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ρ meson decay from the lattice

CP-PACS collaboration:

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We present preliminary results on the ρ meson decay width estimated from the scattering phase shift of the $I = 1$ two-pion system. The phase shift is calculated by the finite size formula for non-zero total momentum frame (the moving frame) derived by Rummukainen and Gottlieb, using the $N_f = 2$ improved Wilson fermion action at $m_\pi/m_\rho = 0.41$ and $L = 2.53$ fm.

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1. Introduction

Lattice study of the ρ meson decay is an important step for understanding of the dynamical aspect of hadron reactions induced by strong interactions. There are already three studies [1, 2, 3]. The earlier two studies employed the quenched approximation ignores the decay into two ghost pions. The most recent work, while using the $N_f = 2$ dynamical configurations, concentrated on the $\rho \to \pi\pi$ transition amplitude rather than the full matrix including the $\rho \to \rho$ and $\pi\pi \to \pi\pi$ amplitudes. All three studies were carried out at an unphysical kinematics $m_\pi/m_\rho > 1/2$.

In this work we attempt to carry out a more rigorous approach. We estimate the decay width from the scattering phase shift for the $I = 1$ two-pion system. The finite size formula presented by Rummukainen and Gottlieb [4] is employed for an estimation of the phase shift. Calculations are carried out with $N_f = 2$ full QCD configuration previously generated for a study of light hadron spectrum with a renormalization group improved gauge action and a clover fermion action at $\beta = 1.8$, $\kappa = 0.14705$ on a $12^3 \times 24$ lattice [5]. The parameters determined from the spectrum analysis are $1/a = 0.92$ GeV, $m_\pi/m_\rho = 0.41$, and $L = 2.53$ fm. All calculations of this work are carried out on VPP5000/80 at the Academic Computing and Communications Center of University of Tsukuba.

2. Method

In order to realize a kinematics such that the energy of the two-pion state is close to the resonance energy m_ρ, we consider the non-zero total momentum frame (the moving frame) [4] with the total momentum $\mathbf{p} = 2\pi/L \cdot \mathbf{e}_3$. The initial ρ meson is assigned a polarization vector parallel to $\mathbf{p}$. One of the final two pions carry the momentum $\mathbf{p}$, while the other pion is at rest. The energies ignoring hadron interactions are then given by $E_1^0 = \sqrt{m_\pi^2 + p^2} + m_\pi$ for the two-pion state and $E_2^0 = \sqrt{m_\rho^2 + p^2}$ for the ρ meson. We neglect higher energy states whose energies are much higher than E_1^0 and E_2^0. On our full QCD configurations, the invariant mass for the two-pion state takes $\sqrt{s} = 0.97 \times m_\rho$, while $1.47 \times m_\rho$ is expected for the zero total momentum. The ρ meson at zero momentum cannot decay energetically, so that it can be used to extract m_ρ.

The hadron interactions shift the energy from E_n^0 to E_n $(n = 1,2)$. These energies E_n are related to the two-pion scattering phase shift $\delta(\sqrt{s})$ through the Rummukainen-Gottlieb formula [4], which is an extension of the Lüscher formula [6] to the moving frame. The formula for the total momentum $\mathbf{p} = p\mathbf{e}_3$ and the $\mathbf{A}_2^-$ representation of the rotation group on the lattice reads

$$\frac{1}{\tan\delta(\sqrt{s})} = \frac{1}{2\pi^2 q\gamma} \sum_{\mathbf{r}\in\Gamma} \frac{1 + (3r_3^2 - r^2)/q^2}{r^2 - q^2} , \tag{2.1}$$

where $\sqrt{s} = \sqrt{E^2 - p^2}$ is the invariant mass, k is the scattering momentum ($\sqrt{s} = 2\sqrt{m_\pi^2 + k^2}$), γ is the Lorentz boost factor ($\gamma = E/\sqrt{s}$), and $q = kL/(2\pi)$. The summation for $\mathbf{r}$ in (2.1) runs over the set

$$\Gamma = \{\mathbf{r} |\ r_1 = n_1\ ,\ r_2 = n_2\ ,\ r_3 = (n_3 + \frac{p}{2}\frac{L}{2\pi})/\gamma\ ,\ \mathbf{n} \in \mathbf{Z}^3\}\ . \tag{2.2}$$

The right hand side of (2.1) can be evaluated by the method described in Ref. [7].

