Calculation of $\Delta I = 3/2$ kaon weak matrix elements including two-pion interaction effects in finite volume

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We calculate $\Delta I = 3/2$ kaon decay matrix elements using domain wall fermions and the DBW2 gauge action at one coarse lattice spacing corresponding to $a^{-1} = 1.3$ GeV. We employ the Lellouch and Lüscher formula and its extention for non-zero total momentum to extract the infinite volume, center-of-mass frame decay amplitudes. The decay amplitudes obtained from the methods correspond to those from the indirect method with full order chiral perturbation theory. We confirm that the result is consistent with the previous result calculated with H-parity (anti-periodic) boundary condition by investigating the relative momentum dependence. We evaluate the decay amplitude ReA_2 at the physical point by a chiral extrapolation with a polynomial function of m_π^2 and the relative momentum as well as the $\Delta I = 3/2$ electroweak penguin contributions to ε'/ε. We found that the result of ReA_2 reasonably agrees with the experiment.

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1. Introduction

It is difficult to directly calculate the $K \to \pi\pi$ weak matrix elements on lattice due to difficulties of calculation of the two-pion state in finite volume. Indirect method [1] is one of the candidates to avoid the difficulty. In the indirect method $K \to \pi\pi$ process is reduced to $K \to \pi$ and $K \to 0$ processes through chiral perturbation theory (ChPT). The RBC [2] and CP-PACS [3] Collaborations calculated full non-leptonic kaon decay processes with the method. Their final results of ε'/ε, however, have the opposite sign to the experiment. In the calculations there are many systematic errors, *e.g.*, calculating at one finite lattice spacing with quenched approximation, and using the reduction with tree level ChPT. The indirect method might cause larger systematic errors than other sources, because the final state interaction of the two-pion is expected to play an important role in the decay process. Thus we have to treat the scattering effect of the two-pion state directly on lattice to eliminate this systematic error.

There are two main difficulties for the direct method, where the two-pion state is calculated on lattice. The one is to extract the two-pion state contribution with non-zero relative momentum in the $K \to \pi\pi$ four-point function, which was pointed out by Maiani and Testa [4]. So far there are several ideas for solving the problem. A method is employed with a proper projection of the $K \to \pi\pi$ four-point functions [5]. In the method we need complicated calculations and analyses, *e.g.*, diagonalization of a matrix of the two-pion correlation functions [6], to treat the two-pion state with non-zero relative momentum on lattice. A simpler idea, where complicated analyses are not required, is to prohibit the zero momentum two-pion state. Recently Kim [7] reported an exploratory study of the idea with H-parity (anti-periodic) boundary conditions in the spatial direction. He succeeded to extract the two-pion state with non-zero relative momentum from the ground state contribution of correlation functions, because the zero momentum two-pion state is prohibited by the boundary condition. We can also forbid the zero momentum two-pion state in center-of-mass (CM) frame by performing the calculation in non-zero total momentum (Lab) frame, $|\vec{P}| \neq 0$. In the frame the ground state of the two-pion is $|\pi(0)\pi(\vec{P})\rangle$, which is related to the two-pion state with the non-zero relative momentum in the CM frame. Thus, we can extract the two-pion state with non-zero momentum from the ground state contributions [8] as well as in the H-parity boundary case.

The other difficulty is related to the finite volume correction due to the two-pion interaction. We have to pay attention to the finite volume effect of the two-pion to obtain matrix elements in the infinite volume, because it is much larger than that of a one-particle state. Lellouch and Lüscher (LL) [9] suggested a solution which is a relation between the finite and infinite volume, center-of-mass frame decay amplitudes. However, this relation is valid only in the CM frame with periodic boundary condition in the spatial direction, so that we need a modified formula when we utilize H-parity boundary condition [7] or Lab frame calculation. Recently, two groups, Kim *et al.* [10] and Christ *et al.* [11], suggested a formula which is an extension of the LL formula for the Lab frame calculation.

Here we attempt to apply these two methods, Lab frame calculation and the extended formula, to the calculation of the $\Delta I = 3/2$ kaon weak matrix elements. Our preliminary result was presented in Ref. [12]. The results presented here are systematically larger than the preliminary results reported in last year lattice conference. During the past year we discovered that larger time

separations were needed to remove excited state contamination.

