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Improving perturbation theory with cactus diagrams

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We study a systematic improvement of perturbation theory for gauge fields on the lattice [1]; the improvement entails resumming, to all orders in the coupling constant, a dominant subclass of tadpole diagrams.
This method, originally proposed for the Wilson gluon action [2], is extended here to encompass all possible gluon actions made of closed Wilson loops; any fermion action can be employed as well. The effect of resummation is to replace various parameters in the action (coupling constant, Symanzik and clover coefficient) by "dressed" values; the latter are solutions to certain coupled integral equations, which are easy to solve numerically.
Some positive features of this method are: a) It is gauge invariant, b) it can be systematically applied to improve (to all orders) results obtained at any given order in perturbation theory, c) it does indeed absorb in the dressed parameters the bulk of tadpole contributions.
Two different applications are presented: The additive renormalization of fermion masses, and the multiplicative renormalization Z_V (Z_A) of the vector (axial) current. In many cases where non-perturbative estimates of renormalization functions are also available for comparison, the agreement with improved perturbative results is consistently better as compared to results from bare perturbation theory.

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1. Introduction

It is well known that quantities measured through numerical simulation are characterized by significant renormalization effects, which must be properly taken into account before making any comparisons to corresponding physical observables.

Although the renormalization procedure can be formally carried out in a systematic way to any given order in perturbation theory, calculations are notoriously difficult, as compared to continuum regularization schemes. Furthermore, the convergence rate of the resulting asymptotic series is often unsatisfactory.

Some years ago, a method was proposed to sum up a whole subclass of tadpole diagrams, dubbed "cactus" diagrams, to all orders in perturbation theory [2, 4]. This procedure is gauge invariant, it can be systematically applied to improve (to all orders) results obtained at any given order in perturbation theory, and it does indeed absorb the bulk of tadpole contributions into an intricate redefinition of the coupling constant. The agreement of available non-perturbative estimates of renormalization coefficients with cactus improved perturbative results is consistently better as compared to results from bare perturbation theory.

In the present work we extend the improved perturbation theory method of Refs. [2, 4], to encompass the large class of actions (including Symanzik improved gluon actions combined with any fermionic action) which are used nowadays in simulations of QCD. In Section II we present our calculation, deriving expressions for a dressed gluon propagator. The methodology can be also applied to dress the gluon and fermion vertices (appears in Ref. [1]). We show how these dressed constituents are employed to improve 1-loop and 2-loop Feynman diagrams coming from bare perturbation theory. In Section III we apply our improved renormalizaton procedure to a number of test cases involving Symanzik gluons and Wilson/clover/overlap fermions.

2. The Method

In this Section, we start illustrating our method by showing how the gluon propagator is dressed by the inclusion of cactus diagrams. We will then explain how this procedure is applied to Feynman diagrams at a given order in bare perturbation theory, concentrating on the 1- and 2-loop case.

2.1 Dressing the propagator

We consider the Symanzik improved gluon action involving Wilson loops with up to 6 links:

$$S_G = \frac{2}{g_0^2}\Bigg[\, c_0 \sum_{\text{plaquette}} \mathrm{Re\,Tr}\,(1-U_{\text{plaquette}}) + c_1 \sum_{\text{rectangle}} \mathrm{Re\,Tr}\,(1-U_{\text{rectangle}}) + c_2 \sum_{\text{chair}} \mathrm{Re\,Tr}\,(1-U_{\text{chair}}) + c_3 \sum_{\text{parallelogram}} \mathrm{Re\,Tr}\,(1-U_{\text{parallelogram}})\Bigg] \qquad (2.1)$$

The coefficients c_i satisfy a normalization condition $c_0 + 8c_1 + 16c_2 + 8c_3 = 1$ which ensures the correct classical continuum limit of the action.

The quantities U_i ($i = 0(\text{plaquette}), 1(\text{rectangle}), 2(\text{chair}), 3(\text{parallelogram})$) in Eq. (2.1) are products of link variables $U_{x,\mu}$ around the perimeter of the closed loop. Using the Baker-Campbell-Hausdorff (BCH) formula, U_i takes the form:

$$U_i = \exp\left(i g_0 F_i^{(1)} + i g_0^2 F_i^{(2)} + i g_0^3 F_i^{(3)} + \mathscr{O}(g_0^4)\right) \tag{2.2}$$

where $F_i^{(1)}$ is simply the sum of the gauge fields on the links of loop i, while $F_i^{(j)}$ $(j > 1)$ are j-th degree polynomials in the gauge fields, constructed from nested commutators.