In order to calculate E_1 and E_2 we construct a 2×2 matrix time correlation function,

$$G(t) = \begin{pmatrix} \langle 0|\ (\pi\pi)^\dagger(t)\ (\pi\pi)(t_s)\ |0\rangle & \langle 0|\ (\pi\pi)^\dagger(t)\ \rho_3(t_s)\ |0\rangle \\ \langle 0|\ \rho_3^\dagger(t)\ (\pi\pi)(t_s)\ |0\rangle & \langle 0|\ \rho_3^\dagger(t)\ \rho_3(t_s)\ |0\rangle \end{pmatrix} . \tag{2.3}$$

Here, $\rho_3(t)$ is an interpolating operator for the neutral ρ meson with the momentum $\mathbf{p} = 2\pi/L \cdot \mathbf{e}_3$ and the polarization vector parallel to $\mathbf{p}$; $(\pi\pi)(t)$ is an interpolating operator for the two pions given by

$$(\pi\pi)(t) = \frac{1}{\sqrt{2}}\Big(\pi^-(\mathbf{p},t)\pi^+(\mathbf{0},t) - \pi^+(\mathbf{p},t)\pi^-(\mathbf{0},t)\Big) , \tag{2.4}$$

which belongs to the $\mathbf{A}_2^-$ and iso-spin representation with $I = 1$, $I_z = 0$.

We can extract the two energy eigenvalues by a single exponential fitting of the two eigenvalues $\lambda_1(t,t_R)$ and $\lambda_2(t,t_R)$ of the normalized matrix $M(t,t_R) = G(t)G^{-1}(t_R)$ with some reference time t_R [8] assuming that the lower two states dominate the correlation function.

In order to construct the meson state with non-zero momentum we introduce a $U(1)$ noise $\xi_j(\mathbf{x})$ in three-dimensional space whose property is

$$\frac{1}{N_R}\sum_{j=1}^{N_R} \xi_j^\dagger(\mathbf{x})\xi_j(\mathbf{y}) = \delta^3(\mathbf{x}-\mathbf{y}) \qquad \text{for } N_R \to \infty \ . \tag{2.5}$$

We calculate the quark propagator

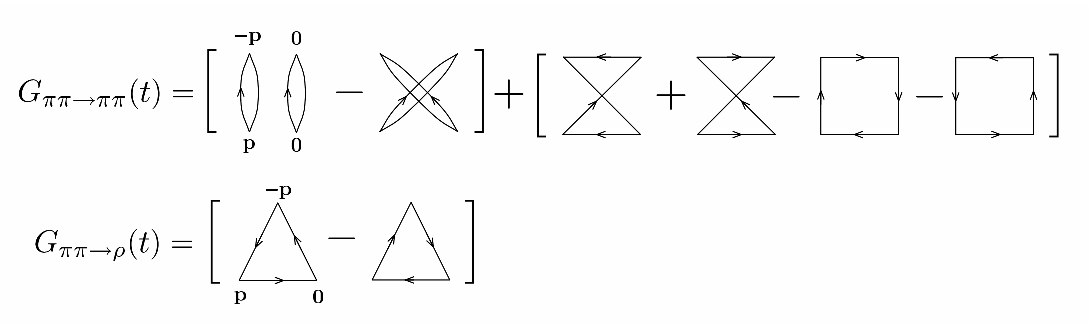
$$Q(\mathbf{x},t|\mathbf{q},t_s,\xi_j) = \sum_{\mathbf{y}}(D^{-1})(\mathbf{x},t;\mathbf{y},t_s)\cdot\Big[\ \mathrm{e}^{i\mathbf{q}\cdot\mathbf{y}}\xi_j(\mathbf{y})\ \Big] , \tag{2.6}$$

regarding the term in the square bracket as the source. The two point function of the meson with the spin content Γ and the momentum $\mathbf{p}$ can be constructed from Q as

$$\frac{1}{N_R}\sum_{j=1}^{N_R}\sum_{\mathbf{x}} \mathrm{e}^{-i\mathbf{p}\cdot\mathbf{x}} \cdot \Big\langle\ \gamma_5\ Q^\dagger(\mathbf{x},t|\mathbf{0},t_s,\xi_j)\ \gamma_5\Gamma^\dagger\ Q(\mathbf{x},t|\mathbf{p},t_s,\xi_j)\ \Gamma\ \Big\rangle , \tag{2.7}$$

where the bracket refers to the trace for color and spin indeces.

