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The rooted staggered determinant in the Schwinger model

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We investigate the continuum limit of the rooted staggered action in the 2-dimensional Schwinger model. We match both the unrooted and rooted staggered determinants with an overlap fermion determinant of two (one) flavors and a local pure gauge effective action by fitting the coefficients of the effective action and the mass of the overlap operator. The residue of this fit measures the difference of the staggered and overlap fermion actions. We show that this residue scales at least as $\mathrm{O}(a^2)$, implying that any difference, be it local or non-local, between the staggered and overlap actions becomes irrelevant in the continuum limit. For the model under consideration here, this observation justifies the rooting procedure for the staggered sea quark action.

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1. Introduction

While staggered fermions offer many computational advantages, their action does not have full chiral symmetry and the chiral limit has to be taken together with the continuum limit. This is no different from other non-chiral actions, but staggered fermions have another, potentially serious problem. In 4 dimensions the staggered action describes four species (or tastes) of fermions and in order to reduce the number of tastes a fractional power of the fermion determinant is taken in the path integral. There is no a priori reason that this rooted determinant corresponds to a local fermionic action belonging to the same universality class as 1–flavor QCD.

Although several analytical and numerical works addressed this question in the last few years [1 – 10], none of them showed evidence that the procedure introduces non–universal errors, i.e. errors that cannot be considered cutoff effects. Recently it has been argued, based on a number of reasonable conjectures, that while the rooted staggered action is non-local at any finite lattice spacing, in the continuum limit the non-local terms become irrelevant [11, 12].

In this paper we present numerical evidence obtained in the massive Schwinger model, showing that the rooted staggered action is in the right universality class. We also show that the staggered action can be considered equivalent to a chiral Ginsparg-Wilson action only when the staggered mass is larger than typical taste symmetry breaking effects, limiting the parameter space where staggered simulations can be expected to approximate continuum QCD. We describe how the masses of the staggered and corresponding overlap actions should be matched to obtain physically equivalent theories when this condition is satisfied. More details on the matching procedure and additional results can be found in [13].

2. The continuum limit of the staggered action

The partition function of the unrooted staggered action is

$$Z = \int D[U\,\bar{\psi}\psi]\, e^{-S_g(U)-\bar{\psi}(M+am_{\rm st})\psi} = \int D[U]\,\det(M+am_{\rm st})\, e^{-S_g(U)}\,, \qquad (2.1)$$

where $S_g(U)$ is a gauge action, M is the staggered Dirac operator and $am_{\rm st}$ is the bare staggered mass. In the $a \to 0$ continuum limit the staggered action describes $n_t = 4$ degenerate fermions in 4, $n_t = 2$ fermions in 2 dimensions. At finite lattice spacing the taste symmetry is broken, the action describes n_t fermion tastes but only with a remnant $U(1)$ taste symmetry. Depending on the staggered quark mass, at finite lattice spacing one has one of the following situations.

- At $am_{st} = 0$ the staggered action's spectrum has a single Goldstone particle and $n_t^2 - 1$ massive pseudoscalars. While $n_t^2 - 2$ of these will become massless as $a \to 0$, at any finite lattice spacing the staggered spectrum is very different from n_t–flavor massless QCD. At small fermion mass $am_{\rm st} \gtrsim 0$ the taste breaking terms dominate and the non–Goldstone pions are heavy compared to the Goldstone one. One does not expect QCD–like behavior.

- $am_{\rm st} \gtrsim 1$ is the cutoff region, again not continuum QCD–like.

- Only in the middle of these extremes would one expect to observe QCD. The $a \to 0$, $am_{\rm st} \to 0$ continuum limit should be approached here.

While staggered fermions formally allow $am_{\rm st} = 0$, physically this limit does not correspond to QCD at any finite lattice spacing [7, 8]. Simulations cannot be trusted at a small fermion mass where taste breaking terms dominate the pseudoscalar sector. However, the taste breaking terms are expected to scale at least with $O(a^2)$, such that at small enough lattice spacing the continuum limit can be approached with any finite fermion mass. Thus the exclusion of $am_{\rm st} = 0$ is not a serious problem for massive fermions.