2. Methods

Through the extended formula [10, 11], the infinite volume decay amplitude $|A|$ in the CM frame is given by

$$|A|^2 = 8\pi\gamma^2 \left(\frac{E_{\pi\pi}}{p}\right)^3 \left\{ p\frac{\partial\delta(p)}{\partial p} + q\frac{\partial\phi_{\vec{P}}(q)}{\partial q} \right\} |M|^2, \qquad (2.1)$$

where $|M|$ is the finite volume, Lab frame decay amplitude, γ is a boost factor, $E_{\pi\pi}$ is the CM two-pion energy, and δ is the scattering phase shift of the final state interaction. The relative momentum p^2 is defined by the two-pion energy, $p^2 = E_{\pi\pi}^2/4 - m_\pi^2$. The function $\phi_{\vec{P}}$ with $\vec{P}$ being the total momentum, derived by Rummukainen and Gottlieb [13], is defined by

$$\tan\phi_{\vec{P}}(q) = -\frac{\gamma q \pi^{3/2}}{Z_{00}^{\vec{P}}(1;q^2;\gamma)}, \qquad (2.2)$$

where $q^2 = (pL/2\pi)^2$, and

$$Z_{00}^{\vec{P}}(1;q^2;\gamma) = \frac{1}{\sqrt{4\pi}} \sum_{\vec{n}\in\mathbb{Z}^3} \frac{1}{n_1^2 + n_2^2 + \gamma^{-2}(n_3 + 1/2)^2 - q^2}, \qquad (2.3)$$

in the $\vec{P} = (0,0,2\pi/L)$ case. The formula eq. (2.1) is valid only for on-shell decay amplitude, *i.e.*, $E_{\pi\pi} = m_K$ as in LL formula [9].

We calculate the four-point function for $\Delta I = 3/2$ $K \to \pi\pi$ decay with total momentum $\vec{P} = \vec{0}$ and $(0,0,2\pi/L)$. The four-point function $G_i(\vec{P};t,t_\pi,t_K)$ is defined by

$$G_i(\vec{P};t,t_\pi,t_K) = \langle 0|[K^0(\vec{P};t_K)]^\dagger O_i^{3/2}(t)\pi^+\pi^-(\vec{P};t_\pi)|0\rangle, \qquad (2.4)$$

where the operators $O_i^{3/2}$ are lattice operators entering $\Delta I = 3/2$ weak decays

$$O_{27,88}^{3/2} = (\bar{s}^a d^a)_L \left[(\bar{u}^b u^b)_{L,R} - (\bar{d}^b d^b)_{L,R} \right] + (\bar{s}^a u^a)_L (\bar{u}^b d^b)_{L,R} \ , \qquad (2.5)$$

$$O_{m88}^{3/2} = (\bar{s}^a d^b)_L \left[(\bar{u}^b u^a)_R - (\bar{d}^b d^a)_R \right] + (\bar{s}^a u^b)_L (\bar{u}^b d^a)_R \ , \qquad (2.6)$$

with $(\bar{q}q)_L = \bar{q}\gamma_\mu(1-\gamma_5)q$, $(\bar{q}q)_R = \bar{q}\gamma_\mu(1+\gamma_5)q$, and a,b being color indices. $O_{27}^{3/2}$ and $O_{88}^{3/2}$ are the operators in the (27,1) and (8,8) representations of $SU(3)_L \otimes SU(3)_R$ with $I = 3/2$, respectively. $O_{m88}^{3/2}$ is identical to $O_{88}^{3/2}$, except the color indices are mixed. We also calculate the four-point function of two pions $G_{\pi\pi}(\vec{P};t,t_\pi)$ and the two-point function of the kaon $G_K(\vec{P};t,t_K)$ with zero and non-zero total momenta, to obtain the needed energies and amplitudes.

3. Simulation parameters

We employ the domain wall fermion action with the domain wall height $M = 1.8$, the fifth dimension length $L_s = 12$ and the DBW2 gauge action with $\beta = 0.87$ corresponding to $a^{-1} = 1.31(4)$ GeV and $m_{\text{res}} = 1.25(3) \times 10^{-3}$. The lattice size is $L^3 \cdot T = 16^3 \cdot 32$, where the physical spatial size corresponds to about 2.4 fm. Our simulation is carried out at four u,d quark masses,
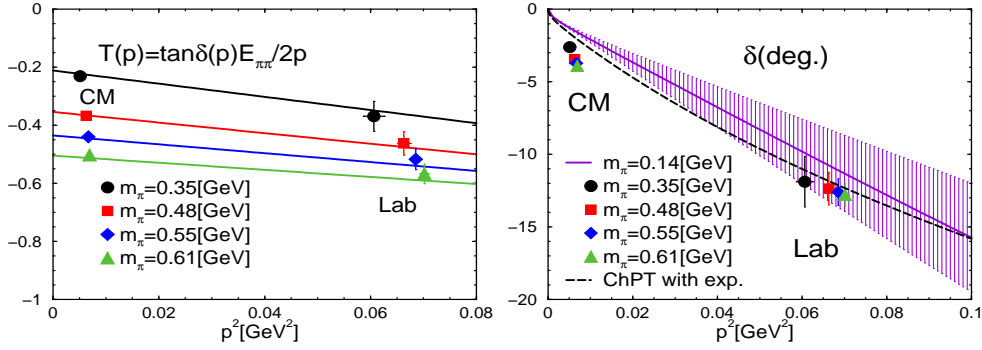
Figure 1: Measured values of scattering amplitude T. and δ obtained from CM and Lab calculations. Solid lines are fit results.