The quark contraction for the $\pi\pi \to \pi\pi$ and the $\pi\pi \to \rho$ components of $G(t)$ are given by



(2.8)

where the four verteces for the $\pi\pi \to \pi\pi$ and three verteces for the $\pi\pi \to \rho$ components refer to the pion or the ρ meson with definite momentum. The time direction is upward in the diagrams, and the $\rho \to \pi\pi$ component is given by changing the time direction.

The first term of the $\pi\pi \to \pi\pi$ component in (2.8) can be calculated by introducing another $U(1)$ noise $\eta_j(\mathbf{x})$ having the same property as $\xi_j(\mathbf{x})$ in (2.5);

$$\frac{1}{N_R}\sum_{j=1}^{N_R}\sum_{\mathbf{x},\mathbf{y}} \mathrm{e}^{-i\mathbf{p}\cdot\mathbf{x}} \cdot \Big\langle Q^\dagger(\mathbf{x},t|\mathbf{0},t_s,\xi_j)\; Q(\mathbf{x},t|\mathbf{p},t_s,\xi_j)\Big\rangle\Big\langle Q^\dagger(\mathbf{y},t|\mathbf{0},t_s,\eta_j)\; Q(\mathbf{y},t|\mathbf{0},t_s,\eta_j)\Big\rangle \,. \tag{2.9}$$

The second term of (2.8) is obtained by exchanging the momentum of the sink in (2.9).

In order to construct the other terms of (2.8) we calculate a quark propagator of another type by the source method,

$$W(\mathbf{x},t|\mathbf{k},t_1|\mathbf{q},t_s,\xi_j) = \sum_{\mathbf{z}} (D^{-1})(\mathbf{x},t;\mathbf{z},t_1)\cdot \Big[\mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}}\gamma_5\; Q(\mathbf{z},t_1|\mathbf{q},t_s,\xi_j)\Big] \,, \tag{2.10}$$

where the term in the square bracket is regarded as the source in solving the propagator. Using W we can construct the third to sixth terms in the $\pi\pi \to \pi\pi$ component of (2.8) by

$$\begin{aligned}
\text{3rd} &= \frac{1}{N_R}\sum_{j=1}^{N_R}\sum_{\mathbf{x}} \mathrm{e}^{-i\mathbf{p}\cdot\mathbf{x}} \cdot \Big\langle\; W^\dagger(\mathbf{x},t|\mathbf{0},t_s|-\mathbf{p},t_s,\xi_j)\; W(\mathbf{x},t|\mathbf{0},t|\mathbf{0},t_s,\xi_j)\;\Big\rangle \,,\\
\text{4th} &= \frac{1}{N_R}\sum_{j=1}^{N_R}\sum_{\mathbf{x}} \mathrm{e}^{-i\mathbf{p}\cdot\mathbf{x}} \cdot \Big\langle\; W(\mathbf{x},t|\mathbf{0},t_s|\mathbf{p},t_s,\xi_j)\; W^\dagger(\mathbf{x},t|\mathbf{0},t|\mathbf{0},t_s,\xi_j)\;\Big\rangle \,,\\
\text{5th} &= \frac{1}{N_R}\sum_{j=1}^{N_R}\sum_{\mathbf{x}} \mathrm{e}^{-i\mathbf{p}\cdot\mathbf{x}} \cdot \Big\langle\; W(\mathbf{x},t|\mathbf{p},t_s|\mathbf{0},t_s,\xi_j)\; W^\dagger(\mathbf{x},t|\mathbf{0},t|\mathbf{0},t_s,\xi_j)\;\Big\rangle \,,\\
\text{6th} &= \frac{1}{N_R}\sum_{j=1}^{N_R}\sum_{\mathbf{x}} \mathrm{e}^{-i\mathbf{p}\cdot\mathbf{x}} \cdot \Big\langle\; W^\dagger(\mathbf{x},t|-\mathbf{p},t_s|\mathbf{0},t_s,\xi_j)\; W(\mathbf{x},t|\mathbf{0},t|\mathbf{0},t_s,\xi_j)\;\Big\rangle \,.
\end{aligned} \tag{2.11}$$

