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The spectrum of SU(N) gauge theories at $\theta \neq 0$

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We study the θ dependence of the spectrum of four-dimensional SU(N) gauge theories, where θ is the coefficient of the topological term in the Lagrangian, for $N \geq 3$ and in the large-N limit. We compute the $O(\theta^2)$ terms of the expansions around $\theta = 0$ of the string tension and the lowest glueball mass, respectively $\sigma(\theta) = \sigma\left(1 + s_2\theta^2 + ...\right)$ and $M(\theta) = M\left(1 + g_2\theta^2 + ...\right)$, where σ and M are the values at $\theta = 0$. For this purpose we use numerical simulations of the Wilson lattice formulation of SU(N) gauge theories for $N = 3,4,6$. The $O(\theta^2)$ coefficients turn out to be very small for all $N \geq 3$. For example, $s_2 = -0.08(1)$ and $g_2 = -0.06(2)$ for $N = 3$. Their absolute values decrease with increasing N. Our results are suggestive of a scenario in which the θ dependence in the string and glueball spectrum vanishes in the large-N limit, at least for sufficiently small values of $|\theta|$. They support the general large-N scaling arguments that indicate $\bar{\theta} \equiv \theta/N$ as the relevant Lagrangian parameter in the large-N expansion.

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1. Introduction

Four-dimensional SU(N) gauge theories have a nontrivial dependence on the angle θ that appears in the Euclidean Lagrangian as

$$\mathscr{L}_\theta = \frac{1}{4} F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) - i\theta \frac{g^2}{64\pi^2} \varepsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x) \tag{1.1}$$

($q(x) = \frac{g^2}{64\pi^2} \varepsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x)$ is the topological charge density). Indeed, the most plausible explanation of how the solution of the so-called U(1)$_A$ problem can be compatible with the $1/N$ expansion requires a nontrivial θ dependence of the ground-state energy density $F(\theta)$,

$$F(\theta) = -\frac{1}{V} \ln \int [dA] \exp\left(-\int d^d x \mathscr{L}_\theta\right), \tag{1.2}$$

to leading order in $1/N$. The large-N ground-state energy is expected to behave as [1, 2]

$$\Delta F(\theta) \equiv F(\theta) - F(0) = \mathscr{A}\,\theta^2 + O\left(1/N^2\right) \tag{1.3}$$

for sufficiently small θ, i.e. $\theta < \pi$. This has been supported by Monte Carlo simulations [3]. Indeed, numerical results for $N = 3,4,6$ are consistent with a scaling behavior around $\theta = 0$:

$$f(\theta) \equiv \sigma^{-2} \Delta F(\theta) = \frac{1}{2} C \theta^2 (1 + b_2 \theta^2 + \ldots), \quad C = C_\infty + c_2/N^2 + \ldots, \quad b_2 = b_{2,2}/N^2 + \ldots \tag{1.4}$$

where σ is the string tension at $\theta = 0$. C is the ratio χ/σ^2 where $\chi = \int d^4x \langle q(x) q(0) \rangle$ is the topological susceptibility at $\theta = 0$. Its large-N limit C_∞ is [3, 4] $C_\infty \approx 0.022$. Estimates of c_2 and $b_{2,2}$ are [3] $c_2 \approx 0.06$, and $b_{2,2} \approx -0.2$ ($b_2 \approx -0.02$ for SU(3)). Eq. (1.4) can be recast as

$$f(\theta) = N^2 \bar{f}(\bar{\theta} \equiv \theta/N), \qquad \bar{f}(\bar{\theta}) = \frac{1}{2} C \bar{\theta}^2 (1 + \bar{b}_2 \bar{\theta}^2 + \ldots), \tag{1.5}$$

where $\bar{b}_2 = b_{2,2} + O(1/N^2) = O(1)$. This is consistent with $1/N$ scaling arguments, indicating $\bar{\theta} \equiv \theta/N$ as the relevant Lagrangian parameter in the large-N limit of the ground-state energy.

The θ dependence of the spectrum is particularly interesting in the large-N limit, which may also be addressed by other approaches, such as AdS/CFT correspondence, see e.g. Ref. [5]. The analysis of glueballs using AdS/CFT suggests that the only effect of θ in the leading large-N limit is that the lowest glueball state becomes a mixed $0^{++}/0^{-+}$ state, but its mass does not change [2].