Let us *define* the cactus diagrams which dress the gluon propagator: These are tadpole diagrams which become disconnected if any one of their vertices is removed; further, each vertex is constructed solely from the $F_i^{(1)}$ parts of the action. A diagrammatic equation for the dressed gluon propagator (thick line) in terms of the bare propagator (thin line) and 1-particle irreducible (1PI) vertices (solid circle) reads:

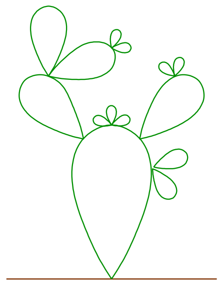
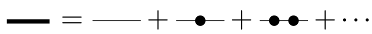
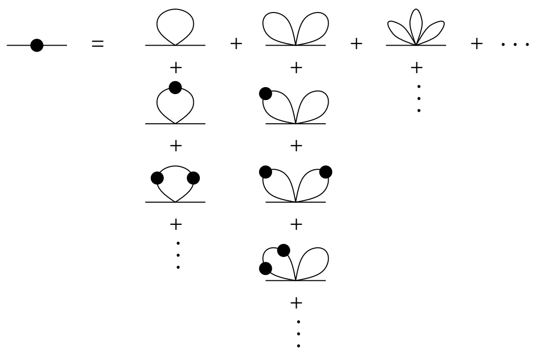


Fig.1: A cactus



(2.3)

The 1PI vertex obeys the following recursive equation:



(2.4)

The bare inverse gluon propagator D^{-1} results from the total gluon action ([1]) and reads

$$D^{-1}_{\mu\nu}(k) = \sum_\rho \left(\hat{k}^2_\rho \delta_{\mu\nu} - \hat{k}_\mu \hat{k}_\rho \delta_{\rho\nu}\right) d_{\mu\rho} + \frac{\hat{k}_\mu \hat{k}_\nu}{1-\xi} \equiv \sum_{i=0,1,2,3} \left(c_i\, G^{(i)}_{\mu\nu}(k)\right) + \frac{\hat{k}_\mu \hat{k}_\nu}{1-\xi} \tag{2.5}$$

where $C_0 = c_0 + 8c_1 + 16c_2 + 8c_3$, $\quad C_1 = c_2 + c_3$, $\quad C_2 = c_1 - c_2 - c_3$. For further definitions of quantities appearing above, the reader can refer to [1]. The matrices $G^{(i)}(k)$ are symmetric and transverse, and originate from a $\mathrm{Tr}\left(F_i^{(1)} F_i^{(1)}\right)$ term of the gluon action. Consequently, the diagrams on the r.h.s. of Eq. (2.4), are a linear combination of $G^{(i)}(k)$; this implies that the 1PI vertex $G^{1\text{PI}}(k)$ (the l.h.s. of Eq. (2.4)) can be written as:

$$G^{1\text{PI}}(k) = \alpha_0\, G^{(0)}(k) + \alpha_1\, G^{(1)}(k) + \alpha_2\, G^{(2)}(k) + \alpha_3\, G^{(3)}(k) \tag{2.6}$$

Each of the quantities α_i will in general depend on N, g_0, c_0, c_1, c_2, c_3, but not on the momentum. Eq. (2.3) leads to the following expression for the inverse dressed propagator $(D^{\text{dr}})^{-1}(k)$ [1]:

$$(D^{\text{dr}})^{-1} = \tilde{c}_0\, G^{(0)} + \tilde{c}_1\, G^{(1)} + \tilde{c}_2\, G^{(2)} + \tilde{c}_3\, G^{(3)} + \frac{1}{1-\xi}\, \hat{k}_\mu \hat{k}_\nu\,, \qquad \tilde{c}_i \equiv c_i - \alpha_i \tag{2.7}$$

We observe that dressing replaces the bare coefficients c_i with improved ones $\tilde{c}_i$, and leaves the longitudinal part intact. This property ensures gauge invariance of the results.