The staggered determinant can always be written as

$$\det(M + am_{\rm st}) = \det{}^{n_t}(D_{1\rm f} + am_{1\rm f})\det(T)\,, \tag{2.2}$$

where $D_{1\rm f} + am_{1\rm f}$ is an arbitrary 1–flavor Dirac operator and $\det(T)$ describes all the terms that are not included in the latter. If the local $D_{1\rm f}$ operator and the mass term $m_{1\rm f}$ could be chosen such that T contains only local gauge terms,

$$\det(T) = e^{-S_{\rm eff}(U)}\,, \tag{2.3}$$

the staggered action would differ from an n_t–flavor degenerate Dirac operator only in cutoff level terms [2]. This is indeed the case for heavy, $am_{\rm st} \gtrsim 1$ fermions.

On the other hand there are several examples [10, 11, 14] that illustrate that at $am_{\rm st} = 0$ the operator T cannot be local at any finite lattice spacing. This, however, does not mean that the staggered operator cannot describe QCD in the continuum limit. If we write the determinant as

$$\det(T) = e^{-S_{\rm eff}(U)}\det(1+\Delta)\,, \tag{2.4}$$

and can choose $S_{\rm eff}$ such that the non–local term Δ is bounded at finite mass and goes to zero as $a \to 0$, the staggered determinant in Eq.(2.2) will describe n_t degenerate flavors in the continuum limit. This is certainly the expected behavior for the unrooted action.

Now we turn our attention to the rooting procedure. With the notation introduced above, the root of the staggered determinant is

$$\det{}^{1/n_t}(M + am_{\rm st}) = \det(D_{1\rm f} + am_{1\rm f})\, e^{-S_{\rm eff}(U)/n_t}\det{}^{1/n_t}(1+\Delta). \tag{2.5}$$

If one could show that

$$\Delta \to 0 \quad \text{as } a \to 0\,, \tag{2.6}$$

the rooted determinant of Eq.(2.5) would correspond to a local 1–flavor action in the continuum limit. Based on renormalization group arguments, in Ref. [4] Shamir showed that this is indeed the case for free fermions. In a recent work [12], based on a number of reasonable assumptions, he argues that the same is true in the interacting theory. Here we present numerical results to support this claim.

In the following we pick an arbitrary Ginsparg-Wilson operator as $D_{1\rm f}$ and ask if $am_{1\rm f}$ and $S_{\rm eff}(U)$ can be chosen such that Eq.(2.6) is satisfied.

3. Matching the fermionic determinants

The actual matching strategy is fairly general and we will describe it for an arbitrary pair of Dirac operators $D_1 + am_1$ and $D_2 + am_2$. We want to know to what extent the determinant of the first Dirac operator can be described by the determinant of the second plus pure gauge terms. To find this we calculate the determinant ratio

$$\det(T) = \frac{\det(D_1 + am_1)}{\det(D_2 + am_2)} \tag{3.1}$$

on configurations generated with the action $S_1 = S_g(U) + \overline{\psi}(D_1 + am_1)\psi$. Next we fit the logarithm of the determinant ratios with a local pure gauge action $S_{\rm eff}$. In practice we use an ultralocal effective action consisting of small Wilson loops. The accuracy of the matching at fixed fermion mass m_2 is characterized by the per flavor/taste residue

$$r(m_2) = \left\langle \left(\log \frac{\det(D_1 + am_1)}{\det(D_2 + am_2)} - S_{\rm eff}(U) \right)^2 \right\rangle^{1/2}. \tag{3.2}$$

The minimum of the residue $r(m_2)$ in terms of m_2 determines the action $D_2 + am_2$ that is *physically closest* to the original $D_1 + am_1$ action. In this sense it defines the mass $\overline{m}_2$ that matches the fermion mass m_1. In the notation of Eq.(2.4) then

$$r(\overline{m}_2) = \left\langle \left((\log \det(1+\Delta) \right)^2 \right\rangle^{1/2}. \tag{3.3}$$

If the two fermion operators describe the same continuum theory the residue has to vanish as $a \to 0$ at fixed volume and quark mass.

4. Schwinger model - numerical results

4.1 Setup and matching tests

The 2–dimensional Schwinger model offers an excellent testing ground for the matching idea as it can be studied with high accuracy and limited computer resources. It is a super–renormalizable theory since the bare gauge coupling g is dimensional, the lattice gauge coupling is $\beta = 1/(ag)^2$. A continuum limit in fixed physical volume can be achieved by keeping the scaling variable $z = Lg$ fixed while increasing the lattice resolution. We choose $z = 6$ and vary the lattice size between $L/a = 12$ and $L/a = 28$. The scaling parameter z characterizes the (physical) volume while we use mL to fix the mass.