We present an exploratory study of θ dependence in the spectrum of SU(N) gauge theories, using numerical simulations of the Wilson action. MC studies are very difficult in the presence of the complex valued θ term, defying simulation. We focus on relatively small θ, where one may expand observable values about $\theta = 0$. For the string tension and the lowest glueball mass we write

$$\sigma(\theta) = \sigma\left(1 + s_2 \theta^2 + \ldots\right), \qquad M(\theta) = M\left(1 + g_2 \theta^2 + \ldots\right) \tag{1.6}$$

where M is the 0^{++} glueball mass at $\theta = 0$. The dimensionless quantities s_2 and g_2 can be computed from correlators at $\theta = 0$; they should approach constants as $a \to 0$, with $O(a^2)$ corrections.

We present results for 4-d SU(N) gauge theories with $N = 3,4,6$. The $O(\theta^2)$ coefficients turn out very small for all $N \geq 3$, e.g., $s_2 = -0.08(1)$ and $g_2 = -0.06(2)$ for $N = 3$. Moreover, their absolute values decrease with N. $O(\theta^2)$ terms are substantially smaller in dimensionless ratios such as $M/\sqrt{\sigma}$ and ratios of independent k strings, $R_k = \sigma_k/\sigma$. Our results suggest a large-N scenario in which θ dependence in the string and glueball spectrum vanishes around $\theta = 0$. They are consistent with arguments indicating $\bar{\theta} \equiv \theta/N$ as the relevant parameter in the large-N limit. We also show a similar situation in the 2-d CP^{N-1} models by an analysis of their $1/N$ expansion.

In Sec. 2 we outline the method to estimate $O(\theta^2)$ terms of the expansion in θ. The results of our numerical study are presented in Sec. 3. In Sec. 4 we discuss the θ dependence of 2-d CP^{N-1} models. Ref. [6] is a longer write-up of this work, with a more complete list of references.

2. Numerical method

2.1 Monte Carlo simulations

In our simulations of lattice gauge theories, using the Wilson formulation, we employed the Cabibbo-Marinari algorithm to upgrade SU(N) matrices by updating their SU(2) subgroups. This was done by alternating microcanonical over-relaxation and heat-bath steps, typically in a 4:1 ratio.

The topological properties of fields defined on a lattice are strictly trivial. Physical topological properties are recovered in the continuum limit. Various techniques have been proposed to associate a topological charge Q to a lattice configuration; the most robust definition of Q uses the index of the overlap Dirac operator. However, due to the computational cost of fermionic methods and the need for very large statistics to measure correlations of Polyakov and plaquette operators with topological quantities, we used the simpler cooling method, implemented as in Ref. [3]. Comparison with a fermionic estimator shows good agreement for SU(3) [3, 7]. Moreover, the agreement among different methods is expected to improve for larger N.

A severe form of critical slowing down affects the measurement of Q, posing a serious limitation for numerical studies, especially at large N. The available estimates of the autocorrelation time τ_Q for topological modes appear to increase as an exponential or a large power of the length scale [3, 8]. This dramatic effect has not been observed in plaquette-plaquette or Polyakov line correlations, suggesting an approximate decoupling between topological and nontopological ones, such as those determining confining properties and the glueball spectrum. But, as we shall see, such a decoupling is not complete. Therefore the strong critical slowing down that is observed in the topological sector will eventually affect also the measurements of nontopological quantities.

2.2 The $O(\theta^2)$ coefficients of the θ expansion

Let us describe how to determine the $O(\theta^2)$ coefficients in Eq. (1.6). We first discuss the string tension; it can be determined from the torelon mass, i.e. the mass describing the large-time exponential decay of wall-wall correlations G_P of Polyakov lines. In the presence of a θ term

$$G_P(t,\theta) = \langle A_P(t)\rangle_\theta = \frac{\int [dU] A_P(t) e^{-\int d^4x \mathscr{L}_\theta}}{\int [dU] e^{-\int d^4x \mathscr{L}_\theta}} , \qquad A_P(t) = \sum_{x_1,x_2} \mathrm{Tr}\, P(0;0)\, \mathrm{Tr}\, P(x_1,x_2;t), \tag{2.1}$$

$P(x_1,x_2;t)$ is the Polyakov line of size L along the x_3 direction. The time separation t is an integer multiple of the lattice spacing a: $t = n_t\, a$. The correlation G_P can be expanded in θ :

$$G_P(t,\theta) = G_P^{(0)}(t) + \frac{1}{2}\theta^2 G_P^{(2)}(t) + O(\theta^4), \tag{2.2}$$

where: $G_P^{(0)}(t) = \langle A_P(t)\rangle_{\theta=0}$, $\qquad G_P^{(2)}(t) = -\langle A_P(t) Q^2\rangle_{\theta=0} + \langle A_P(t)\rangle_{\theta=0}\langle Q^2\rangle_{\theta=0}$.