In terms of the dressed propagator, Eq. (2.4) can be *drawn* as:

(2.8)

A typical diagram on the r.h.s. of Eq. (2.8) is the sum of 4 terms, and has $(n-2)/2$ 1-loop integrals in the diagram (coming from the contraction of two powers of $F_i^{(1)}$ via a dressed propagator), and will contribute one power of $\beta_i(\tilde{c}_0, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3)$, where:

$$\begin{aligned}
\beta_0 &= \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \Big(2\,\hat{q}_\mu^2\, D^{\rm dr}_{\nu\nu}(q) - 2\,\hat{q}_\mu\,\hat{q}_\nu\, D^{\rm dr}_{\mu\nu}(q)\Big)\\
\beta_1 &= \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \Big((4\hat{q}_\nu^2 - \hat{q}_\nu^4)\, D^{\rm dr}_{\mu\mu}(q) + \hat{q}_\mu^2(4-\hat{q}_\nu^2)\, D^{\rm dr}_{\nu\nu}(q) - 2\,\hat{q}_\mu\,\hat{q}_\nu(4-\hat{q}_\nu^2)\, D^{\rm dr}_{\mu\nu}(q)\Big)\\
\beta_2 &= \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \Big(\hat{q}_\mu^2(8-\hat{q}_\nu^2)\, D^{\rm dr}_{\rho\rho}(q)/2 - \hat{q}_\mu\,\hat{q}_\rho(8-\hat{q}_\nu^2)\, D^{\rm dr}_{\mu\rho}(q)/2\Big)\\
\beta_3 &= \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \Big(3\,\hat{q}_\mu^2(4-\hat{q}_\nu^2)\, D^{\rm dr}_{\rho\rho}(q)/2 - 3\,\hat{q}_\mu\,\hat{q}_\nu\,(4-\hat{q}_\rho^2)\, D^{\rm dr}_{\mu\nu}(q)/2\Big)
\end{aligned} \tag{2.9}$$

(μ, ν, ρ assume distinct values; no summation implied). We note that β_i are gauge independent, since the longitudinal part cancels in the loop contraction.

In order to set Eq. (2.4) in a mathematical form, we need to evaluate $F(n;N)$ which is the sum over all complete pairwise contractions of $\mathrm{Tr}\{T^{a_1}T^{a_2}\dots T^{a_n}\}$. Use of $F(n;N)$, along with the integrals (2.9), allows us to resum (2.4), leading to [1]:

$$\frac{c_i-\alpha_i}{c_i}\,(N^2-1) = e^{-\beta_i\, g_0^2\,(N-1)/(4N)}\,\left(\frac{N-1}{N}\,L^1_{N-1}(g_0^2\,\beta_i/2) + 2\,L^2_{N-2}(g_0^2\,\beta_i/2)\right) \tag{2.10}$$

(L^α_β(x): Laguerre polynomials). Eqs. (2.10) are 4 separate equations where unknown quantities are the coefficients α_i; they appear on the l.h.s., as well as inside the integrals β_i of the r.h.s, by virtue of Eqs. (2.9, 2.7). It is worth mentioning that all combinatorial weights are correctly incorporated in the procedure.

Eqs. (2.10) can be solved numerically and each choice of values for (c_i, g_0, N) leads to a set of values for $\tilde{c}_i \equiv c_i - \alpha_i$ that are no longer normalized; one may express the results of our procedure in terms of a normalized set of improved coefficients, $\tilde{c}_i/\tilde{C}_0$ and an improved coupling constant $\tilde{g}_0^2 = g_0^2/\tilde{C}_0$, where: $\tilde{C}_0 = \tilde{c}_0 + 8\tilde{c}_1 + 16\tilde{c}_2 + 8\tilde{c}_3$. For reasons of simplicity we define rescaled quantities:

$$\gamma_i \equiv \frac{c_i}{g_0^2}\,, \qquad \tilde{\gamma}_i \equiv \frac{\tilde{c}_i}{g_0^2}\,, \qquad \tilde{\beta}_i(\tilde{c}_0,\tilde{c}_1,\tilde{c}_2,\tilde{c}_3) \equiv g_0^2\,\beta_i(\tilde{c}_0,\tilde{c}_1,\tilde{c}_2,\tilde{c}_3) = \beta_i(\tilde{\gamma}_0,\tilde{\gamma}_1,\tilde{\gamma}_2,\tilde{\gamma}_3) \tag{2.11}$$

$\tilde{\gamma}_i$ must now satisfy the coupled equations:

$$\tilde{\gamma}_i = \frac{1}{N^2-1}\,\gamma_i\, e^{-\tilde{\beta}_i\,(N-1)/(4N)}\,\left(\frac{N-1}{N}\,L^1_{N-1}(\tilde{\beta}_i/2) + 2\,L^2_{N-2}(\tilde{\beta}_i/2)\right) \tag{2.12}$$

For $SU(2)$ and $SU(3)$ the Laguerre polynomials have a simple form and Eqs. (2.12) can be written explicitly [1].

