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The effect of reduced spatial symmetries on lattice states: results for non-zero linear momentum

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The irreducible representations of the cubic space group are described and used to determine the mapping of continuum states to lattice states with non-zero linear momentum. The Clebsch-Gordan decomposition is calculated from the character table for the cubic space group. These results are used to identify multiparticle states which appear in the hadron spectrum on the lattice.

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1. Introduction

A question of great interest in lattice QCD is how rotational states on the lattice correspond to states of definite angular momentum in the continuum limit. This problem has been discussed in several contexts previously – in solid state physics, the "cubic harmonics" are formed by a projection of the continuum spherical harmonics onto the lattice, e.g. [1]. In lattice QCD, the reduction of continuum states to the hypercubic lattice was given by Mandula *et al.* [2], and the reduction of the full continuum symmetry group (including Poincaré, color, flavor, and baryon number symmetries) to the lattice for the case of staggered fermions was given by Golterman and Smit [3, 4] and expanded to include non-zero momentum by Kilcup and Sharpe [5].

We are interested in the classes of lattice actions with unbroken flavor symmetries (i.e. Wilson and overlap). In particular, we focus on the IRs of the symmetry group of the lattice Hamiltonian. Johnson [6] considered the mapping of continuum SU(2) states to the octahedral group and its double cover, and Basak *et al.* [7, 8] considered the inclusion of parity in these groups. We expand this work to include states at non-zero momentum [9].

In addition, we consider multiparticle states on the lattice by calculating the Clebsch-Gordan decomposition of direct products of lattice IRs. In many lattice calculations, the operators used to compute correlation functions are constructed to transform irreducibly under the symmetry group of continuum QCD Hamiltonian. As above, it is well known that these operators need not transform irreducibly under the symmetry group of the lattice QCD Hamiltonian. Thus, it is important to understand how such continuum operators transform on the lattice in order to correctly determine what multiparticle states will appear in the lattice spectrum. When calculating ground state masses, ignoring this fact usually does not lead to confusion. One possible exception is the $I(J^P) = \frac{1}{2}(\frac{3}{2}^+)$ and $\frac{1}{2}(\frac{5}{2}^+)$ baryons, whose lowest-lying resonances should correspond to the experimentally observed $N(1720)\ P_{13}$ and $N(1680)\ F_{15}$, respectively [10].

We demonstrate for each set of quantum numbers in the center-of-mass frame what two-particle decompositions are possible, including states with non-zero relative momentum.

2. The symmetry groups and their representations

The continuum spacetime symmetries of the QCD action are given by the Poincaré group, $\mathscr{P}$. The spatial symmetries of the QCD Hamiltonian correspond to a subgroup of $\mathscr{P}$ which is the semidirect product of the group of orthogonal transformations in three dimensions and the group of translations, $\mathscr{T}^3 \rtimes \mathrm{O}(3)$ (pure time inversion is also a symmetry element, but its addition is trivial since it commutes with all other group elements). On the lattice, the symmetry group is $\mathscr{T}^3_{lat} \rtimes \mathrm{O}_h$, where the rotational group is reduced to a subgroup of O(3) with only a finite number of rotations and reflections, and where the subgroup of $\mathscr{T}^3_{lat}$ contains only lattice translations. We consider the double covers of the rotational groups since the double valued IRs correspond to fermionic states in the continuum.

The group of proper rotations of a cube in three dimensions is the octahedral group, O. We are also interested in its double cover, $\mathrm{O^D}$, and we consider the inclusion of parity by forming the group $\mathrm{O}^\mathrm{D}_h = \mathrm{O^D} \times \mathrm{C}_2$, where C_2 consists of the identity and a parity element I_s, corresponding to