The two terms of $\pi\pi \to \rho$ of (2.8) can be similarly constructed by

$$\begin{aligned}
\text{1st} &= \frac{1}{N_R}\sum_{j=1}^{N_R}\sum_{\mathbf{x}} \mathrm{e}^{-i\mathbf{p}\cdot\mathbf{x}} \cdot \Big\langle\; W^\dagger(\mathbf{x},t|-\mathbf{p},t_s|\mathbf{0},t_s,\xi_j)\; (\gamma_5\gamma_3)\; Q(\mathbf{x},t|\mathbf{0},t_s,\xi_j)\;\Big\rangle \,,\\
\text{2nd} &= \frac{1}{N_R}\sum_{j=1}^{N_R}\sum_{\mathbf{x}} \mathrm{e}^{-i\mathbf{p}\cdot\mathbf{x}} \cdot \Big\langle\; Q^\dagger(\mathbf{x},t|\mathbf{0},t_s,\xi_j)\; (\gamma_5\gamma_3)\; W(\mathbf{x},t|\mathbf{p},t_s|\mathbf{0},t_s,\xi_j)\;\Big\rangle \,.
\end{aligned} \tag{2.12}$$

In this work we set the source at $t_s = 4$ and impose the Dirichlet boundary condition in the time direction. We calculate the Q-type propagators for four sets of $\mathbf{q}$ and the $U(1)$ noise in (2.6) : $(\mathbf{q},\text{noise}) = \{(\mathbf{0},\xi),(\mathbf{0},\eta),(\mathbf{p},\xi),(-\mathbf{p},\xi)\}$. The W-type propagators are calculated for 22 sets of $\mathbf{k}$, t_1 and $\mathbf{q}$ in (2.10) : $(\mathbf{k},t_1|\mathbf{q}) = \{(\mathbf{p},t_s|\mathbf{0}),(-\mathbf{p},t_s|\mathbf{0}),(\mathbf{0},t_s|\mathbf{p}),(\mathbf{0},t_s|-\mathbf{p}),(\mathbf{0},t_1 = 4-21|\mathbf{0})\}$, with the same $U(1)$ noise ξ. All diagrams for the time correlation function can be calculated with combinations of these propagators. We choose $N_R = 10$ for the number of $U(1)$ noise. We carry out additional measurements to reduce statistical errors using the source operator is located at $t_s + T/2$ and the Dirichlet boundary condition is imposed at $T/2$. We average over the two measurements for the analysis. Thus we calculate 520 quark propagators for each configuration. The total number of configurations analyzed are 800 separated by 5 trajectories [5].

3. Results

In Fig. 1 we plot the real part of the diagonal components ($\pi\pi \to \pi\pi$ and $\rho \to \rho$) and the imaginary part of the off-diagonal components ($\pi\pi \to \rho$, $\rho \to \pi\pi$) of $G(t)$. Our construction of