We produced gauge configurations using a global heatbath for the plaquette gauge action and in the data analysis the measurements are reweighted with the appropriate power of the fermion determinant to obtain the observables in the full dynamical theory. On the gauge configurations we measure a set of Wilson loops $\mathscr{C}_l$ as well as the complete spectra of the Dirac operators under consideration. For the matching we use 9 loops up to length 10 in $S_{\rm eff}$. With a maximal extension of four lattice units $S_{\rm eff}$ is very localized even on our coarsest lattices and in particular we do *not* increase the size or number of loops as we approach the continuum. In the following we concentrate on the matching of the staggered action to a smeared overlap action. For more details see [13].

4.2 The unrooted staggered action

We start our investigation with the unrooted action, which in 2 dimensions corresponds to two fermion tastes. In the continuum limit it is expected to describe two degenerate flavors and thus it should differ from a degenerate 2–flavor overlap action in irrelevant terms only. Fig.1 shows the matching of the $n_t = 2$ staggered determinant with the $N_f = 2$ flavor-degenerate overlap determinant at $z = 6$ on $L/a = 20$ lattices ($\beta \simeq 11.11$). The quenched configurations were reweighted to the dynamical staggered ensemble at $am_{\rm st} = 0.025$. The residue of the matching (Eq.3.2) has a well defined minimum at $a\overline{m}_{\rm ov} = 0.0317(3)$.

By repeating the matching at different values of the staggered masses $am_{\rm st}$ we can find the matching overlap masses at the given lattice spacing as shown by the blue dots in Fig.2. For

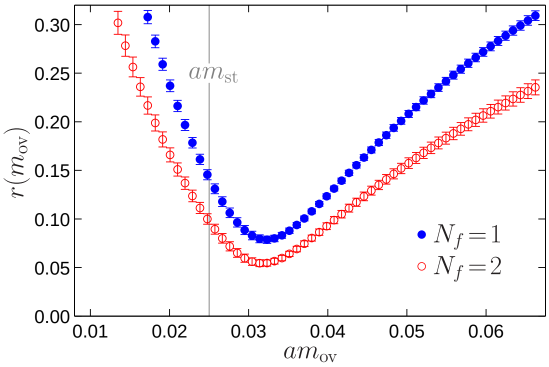

Figure 1: The residue of Eq.(3.2) as the function of the matching mass on $L/a = 20$ configurations generated at $am_{\rm st} = 0.025$.

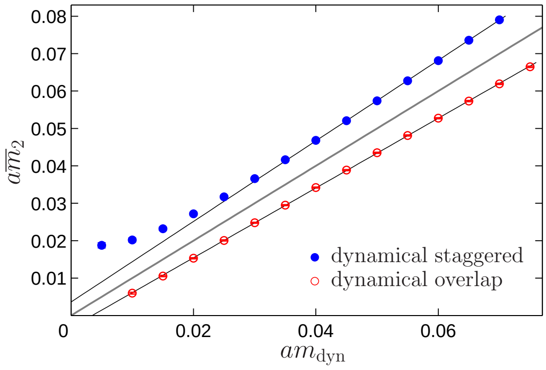

Figure 2: The matching mass as the function of the dynamical action mass at fixed lattice spacing for 2 tastes/flavors.

larger masses the data show a linear dependence with a constant offset. For small masses, below $am_{\rm st} \approx 0.02$, there is a clear deviation from the linear behavior. The residue of the fit increases from 0.02 at the heaviest mass to 0.09 at the lightest one, indicating that the matching is no longer meaningful. According to the discussion in Sect.2 we interpret this as the staggered action being QCD–like for $am_{\rm st} \gtrsim 0.02$ and not QCD–like below. As a consistency check we repeated the same matching on a dynamical overlap background. The result, shown by the red circles in Fig.2, is the mirror image of the staggered with overlap matching data up to the point where the latter matching breaks down. This is the expected behavior if the two actions differ only by lattice artifacts.

By restricting the configurations to the sector of trivial topology we could verify that the difference between the matching on the two ensembles and also most of the residue can be ascribed to configurations with non–vanishing topological charge.