Since Eqs. (2.12) have the form $x = f(x)$, they can be numerically solved using a fixed point procedure [1]. A unique solution for $\tilde{\gamma}_i$ always exists for all physically interesting values of c_i, and for all values of g_0 well inside the strong coupling region. The convergence of the procedure has been verified in a number of extreme cases.

2.2 Numerical values of improved coefficients

We now present the values of the dressed coefficients for several gluon actions of interest. In Figs. 2-5, one can see the improved coefficients for Plaquette, Tree-level Symanzik, Iwasaki and Tadpole improved Lüscher-Weisz actions. Results for DBW2 action are listed in Table I.

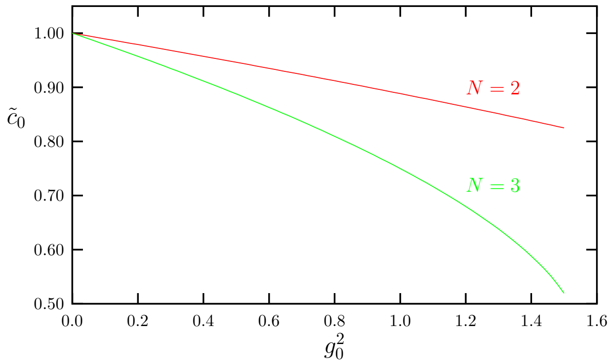


Fig.2: Improved coefficient $\tilde{c}_0$ for N=2 and N=3 (plaquette action)

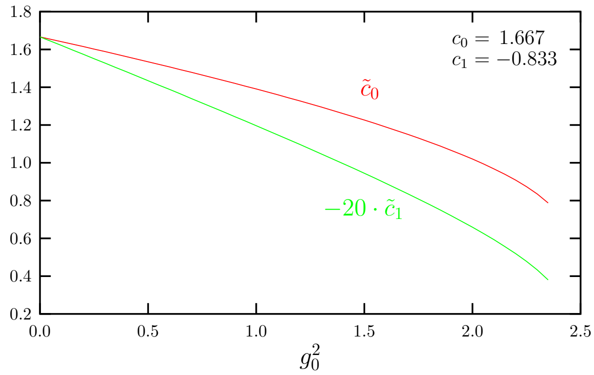


Fig.3: Improved coefficients $\tilde{c}_0$ and $\tilde{c}_1$ (tree-level Symanzik improved action, $N = 3$)

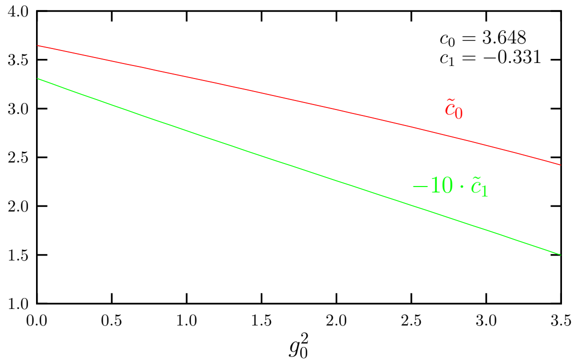


Fig.4: Improved coefficients $\tilde{c}_0$ and $\tilde{c}_1$ (Iwasaki action, $N = 3$)

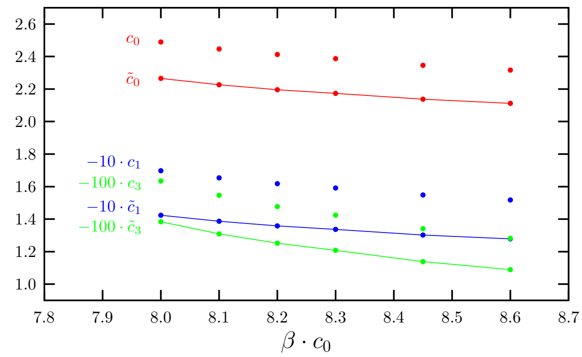


Fig.5: Coefficients c_i and their dressed counterparts $\tilde{c}_i$ for different values of $\beta\, c_0 = 6\, c_0/g_0^2$ (TILW actions, $N = 3$)