$m_u = 0.015, 0.03, 0.04$ and 0.05, for the chiral extrapolation of the decay amplitudes, using 252 gauge configurations, except for the lightest mass where we use 370 configurations. For interpolations of the amplitudes to the on-shell point, we calculate the decay amplitudes with six strange quark masses, $m_s = 0.12, 0.18, 0.24, 0.28, 0.35$ and 0.44, at the heavier three m_u, while we use the three lighter m_s at the lightest m_u. It is enough for the on-shell interpolation with the three m_s in the lightest m_u case. We will see the point in a later section.

We fix the two-pion operator at $t_\pi = 0$, and the kaon operator $t_K = 20$ to avoid contaminations from excited states as much as possible. A quark propagator is calculated by averaging quark propagators with periodic and anti-periodic boundary conditions for the time direction to obtain a propagator with $2T$ periodicity. We calculate the quark propagators with Coulomb gauge-fixed wall and momentum sources.

4. Results

The final state of the $\Delta I = 3/2$ kaon decay is the S-wave isospin $I = 2$ two-pion state. The scattering phase shift of the two-pion state can be obtained by the finite volume formulae for the CM [14] and the Lab frames [13] with the two-pion energy in each frame. We define "scattering amplitude" $T(p) = \tan \delta(p) E_{\pi\pi}/2p$, where $E_{\pi\pi}$ is the two-pion energy in the CM frame. The scattering amplitude is used for the chiral extrapolation of the phase shift with a global fitting for m_π^2 and p^2 with a polynomial function $a_{10}m_\pi^2 + a_{20}m_\pi^4 + a_{01}p^2 + a_{11}m_\pi^2 p^2$. While p^4 and m_π^4 should both be treated as second-order in ChPT, for our calculation we have only two different relative momenta. For that reason we omit the additional p^4 term. The figure 1 shows the result of $T(p)$ and the fit results at each pion mass in the left figure, and the measured values of $\delta(p)$ in the right figure. The phase shift at the physical pion mass $m_\pi = 0.14$ GeV, plotted by solid line with the error band in the right graph, is comparable with the prediction from ChPT with experiment. In order to utilize the extension eq.(2.1) of the LL formula, we evaluate the derivative of the phase shift from the fit result, while the derivative of the function $\phi_{\vec{P}}$ is obtained by a numerical derivative.