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m_j	Dic_4	Dic_3	Dic_2	C_4	C_2
0^+	A_1	A_1	A_1	A	A
0^-	A_2	A_2	A_2	B	A
$\frac{1}{2}$	E_1	E_1	E	E	$2B$
1	E_2	E_2	$B_1 \oplus B_2$	$A \oplus B$	$2A$
$\frac{3}{2}$	E_3	$B_1 \oplus B_2$	E	E	$2B$
2	$B_1 \oplus B_2$	E_2	$A_1 \oplus A_2$	$A \oplus B$	$2A$
$\frac{5}{2}$	E_3	E_1	E	E	$2B$
3	E_2	$A_1 \oplus A_2$	$B_1 \oplus B_2$	$A \oplus B$	$2A$
$\frac{7}{2}$	E_1	E_1	E	E	$2B$
4	$A_1 \oplus A_2$	E_2	$A_1 \oplus A_2$	$A \oplus B$	$2A$

Table 1: Reduction of the double cover of O(2) to the possible little groups.

inversion of all three coordinate axes through the origin. The inclusion of parity is straightforward because I_s commutes with all proper rotations in three dimensions.

For $\mathscr{T}^3_{lat} \rtimes \text{O}^\text{D}_h$, we can then write the group elements $\{R_i, \mathbf{n}\}$, where R_i denotes a rotation in one of the lattice rotation groups discussed above, followed by a lattice translation by $\mathbf{n}$. The subgroup of translations, $\mathscr{T}^3_{lat}$, is normal, so we can easily use this subgroup to induce the IRs of the full group. We can construct the characters for the IRs of the continuous group analogously. We then subduce a representation of the cubic space group by considering the IRs of the continuum group restricted to the subgroup of elements which are in the lattice group. By the orthogonality properties of characters for finite groups, we can decompose the subduced continuum representation into lattice IRs. However, the group $\mathscr{T}^3_{lat} \rtimes \text{O}^\text{D}_h$ allows arbitrarily large translations, so we must consider the 3-torus formed by the boundary conditions $\mathbf{r} + \mathbf{N} = \mathbf{r}$ for all vectors $\mathbf{r}$ and some constant vector $\mathbf{N} = (N, N, N)$.

For finite lattices, we find that the projection formula reduces to the projection of the continuous rotation group to the little group given by $\mathbf{k}$, independent of the lattice size. We also find that representations labeled by different stars are orthogonal. Therefore, the reduction of an arbitrary continuum IR labeled by $(\mathbf{k}, m_j)$ contains IRs of the discrete group labeled by $\mathbf{k}$, and by α which correspond to the reduction of $\text{O}^\text{D}(2)$ to the little group [9].

If $\mathbf{k} = 0$, one reduces $\text{O}^\text{D}(3)$ to O^D_h. These results can be read off those given by Johnson [6] using the result that IRs of $\text{O}^\text{D}(3)$ with positive parity, $\pi = +1$, correspond to the “gerade" IRs (*e.g.* A_{1g}) of O^D_h only, and those with $\pi = -1$ correspond to the “ungerade" IRs (*e.g.* A_{1u}) only .

3. Clebsch-Gordan Decomposition

Additionally, we wish to calculate the decomposition of the direct product of IRs of $\mathscr{T}^3_{lat} \rtimes \text{O}^\text{D}_h$ into a direct sum of IRs. Using the character table for $\mathscr{T}^3_{lat} \rtimes \text{O}^\text{D}_h$, the character of a group element $g \in \mathscr{T}^3_{lat} \rtimes \text{O}^\text{D}_h$ in the direct product representation $\Gamma_i \otimes \Gamma_j$ is given as $\chi^{\Gamma_i,\Gamma_j}(g) = \chi^{\Gamma_i}(g)\chi^{\Gamma_j}(g)$. The direct product representations are then decomposed into lattice IRs using their orthogonality properties.

As we expect, we see that linear momentum is conserved, i.e. the product of two representations with momenta $|\mathbf{k_1}|$ and $|\mathbf{k_2}|$ gives only representations labeled by $|\mathbf{k}|$ which are the sum of some vector in the star of $\mathbf{k_1}$ and some vector in the star of $\mathbf{k_2}$. Thus, the direct product of two IRs of $\mathscr{T}^3_{lat} \rtimes \mathrm{O}^{\mathrm{D}}_h$ contain IRs labeled by $|\mathbf{k}| = 0$ (the IRs of $\mathrm{O}^{\mathrm{D}}_h$) if and only if $|\mathbf{k_1}| = |\mathbf{k_2}|$. Complete tables of the Clebsch-Gordan decomposition for each of the possible lattice momenta are given in [11].