$\beta = 6/g_0^2$	c_0	c_1	$\tilde{c}_0$	$\tilde{c}_1$
1.1636	5.29078	-0.53635	3.39826	-0.22528
0.6508	12.2688	-1.4086	8.8070	-0.7313

TABLE I. Improved coefficients $\tilde{c}_0$ and $\tilde{c}_1$ in the DBW2 action (c_0 and c_1 are obtained *starting* from $\beta\, c_0 = 6.0$ and 6.3)

3. Applications

We now turn to two different applications of cactus improvement: The additive mass renormalization for clover fermions and the 1-loop renormalization of the axial and vector currents using the overlap action. Both cases employ Symanzik improved gluons; hence, our results are presented for various sets of Symanzik coefficients.

3.1 Critical mass of clover fermions

It is well known that an ultra-local discretization of the fermion action without doubling breaks chirality. Consequently, we must demand a zero renormalized fermion mass, in order to ensure chiral symmetry while approaching the continuum limit. For this purpose, the bare mass is additively renormalized from its zero tree-level value to a critical value dm.

We calculate the 1-loop result for the critical mass $dm_{1-\mathrm{loop}}$ using clover fermions and Symanzik improved gluons. The result is then dressed with cactus diagrams in order to get the improved value $dm^{\mathrm{dr}}_{1-\mathrm{loop}}$. Details on the definition of dm as well as a 2-loop calculation of dm with the same actions can be found in Ref. [6, 7, 8]. The result of the 1-loop diagrams contributing to $dm_{1-\mathrm{loop}}$ can be written as a polynomial in the clover parameter, and is independent of the number of fermion flavors N_f. Some numerical values for $dm_{1-\mathrm{loop}}$ corresponding to the plaquette and Iwasaki actions ($N=3$) appear in Ref. [1].

Using the critical mass, one can evaluate the critical hopping parameter, $\kappa_{\mathrm{cr}} \equiv 1/(2\,dm + 8\,r)$ (r is the Wilson parameter). Estimates of κ_{cr} from numerical simulations exist in the literature for the plaquette action [9, 10] ($N_f = 0$), [11] ($N_f = 2$), and also the Iwasaki action [12] ($N_f = 2$). Perturbative (unimproved and dressed) and non-perturbative results are listed in Table II for specific values of c_{SW}. It is clear that cactus dressing leads to results for κ_{cr} which are much closer to values obtained from simulations.

Action	N_f	β	c_{SW}	$\kappa_{\mathrm{cr},1-\mathrm{loop}}$	$\kappa^{\mathrm{dr}}_{\mathrm{cr},1-\mathrm{loop}}$	$\kappa^{\mathrm{non-pert}}_{\mathrm{cr}}$
Plaquette	0	6.00	1.479	0.1301	0.1362	0.1392
Plaquette	0	6.00	1.769	0.1275	0.1337	0.1353
Plaquette	2	5.29	1.9192	0.1262	0.1353	0.1373
Iwasaki	2	1.95	1.53	0.1292	0.1388	0.1421

TABLE II. 1-loop results and non-perturbative values for κ_{cr}

3.2 One-loop renormalization of fermionic currents

As a second application of cactus improvement, we investigate the renormalization constant Z_V (Z_A) of the flavor non-singlet vector (axial) current in 1-loop perturbation theory. Overlap fermions and Symanzik improved gluons are employed. Bare 1-loop results for $Z_{V,A}$ have been computed in the literature [13, 5, 14]; they depend on the overlap parameter ρ ($0 < \rho < 2$).

The renormalization constants Z_V and Z_A are equal [13] when using the overlap action and in the $\overline{MS}$ scheme. Table III of Ref. [1] gives the values of $Z_{V,A}$ and $Z^{\mathrm{dr}}_{V,A}$ for different sets of the Symanzik coefficients, choosing $\rho = 1.0$, $\rho = 1.4$. The dependence of $Z_{V,A}$ and $Z^{\mathrm{dr}}_{V,A}$ on the overlap parameter ρ is shown in Fig. 6, where we plot our results for three actions: Plaquette, Iwasaki and TILW. Note that improvement is more apparent for the case of the plaquette action. This was to be expected, since improved gluon actions were constructed in a way as to reduce lattice artifacts, in the first place. A comparison between our improved $Z_{V,A}$ values and some non-perturbative estimates [15], shows that improvement moves in the right direction.