Next we consider the continuum limit of the matching at fixed physical mass [1] and volume $z = 6$. Fig.3a shows the residue of matching the 2-taste staggered determinant with the 2-flavor overlap determinant at different masses as a function of a^2g^2. For the smallest mass, $m_{\rm ov}L = 0.4$ the data stops around $a^2g^2 = 0.11$ - on coarser lattices the two actions cannot be matched, the residue of Eq.(3.2) has no minimum. Nevertheless matching is possible at smaller lattice spacing and the residue at fixed $m_{\rm ov}L$ approaches zero at least quadratically in a. The continuum limit can be approached with any fermion mass and the staggered determinant can be described as a 2–flavor chiral determinant plus pure gauge terms. This is the behavior we expected from universality.

4.3 The rooted staggered action

Now we repeat the analysis of the previous section for the rooted staggered action. The configurations are reweighted with the square root of the staggered determinant and the rooted determinant is matched with the 1-flavor overlap determinant plus pure gauge terms, according to Eq.(3.2).

The quality of the matching is very similar to the unrooted/2–flavor case as the open circles in Fig.1 show. In fact, even the matched mass $a\overline{m}_{\rm ov} = 0.0322(2)$ hardly differs from the 2–flavor case. The 1–flavor data in Fig.2 is indistinguishable from the shown 2–flavor data.

[1] We fix the physical mass by keeping $m_{\rm ov}L$ constant and vary the staggered sea quark mass to achieve the matching.

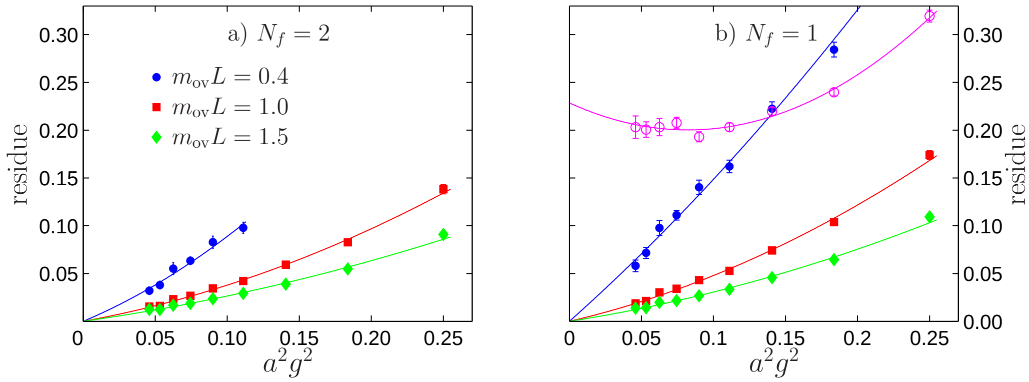

Figure 3: Residue of the matching as a function of the (squared) lattice spacing at different physical masses. a) unrooted staggered/2–flavor overlap; b) rooted staggered/1–flavor overlap matching. The open circles in b) show the residue from matching a 1–flavor overlap ($m_{ov}L = 0.5$) to an *unrooted* staggered action.

Fig.3b is the important plot for the rooted staggered action as it shows the residue at fixed physical masses as the continuum limit is approached. While the residue for the 1–flavor rooted determinant is larger than in the unrooted case, the continuum approach is identical, at least quadratic in a. The taste violating term Δ in Eqs.(2.4) and (3.3) becomes irrelevant in the continuum limit, a result that justifies the rooting procedure. As a test of the sensitivity of our matching method we also show the residue of an attempted matching of a 1–flavor overlap action with an unrooted staggered action, which clearly does *not* vanish in the continuum limit.

4.4 Application

We can now apply our knowledge of the matching overlap mass in mixed action simulations. The first observable we consider is the topological susceptibility $\langle Q^2 \rangle / z^2$, as it is very sensitive to the sea quarks. We define the topology through the zero–modes of the smeared overlap operator used in the matching and evaluate it on gauge ensembles generated with two and one flavor/taste staggered and overlap actions at various masses. This is the simplest case of a mixed action simulations as the observable does not depend on the valance quark mass. Possibly more interesting is the scalar condensate $\langle \bar{\psi}\psi \rangle$, which diverges in the limit of vanishing staggered mass due to insufficient suppression of topologically non–trivial configurations [5].

Results from $L/a = 20$ lattices are shown in Figs.4 and 5, where the difference of the staggered and overlap ensemble at the same bare fermion mass is very evident, especially at small masses. After shifting the staggered data to the matching overlap mass, excellent agreement is achieved. One should note that the agreement for the 2 taste/flavor case, where the difference between the actions is entirely due to local lattice artifacts, is not better or significantly different than the rooted 1 taste/flavor case. The agreement on our finer lattices is equally good and extends to smaller quark masses.