The correlation function G_P is expected to have a large-t exponential behavior

$$G_P(t,\theta) \approx B(\theta) e^{-E(\theta)t}, \tag{2.3}$$

where $E(\theta)$ is the energy of the lowest state, and $B(\theta)$ is the overlap of the source with this state. If L is sufficiently large, the lowest-energy states should be those of a string-like spectrum.

$$E(\theta) = \sigma(\theta)L - \pi/(3L) \tag{2.4}$$

We expand the large-t behavior (2.3) of $G(t,\theta)$ as

$$G_P(t,\theta) \approx B_0 e^{-E_0 t}\left[1 + \theta^2 h(t) + \ldots\right] \tag{2.5}$$

where: $B(\theta) = B_0 + \theta^2 B_2 + \ldots$, $\qquad E(\theta) = E_0 + \theta^2 E_2 + \ldots$, $\qquad h(t) = (B_2/B_0) - E_2 t$.

Comparing Eq. (2.5) with Eq. (2.2), we find that

$$h(t) = G_2^{(2)}(t)\,/\,(2G^{(0)}(t)) \tag{2.6}$$

Thus E_2 can be estimated from the difference: $\Delta h(t) = h(t) - h(t+a)$. Indeed, $\lim_{t\to\infty} \Delta h(t) = E_2\, a$. Corrections are exponentially suppressed as $\exp[-(E_0^* - E_0)t]$, where E_0^* is the mass of the first excited state at $\theta = 0$. Assuming the free-string spectrum, $E_0^* - E_0 = 4\pi/L$. Since we choose the lattice size L so that $l_\sigma \equiv \sqrt{\sigma}L \approx 3$, $(E_0^* - E_0)/E_0 \approx 4\pi/l_\sigma^2 \approx 1.4$.

Finally, the dimensionless scaling coefficient s_2 of the $O(\theta^2)$ term in (1.6) is obtained by

$$s_2 = \frac{E_2}{\sigma L} \tag{2.7}$$

s_2 is expected to approach a constant in the continuum limit, with $O(a^2)$ scaling corrections.

An analogous procedure can be used for the lowest 0^{++} glueball mass $M(\theta)$. We employ wall-wall correlators of Wilson loops with up to 6 spatial links. We define: $\Delta k(t) \equiv k(t) - k(t+a)$, where $k(t)$ is analogous to $h(t)$, using glueball correlators. Then, g_2 in (1.6) is obtained by

$$g_2 = \frac{1}{aM} \lim_{t\to\infty} \Delta k(t) \tag{2.8}$$

In order to improve the efficiency of the measurements we used smearing and blocking procedures to construct operators with better overlaps. Our implementation is described in Ref. [9].

3. Results

Table 1 contains some information on our MC runs for $N = 3,4,6$ on lattices $L^3 \times T$. Since the coefficients of the θ expansions are computed from connected correlation functions, and turn out to be quite small, high statistics is required to distinguish their estimates from zero: Our runs range from 9 to 25 million sweeps, with measures taken every 20-50 sweeps. This requirement represents a serious limitation to the possibility of performing runs for large lattices and in the continuum limit, especially for large N, due also to the severe critical slowing down. For all values

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N	β	lattice	stat	$a^2\sigma$	$aM_{0^{++}}$	$M_{0^{++}}/\sqrt{\sigma}$
3	5.9	$12^3 \times 18$	25M/20	0.0664(6)	0.80(1)	3.09(4)
3	6.0	$16^3 \times 36$	25M/40	0.0470(3)	0.70(1)	3.23(4)
4	10.85	$12^3 \times 18$	16M/50	0.0646(6)	0.76(1)	2.99(5)
6	24.5	$8^3 \times 12$	9M/50	0.114(2)	0.83(1)	2.46(4)

Table 1: Information on our MC simulations. Estimates of σ are obtained using Eq. (2.4).

of β considered, τ_Q satisfies $\tau_Q \lesssim 100$ [3]. Furthermore, β values were chosen in the weak-coupling region (see Ref. [9] for a more detailed discussion of this point). The lattice size L was chosen so that $l_\sigma \equiv \sqrt{\sigma}L \gtrsim 3$ (see, e.g., Refs. [4, 9]). Due to these limitations, in particular for $N = 4,6$, we could afford only one value of β, so that no stringent checks of scaling could be performed. For this reason our study should be still considered as a first exploratory investigation.