All the off-shell decay amplitudes are determined by the ratio of correlation functions with $i = 27, 88$ and $m88$, $\sqrt{3}G_i(\vec{P}; t, t_\pi, t_K)Z_{\pi\pi}Z_K/G_{\pi\pi}(\vec{P}; t, t_\pi)G_K(\vec{P}; t, t_K)$, where $Z_{\pi\pi}$ and Z_K are the overlap of the relevant operator with each state. We determine the off-shell amplitude in the flat
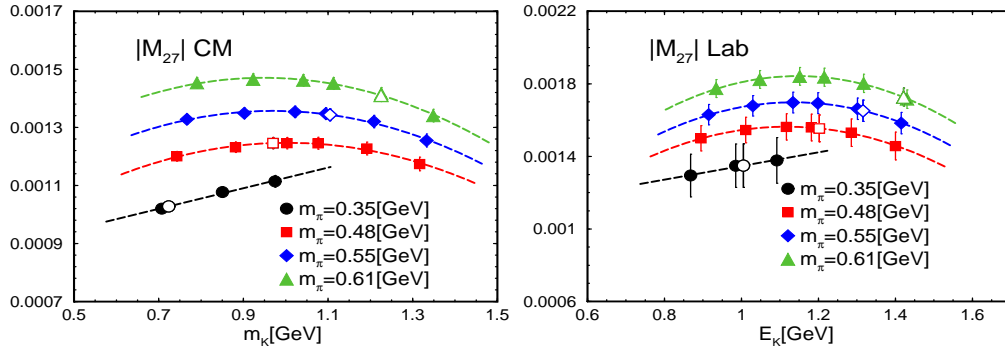
Figure 2: Interpolation of off-shell decay amplitude to on-shell in CM and Lab frames. Closed(open) symbol is off-shell(on-shell) amplitude.
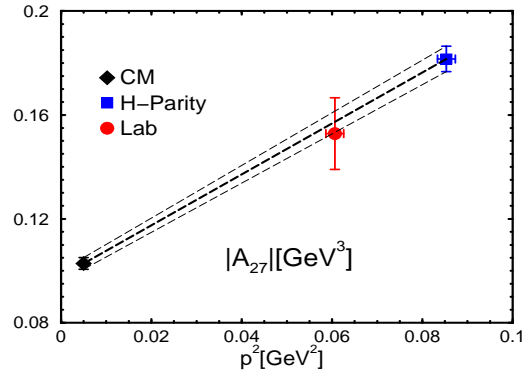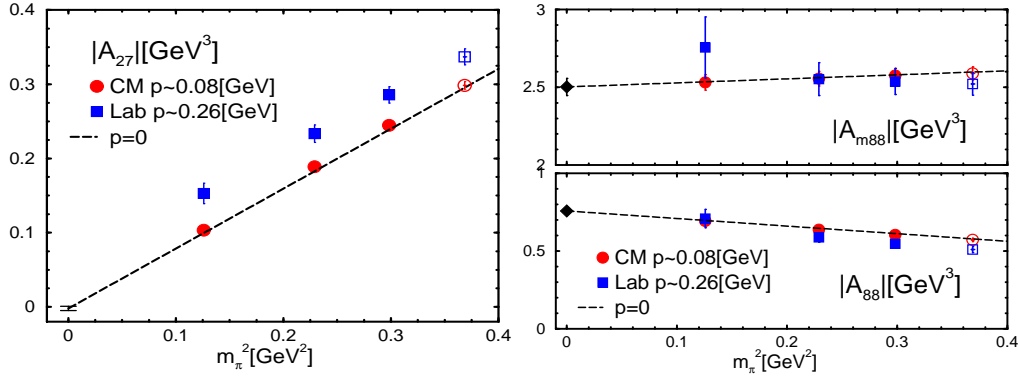

Figure 3: Infinite volume decay amplitude of 27 operator obtained from different calculation methods.

region of the ratio as a function of t, because the ratio will be a constant for those values of t when these correlation functions are dominated by each ground state.

The figure 2 shows the off-shell decay amplitude of the 27 operator in finite volume and its interpolation to the on-shell $E_K = E_{\pi\pi}$ in the cases of both the frames. In the lightest pion mass case, we can linearly interpolate the data with the three kaon energies. On the other hand, the off-shell decay amplitudes at the heavier pion masses have large curvature for the kaon energy, so that we fit the data with a quadratic function. In all the pion mass cases, the interpolated, on-shell decay amplitude and the fit line are plotted in the figure.

The decay amplitudes on finite volume in each frame are converted to those of the CM frame in the infinite volume through the LL formula [9] and its extended formula eq.(2.1) using the derivatives of δ and $\phi_{\vec{P}}$ obtained in the above. In Fig. 3 we plot the infinite volume decay amplitude of the 27 operator obtained from the different frames at the lightest pion mass as a function of p^2. The previous result obtained from H-parity boundary calculation is also plotted to compare with these results. The amplitude obtained from the Lab frame is consistent with the line determined from those of the CM frame and H-parity boundary calculations.

The infinite volume decay amplitudes of all the operators are plotted in Fig. 4. The result of the 27 operator seems to vanish at the chiral limit with zero relative momentum, while the other elements remain a constant in the limit. These trends of the pion mass dependence are reasonably consistent with the prediction of ChPT at leading order [8]. In order to investigate the m_π^2 and relative momentum p^2 dependences, we carry out a global fitting for each decay amplitude for m_π^2