5. Conclusion

The rooted staggered action is likely non–local in the physically relevant range of small quark masses. However, this does not invalidate the rooted action as long as the non–local terms are

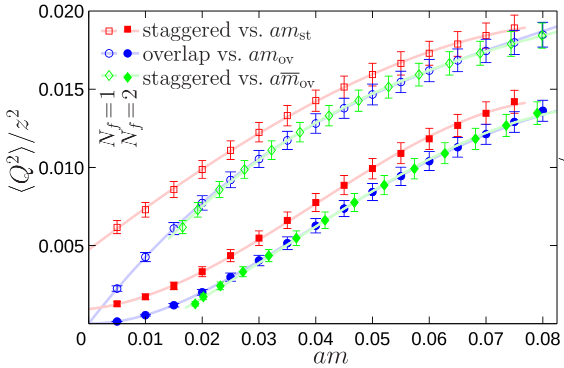

Figure 4: The topological susceptibility on the $L/a = 20$ ensemble. After shifting the staggered data to the matched overlap mass, almost perfect agreement with the overlap data is achieved.

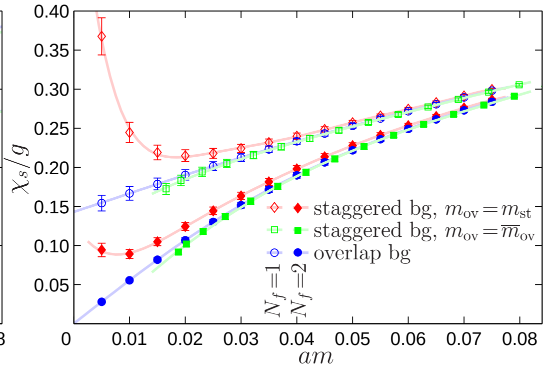

Figure 5: Also the scalar condensate χ_s on a staggered background agrees with the overlap result when it is evaluated using the matched overlap mass.

irrelevant and scale away in the continuum limit. Here we demonstrated that this is indeed the case in the 2–dimensional Schwinger model. We studied how the staggered action differs from a chiral overlap action along a line of constant physics as the continuum is approached. For both the unrooted (as expected) and rooted staggered action we found that the difference reduces to irrelevant operators plus local pure gauge terms. Nevertheless care is required in taking the continuum limit of staggered fermions such that the non QCD–like region is avoided.

References

[1] B. Bunk, M. Della Morte, K. Jansen, and F. Knechtli, Nucl. Phys. **B697** (2004) 343–362, [`hep-lat/0403022`].

[2] D. H. Adams, Phys. Rev. **D72** (2005) 114512, [`hep-lat/0411030`].

[3] F. Maresca and M. Peardon, *A path-integral representation of the free one-flavour staggered-fermion determinant*, `hep-lat/0411029`.

[4] Y. Shamir, Phys. Rev. **D71** (2005) 034509, [`hep-lat/0412014`].

[5] S. Dürr and C. Hoelbling, Phys. Rev. **D69** (2004) 034503, [`hep-lat/0311002`].

[6] H. Neuberger, Phys. Rev. **D70** (2004) 097504, [`hep-lat/0409144`].

[7] S. Dürr and C. Hoelbling, Phys. Rev. **D71** (2005) 054501, [`hep-lat/0411022`].

[8] C. Bernard, Phys. Rev. **D71** (2005) 094020, [`hep-lat/0412030`].

[9] C. Bernard, et al. *PoS* **LAT2005** (2005) 114, [`hep-lat/0509176`].

[10] A. Hasenfratz, *Universality and quark masses of the staggered fermion action*, `hep-lat/0511021`.

[11] C. Bernard, M. Golterman, and Y. Shamir, Phys. Rev. **D73** (2006) 114511, [`hep-lat/0604017`].

[12] Y. Shamir, *Renormalization-group analysis of the validity of staggered-fermion QCD with the fourth-root recipe*, `hep-lat/0607007`.

[13] A. Hasenfratz and R. Hoffmann, Phys. Rev. **D74** (2006) 014511, [`hep-lat/0604010`].

[14] J. B. Kogut and D. K. Sinclair, Phys. Rev. **D73** (2006) 074512, [`hep-lat/0603021`].