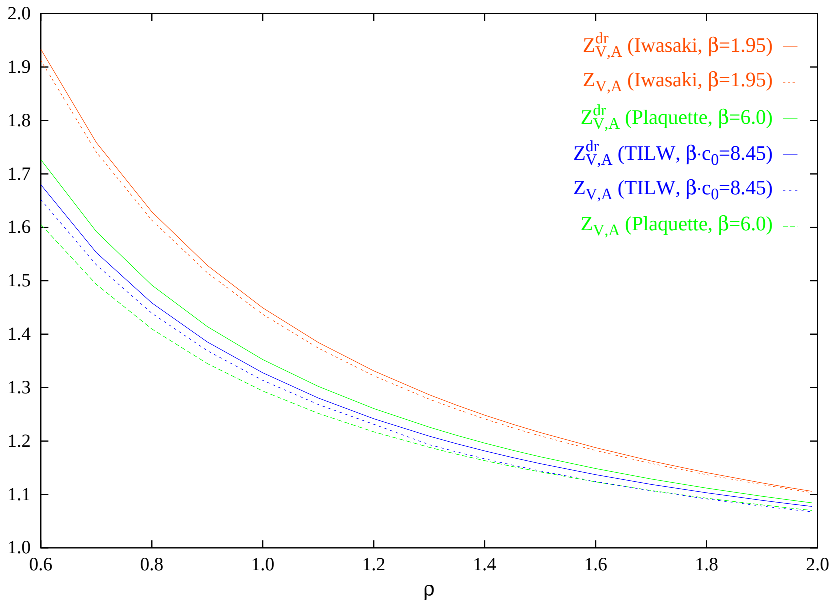


Fig. 6: Plots of $Z_{V,A}$ and $Z^{dr}_{V,A}$ for the plaquette, Iwasaki and TILW actions. Labels have been placed in the same top-to-bottom order as their corresponding curves.

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References

[1] M. Constantinou, H. Panagopoulos and A. Skouroupathis, Phys. Rev. **D** (2006) 074503 [hep-lat/0606001].

[2] H. Panagopoulos and E. Vicari, Phys. Rev. **D58** (1998) 114501 [hep-lat/9806009].

[3] G. P. Lepage and P. B. Mackenzie, Phys. Rev. **D48** (1993) 2250 [hep-lat/9209022].

[4] H. Panagopoulos and E. Vicari, Phys. Rev. **D59** (1999) 057503 [hep-lat/9809007].

[5] R. Horsley et al., Nucl. Phys. **B693** (2004) 3; Erratum-ibid. **B713** (2005) 601 [hep-lat/0404007].

[6] E. Follana and H. Panagopoulos, Phys. Rev. **D63** (2001) 017501 [hep-lat/0006001].

[7] H. Panagopoulos and Y. Proestos, Phys. Rev. **D65** (2002) 014511 [hep-lat/0108021].

[8] A. Skouroupathis, M. Constantinou and H. Panagopoulos, [hep-lat/06xxxxx].

[9] UKQCD Collaboration (K. C. Bowler et al.), Phys. Rev. **D62** (2000) 054506 [hep-lat/9910022].

[10] M. Lüscher, S. Sint, R. Sommer, P. Weisz and U. Wolff, Nucl. Phys. **B491** (1997) 323 [hep-lat/9609035].

[11] K. Jansen and R. Sommer, Nucl. Phys. **B530** (1998) 185; Erratum-ibid. **B643** (2002) 517 [hep-lat/9803017].

[12] CP-PACS Collaboration (A. Ali Khan et al.), Phys. Rev. **D65** (2002) 054505; Erratum-ibid. **D67** (2003) 059901 [hep-lat/0105015].

[13] C. Alexandrou, E Follana, H. Panagopoulos and E. Vicari, Nucl.Phys. **B580** (2000) 394 [hep-lat/0002010].

[14] M. Ioannou and H. Panagopoulos, Phys. Rev. **D73** (2006) 054507 [hep-lat/0601020].

[15] L. Giusti, C. Hoelbling and C. Rebbi, Nucl. Phys. (Proc. Suppl.) **106** (2002) 739 [hep-lat/0110184].