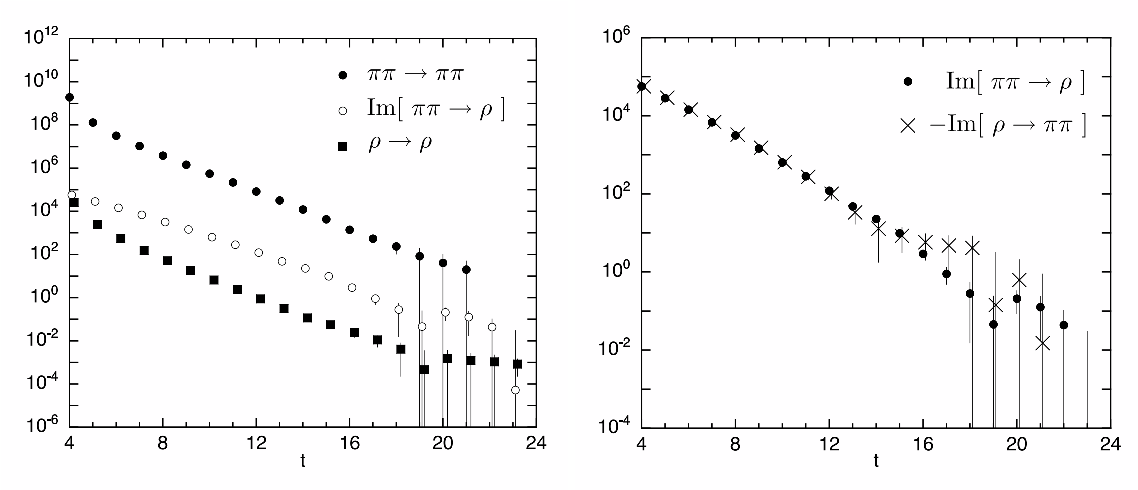


Figure 1: $G(t)$

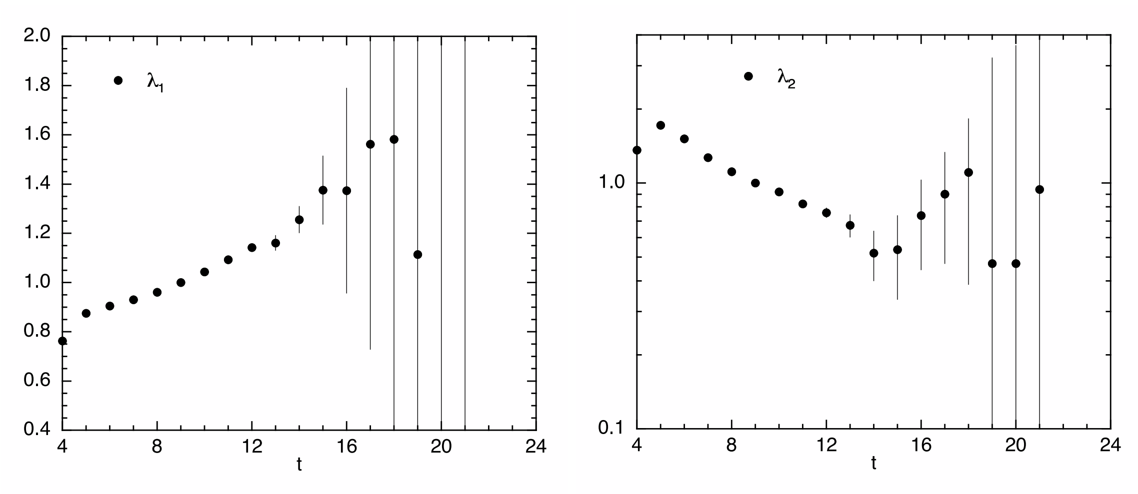


Figure 2: Normalized eigenvalues $\lambda_1(t,t_R)$ and $\lambda_2(t,t_R)$.

$G(t)$ is such that the sink and source operators are identical for a sufficiently large number of the $U(1)$ noise. In this case we can prove that $G(t)$ is an Hermitian matrix and the off-diagonal parts are pure imaginary from P and CP symmetry. We find that this is valid within statistics, but the statistical errors of the $\rho \to \pi\pi$ component is larger than those of $\pi\pi \to \rho$ in Fig. 1. In the following analysis we substitute $\rho \to \pi\pi$ by $\pi\pi \to \rho$ to reduce the statistical error.

The two eigenvalues $\lambda_1(t,t_R)$ and $\lambda_2(t,t_R)$ for the matrix $M(t,t_R) = G(t)G^{-1}(t_R)$ are shown in Fig. 2. We set the reference time $t_R = 9$ and normalize the eigenvalues by the correlation function for the free two-pion system, $\langle 0|\pi(-\mathbf{p},t)\pi(\mathbf{p},t_s)|0\rangle\langle 0|\pi(\mathbf{0},t)\pi(\mathbf{0},t_s)|0\rangle$. Thus the slope of the figure corresponds to the energy difference $\Delta E_n = E_n - E_1^0$. We observe that the energy difference for λ_1 is negative and that for λ_2 is positive. This means that the two-pion scatting phase shift is positive for the lowest state and negative for the next higher state.

We extract the energy difference ΔE_n for both states by a single exponential fitting of the normalized eigenvalues λ_1 and λ_2 for the time range $t = 10-16$. Then we reconstruct the energy E_n in the moving frame by adding the energy of the two free pions, *i.e.*, $E_n = \Delta E_n + E_1^0$, and convert it

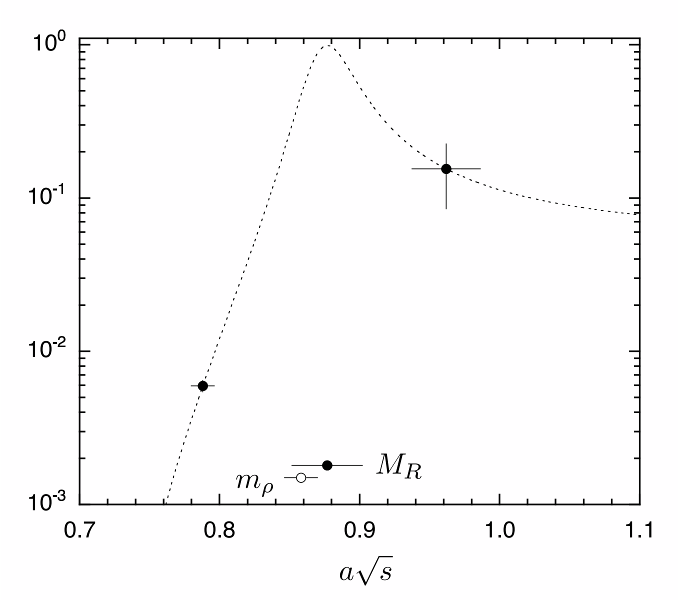


Figure 3: $\sin^2\delta(\sqrt{s})$, position of m_ρ and resonance mass M_R.