Figure 4: Infinite volume decay amplitudes obtained at different relative momenta. Diamond symbols are fit results with $m_\pi^2 = p^2 = 0$. Open symbols are omitted in global fit.
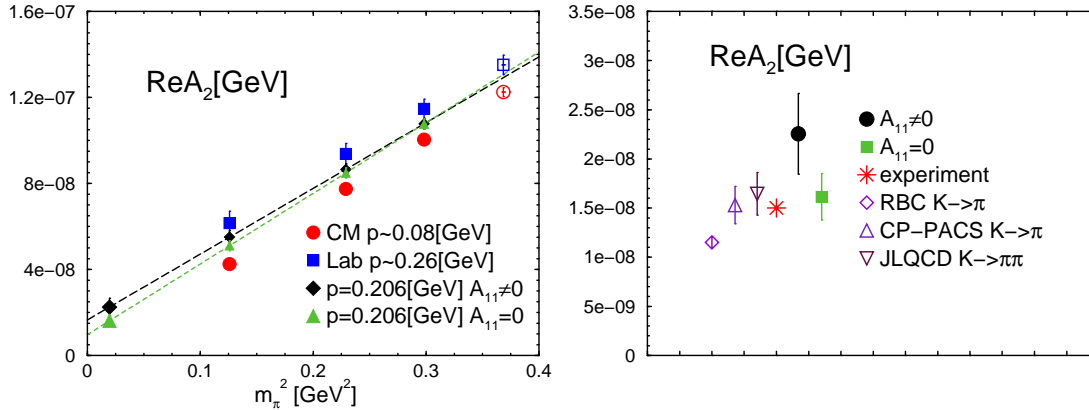
Figure 5: Measured ReA_2 and result from previous works. Open symbols are omitted in global fit. In left panel dashed lines denote fit results.

and p^2 assuming a simple polynomial form as

$$A_{00} + A_{10}m_\pi^2 + A_{01}p^2 + A_{11}m_\pi^2 p^2. \tag{4.1}$$

The χ^2/d.o.f. of the 27 operator is larger than 5 with all the four pion mass data, so that we exclude the heaviest pion mass data in all the operators from the following analysis to make the χ^2/d.o.f. reasonable. The constant term of the 27 operator is consistent with zero within the error, $A_{00}^{27} = -0.0020(29)$, as expected. In other operators, we employ the same fitting function as the 27 operator except $A_{11} = 0$. The fit results are plotted in the figures, and those at the both limits, $m_\pi^2 = p^2 = 0$, are represented by diamond symbols. It should be noted that while the linear fit to $|A_{27}|$ is the correct leading CHPT behavior, it is being used in a region of quite large mass. Likewise the linear fit to $|A_{88}|$ and $|A_{m88}|$ omits possible logarithm terms which are of the same order.

The weak matrix elements are determined using non-perturbative matching factors previously calculated with the regularization independent (RI) scheme in Ref. [7]. The electroweak contributions of ε'/ε, $\langle Q_7 \rangle_2$ and $\langle Q_8 \rangle_2$, are evaluated from $|A_{88}|$ and $|A_{m88}|$. At the physical pion mass $m_\pi = 0.14$ GeV and relative momentum $p = 0.206$ GeV, we obtain the matrix elements $\langle Q_7 \rangle_2(\mu) = 0.2473(64)$ and $\langle Q_8 \rangle_2(\mu) = 1.160(31)$ GeV^3 at the scale $\mu = 1.44$ GeV using the same fitting form as in the $|A_{88}|$ and $|A_{m88}|$ cases.

We calculate ReA_2 from the weak matrix elements in the RI scheme with the Wilson coefficients evaluated by NDR scheme calculated in Ref. [7]. The left figure in Fig. 5 shows ReA_2 obtained from the CM and Lab frame calculations. The measured values of ReA_2 are almost same as those of the 27 decay amplitude, in Fig. 4, apart from the overall constant. The reason is that the main contribution of ReA_2 comes from the 27 amplitude. We carry out a global fit using the same polynomial assumption eq.(4.1) except the constant term $A_{00} = 0$ to evaluate the result at the physical pion mass $m_\pi = 0.14$ GeV and momentum $p = 0.206$ GeV. In the fitting we omit the data at the heaviest pion mass because of the same reason mentioned in the above. We can also carry out a reasonable fitting, $\chi^2/\mathrm{d.o.f.} = 1.2$, without A_{11}. The fit results at each pion mass and the physical point are plotted in the figure.

We plot the results of ReA_2 at the physical point in the right panel of Fig. 5 together with those of the previous works using the indirect method [2, 3] and direct calculation with evaluating the finite volume, two-pion interaction effect through ChPT [15]. We estimate Re$A_2 = 2.26(41)$ and $1.61(24) \times 10^{-8}$ GeV at the physical point with and without A_{11} term, respectively. These results agree with each other and the experiment within two standard deviations. The result is encouraging, albeit it includes systematic errors due to quenched approximation, finite lattice spacing effects, and heavy pion and kaon masses.

Acknowledgments

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