4. Multiparticle States

Since continuum IRs with $J \leq \frac{3}{2}$ remain irreducible under the reduced symmetries of the lattice, the continuum relations are recovered for low spins. For example, the π with $J^P = 0^-$ lies in the irreducible representation A_{1u}, and the $\pi\pi$ multiparticle state lies in $A_{1u} \otimes A_{1u} = A_{1g}$, which as expected corresponds to $J^P = 0^+$.

Higher spin states are less straightforward since multiple lattice IRs appear for each spin. A spin 2 continuum state could lie in either the E or T_2 representations on the lattice, which leads to different possibilities for multiparticle states depending on the particular decomposition of the spin 2 continuum IR into lattice IRs.

When combining states with non-zero momentum, the continuum relations are not as easily recovered. Here, the continuum representations are no longer labeled by J, but by the projection of J along the momentum vector, m_j. However, we know that a particle with a given m_j has $J \geq m_j$, so the lowest spin state for a given irreducible representation of $\mathscr{T}^3_{lat} \rtimes \mathrm{O}^{\mathrm{D}}_h$ will be $J = m_j$. In addition, the reduced symmetry of the little groups leads to fewer distinct lattice IRs than at zero momentum. Since the continuum IRs are mapped to fewer lattice representations, it is more difficult to assign a particular spin to a given lattice irreducible representation.

Thus, for low spins on the lattice we expect to see the same multiparticle states as we would in the continuum, but as we go to higher spins or non-zero momentum deviations will occur from the continuum behavior. For example, from Tab. 1 a $J^P = 2^+$ state in the continuum can lie in either E_g, T_{2g}, or some combination of both on the lattice. In the continuum, an $f_2(1270)$ meson with $J^P = 2^+$ has the decay modes $\pi\pi$, 4π, and $K\bar{K}$ [12]. Thus we expect to see multiparticle states in the lattice spectrum corresponding to these decay modes. From the Clebsch-Gordan decomposition [11], we see that the multiparticle state $\{n,0,0\};A_2 \otimes \{n,0,0\};A_2$ corresponding to these decay modes occurs in the E_g channel, but not in the T_{2g} channel. We must calculate exactly how the particular spin 2 continuum state we are interested in subduces to the lattice to determine whether these multiparticle states will appear. In this case, it is possible that states we expect from the continuum rules for addition of angular momentum would be absent from the lattice spectrum.

Another example where the lattice multiparticle states cannot be predicted from the continuum behavior occurs for $J^P = \frac{5}{2}^-$. As for spin 2, a continuum spin $\frac{5}{2}$ state can lie in more than one lattice representation, either G_{2u} or H_u. However, if we consider multiparticle states in the H_u channel, then both states which go to spin $\frac{3}{2}$ in the continuum limit and states which go to spin $\frac{5}{2}$ in the continuum limit will appear, since at any finite lattice spacing these states may have the same lattice quantum numbers (i.e. correspond to the H_u irreducible representation). Again, we must know how our $J^P = \frac{5}{2}$ state subduces to the lattice in order to determine exactly which multiparticle states will occur in the lattice spectrum. Here, multiparticle states are present in the lattice spectrum which

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we would not predict from the continuum states. As we consider higher spins, these types of ambiguities become common.

5. Conclusion

We have found the mapping of continuum IRs to lattice states by decomposing subduced continuum representations into the discrete lattice IRs. In general, we find that a single continuum IR decomposes into multiple IRs under the reduced symmetry of the lattice.

These results were used to calculate the Clebsch-Gordan decomposition and determine what multiparticle states can appear on the lattice. For low spins and zero momentum, we recover the continuum behavior. However, as we go to higher spins or non-zero momentum, it is important to understand how our operators transform irreducibly on the lattice since deviations from the continuum behavior can occur.

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