to the invariant mass $\sqrt{s}$. Substituting $\sqrt{s}$ into the Rummukainen-Gottlieb formula (2.1) we obtain the scattering phase shift :

$a\sqrt{s}$	$\tan\delta(\sqrt{s})$
0.7880 ± 0.0082	0.0773 ± 0.0033
0.962 ± 0.024	-0.43 ± 0.12

(3.1)

The ρ meson mass obtained at zero momentum is $am_\rho = 0.858 \pm 0.012$. Hence the sign of the scattering phase shifts at $\sqrt{s} < m_\rho$ is positive (attractive interaction) and that at $\sqrt{s} > m_\rho$ is negative (repulsive interaction) as expected. The corresponding results for $\sin^2\delta(\sqrt{s})$, which is proportional to the scattering cross section of the two-pion system, are plotted in Fig. 3 together with the position of m_ρ.

In order to estimate the ρ meson decay width at the physical quark mass we parameterize the scattering phase shift by the effective $\rho \to \pi\pi$ coupling constant $g_{\rho\pi\pi}$,

$$\tan\delta(\sqrt{s}) = \frac{g_{\rho\pi\pi}^2}{6\pi} \cdot \frac{k^3}{\sqrt{s}(M_R^2 - s)} , \tag{3.2}$$

with $g_{\rho\pi\pi}$ defined by the effective Lagrangian,

$$L_{\text{eff.}} = g_{\rho\pi\pi} \cdot \varepsilon_{abc}(k_1 - k_2)_\mu \rho_\mu^a(p)\pi^b(k_1)\pi^c(k_2) , \tag{3.3}$$

where k is the scattering momentum and M_R is the resonance mass. The coupling $g_{\rho\pi\pi}$ generally depends on the quark mass and the energy, but our present data at a single quark mass do not provide this information. Here we assume that these dependences are small and try to estimate $g_{\rho\pi\pi}$ and M_R from our results in (3.1). We also estimate the ρ meson decay width at the physical quark mass from

$$\Gamma_\rho = \frac{g^2_{\rho\pi\pi}}{6\pi} \cdot \frac{\bar{k}^3_\rho}{\bar{m}^2_\rho} = g^2_{\rho\pi\pi} \times 4.128 \ \ \text{MeV} \ , \tag{3.4}$$

where $\bar{m}_\rho$ is the ρ meson mass at the physical quark mass and $\bar{k}_\rho$ is the scattering momentum at $\sqrt{s} = \bar{m}_\rho$.

Our final results are as follows.

$$\begin{aligned} aM_R &= 0.877 \pm 0.025 \\ g_{\rho\pi\pi} &= 6.01 \pm 0.63 \\ \Gamma_\rho &= 149 \pm 31 \ \ \text{MeV} \ . \end{aligned} \tag{3.5}$$

The resonance mass M_R obtained from the scattering phase shift is consistent with $am_\rho = 0.858 \pm 0.012$ obtained from the ρ meson with zero momentum. The ρ meson decay width Γ_ρ at the physical quark mass is consistent with experiment (150 MeV). In Fig. 3 we indicate the position of M_R and draw the line given by (3.2) with $g_{\rho\pi\pi}$ and M_R in (3.5).

4. Summary

We have shown that a direct calculation of the ρ meson decay width from the scattering phase shift for the $I = 1$ two-pion system is possible with present computing resources. However, several issues remain which should be investigated in future work. The most important issue is a proper evaluation of the quark mass and energy dependence of the effective $\rho \to \pi\pi$ coupling constant $g_{\rho\pi\pi}$. This constant is used to obtain the physical decay width at $m_\pi/m_\rho = 0.18$ from our results at $m_\pi/m_\rho = 0.41$ by a long chiral extrapolation. In principle we can estimate the decay width from the scattering phase shift without such a parameterization, if we have data for several energy values at or near the physical quark mass. This will be our goal toward the lattice determination of the ρ meson decay.

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