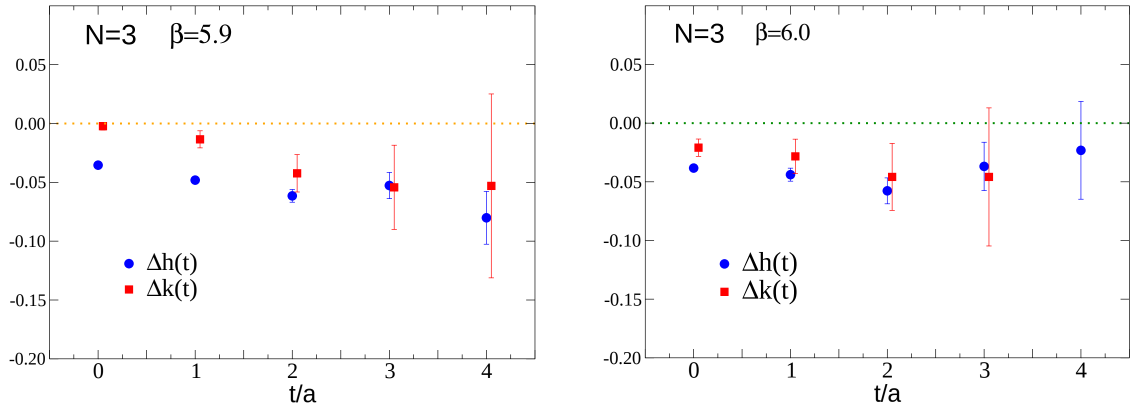

Figure 1: Plot of $\Delta h(t)$ and $\Delta k(t)$ for $N = 3$ at $\beta = 5.9$ and $\beta = 6.0$.

Figs. 1 and 2 show the results for the discrete differences $\Delta h(t)$ and $\Delta k(t)$ for $N = 3$ at $\beta = 5.9, 6.0$ and for $N = 4,6$ respectively. As expected, the signal degrades rapidly with increasing t. Anyway, they appear rather stable already for small values of t. In the case $N = 3$ the data at $\beta = 6$ appear to approach the asymptotic behavior more rapidly than at $\beta = 5.9$. This should be due to the fact that more effective blocking can be applied when $L = 16$, rather than $L = 12$.

We estimate s_2 and g_2 in (1.6) from $\Delta h(t)$ and $\Delta k(t)$, taking the data at $t/a = 2$ in the $N = 3,4$ runs, and at $t/a = 1$ for $N = 6$. In Table 2 we report the results. The estimates of s_2 and g_2 are small in all cases, and decrease with increasing N. For $N = 3$ the results at $\beta = 5.9$ and $\beta = 6.0$ are consistent, supporting the expected scaling behavior. As final estimate one may consider

$$s_2 = -0.08(1), \qquad g_2 = -0.06(2) \qquad \text{for } N = 3 \tag{3.1}$$

One may also consider the the scaling ratio: $M(\theta)/\sqrt{\sigma(\theta)} = (M/\sqrt{\sigma})(1 + c_2\theta^2 + \ldots)$, where $c_2 = g_2 - s_2/2$. Using the results of Table 2, we see that the $O(\theta^2)$ terms tend to cancel in the ratio. Indeed, we find $c_2 = -0.02(2), -0.01(3), -0.01(2)$ respectively for $N = 3,4,6$.

For $N > 3$ there are additional independent k-strings associated with representations of higher n-ality. One may consider the ratio $R_k(\theta) = \sigma_k(\theta)/\sigma(\theta) = R_k(1 + r_{k,2}\,\theta^2 + \ldots)$, where σ_k is the k-string tension (see e.g. Refs. [4, 9, 10, 11]). For $N = 4$ there is one additional k string, σ_2 ; for $N = 6$ there are two. Our results for $k > 1$ strings are less stable. We obtained sufficiently precise results only for $N = 4$. They suggest a very weak θ-dependence in R_2, i.e. $|r_{2,2}| \lesssim 0.02$.

