+1$ dimensions. We find a very good match with the spectra of Wilson loops in $1+1$ dimensions. This match appears to hold for loops of all sizes, for all N, and at all values of the coupling, as long as the traces of the loops are matched. Furthermore, the matching also works for Polyakov loops. This surprising results implies that the eigenvalues are not really independent degrees of freedom. As a corrolary, it immediately follows that in $D=2+1$ at $N=\infty$ a gap will form in the eigenvalue spectrum of a Wilson loop at a critical coupling that depends on the size of the loop. However, the physical consequences of this non–analyticity are unclear since the partition function remains analytic at the non–analyticity, and it occurs at zero physical area in the continuum limit.

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