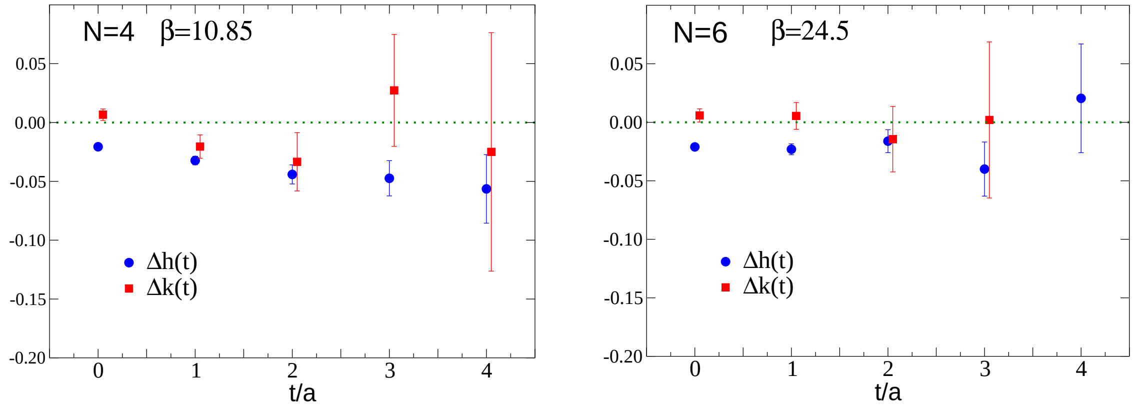

Figure 2: Plot of $\Delta h(t)$ and $\Delta k(t)$ for $N=4$ at $\beta=10.85$ and $N=6$ at $\beta=24.5$.

N	β	s_2	g_2
3	5.9	$-0.077(8)$	$-0.05(2)$
3	6.0	$-0.077(15)$	$-0.07(4)$
4	10.85	$-0.057(10)$	$-0.04(3)$
6	24.5	$-0.025(5)$	$0.006(15)$

Table 2: Results for s_2 and g_2, as derived from the discrete differences at $t/a=2$ for $N=3,4$, and at $t/a=1$ for $N=6$.

In conclusion, the above results show that $O(\theta^2)$ terms in the spectra of SU(N) gauge theories are very small, especially in dimensionless ratios. Moreover, they decrease with increasing N, and the coefficients do not appear to converge to a nonzero value. This suggests a scenario in which the θ dependence of the spectrum disappears at large N. General arguments indicate $\bar{\theta} \equiv \theta/N$ as the relevant parameter, implying that $O(\theta^2)$ coefficients in the spectrum should decrease as $1/N^2$. This is roughly verified by our results, barring possible scaling corrections, especially for $N=4,6$. For example, in the case of the string tension, $s_2 \approx s_{2,2}/N^2$ with $s_{2,2} \approx -0.9$. Of course, further investigations are required to put this scenario on a firmer ground.

Recent studies [12, 13] at finite temperature have shown that in the large-N limit the topological properties remain substantially unchanged up to the first-order transition point.

4. θ dependence in the two-dimensional CP^{N-1} model

Issues concerning the θ dependence can also be discussed in two-dimensional CP^{N-1} models [14], which present several features of QCD and, in addition, are amenable to a systematic $1/N$ expansion around the large-N saddle-point solution [14, 15].

One may expand the ground state energy $F(\theta)$ about $\theta=0$. Defining a scaling quantity $f(\theta)$:

$$f(\theta) \equiv M^{-2}[F(\theta)-F(0)] = \frac{1}{2}C\theta^2(1+\sum_{n=1} b_{2n}\theta^{2n}) \tag{4.1}$$

M is the “zero momentum” mass at $\theta=0$, and C is the ratio χ/M^2 at $\theta=0$, where χ is the topological susceptibility. Within the $1/N$ expansion: $C=\chi/M^2 = 1/(2\pi N)+O(1/N^2)$. One obtains b_{2n} from correlation functions of $q(x)$. The analysis of the $1/N$-expansion Feynman diagrams shows that b_{2n} is suppressed as: $b_{2n} = O(1/N^{2n})$. Thus the ground-state energy becomes

$$f(\theta) = N\bar{f}(\bar{\theta} \equiv \theta/N), \qquad \bar{f}(\bar{\theta}) = \frac{1}{2}\bar{C}\bar{\theta}^2(1+\sum_{n=1}\bar{b}_{2n}\bar{\theta}^{2n}), \tag{4.2}$$

where $\overline{C} \equiv NC$ and $\overline{b}_{2n} = N^{2n} b_{2n}$ are $O(1)$ in the large-N limit. Note the analogy with SU(N) gauge theories. The calculation of b_{2n} is rather cumbersome; we obtain

$$\overline{b}_2 = -27/5, \qquad \overline{b}_4 = -1830/7. \tag{4.3}$$

Within the $1/N$ expansion one may also study the dependence of M on θ. We write: $M(\theta) = M\left(1 + m_2\theta^2 + ...\right)$. A diagrammatic analysis indicates that m_2 is suppressed as: $m_2 = O(1/N^2)$. This confirms the arguments indicating $\overline{\theta} \equiv \theta/N$ as the relevant parameter in the large-N limit.

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