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Neutrino Oscillations

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The experiments with solar, atmospheric, reactor and accelerator neutrinos have provided compelling evidences for oscillations of neutrinos caused by nonzero neutrino masses and neutrino mixing. The data imply the existence of 3-neutrino mixing in vacuum. We review the theory of neutrino oscillations, the phenomenology of 3-neutrino mixing, and the current data on the 3-neutrino mixing parameters. The open questions and the main goals of future research in the field of neutrino mixing and oscillations are outlined.

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1. Introduction

There has been a remarkable progress in the studies of neutrinos in the last several years. The experiments with solar, atmospheric and reactor neutrinos [1, 2, 3, 4, 5] have provided compelling evidences for existence of neutrino oscillations - transitions in flight between different flavour neutrinos, caused by nonzero neutrino masses and neutrino mixing.

The hypothesis of neutrino mixing and oscillations was formulated in [6, 7]. In [8] it was predicted that the ν_e oscillations will cause a "disappearance" of solar (ν_e) neutrinos on their way to the Earth. The evidences of solar ν_e "disappearance", obtained first in the Homestake experiment and strengthened by the results of Kamiokande, SAGE and GALLEX/GNO experiments [1, 9], were made compelling by the data of Super-Kamiokande (SK) and SNO experiments [2, 3]. The hypothesis of solar ν_e oscillations, which (in one variety or another) were considered from $\sim$1970 on as the most natural solution of the solar neutrino "puzzle" (see, e.g., refs. [10, 11, 12, 13, 14]), has received a convincing confirmation from the measurement of the solar neutrino flux through the neutral current reaction on deuterium by the SNO experiment [3], and by the first results of the KamLAND experiment [5]. The combined analysis of the solar neutrino data obtained by Homestake, SAGE, GALLEX/GNO, Super-Kamiokande and SNO experiments, and of the KamLAND reactor $\bar{\nu}_e$ data [5], established the large mixing angle (LMA) MSW oscillations/transitions [11, 12] as the dominant mechanism at the origin of the observed solar ν_e deficit (see, e.g., [15]). The Kamiokande experiment [9] provided the first evidences for oscillations of atmospheric ν_μ and $\bar{\nu}_\mu$, while the data of the Super-Kamiokande experiment made the case of atmospheric neutrino oscillations convincing [4, 16, 17]. Evidences for oscillations of neutrinos were obtained also in the long baseline accelerator neutrino experiments K2K [18] and MINOS [19]. Indications for ν-oscillations were reported by the LSND collaboration [20].

A beautiful confirmation of the oscillations of atmospheric ν_μ ($\bar{\nu}_\mu$) and reactor $\bar{\nu}_e$ neutrinos was provided by the Super-Kamiokande data on the L/E-dependence of the μ-like atmospheric neutrino events [16], L and E being the distance traveled by neutrinos and the neutrino energy, and the spectrum data of the KamLAND and K2K experiments [21, 22]. For the first time the data exhibit directly the effects of the oscillatory dependence on L/E and E of the probabilities of ν-oscillations in vacuum [23]. As a result of these developments, the oscillations of solar ν_e, atmospheric ν_μ and $\bar{\nu}_\mu$, accelerator ν_μ (at $L \sim 250$ km and $L \sim 730$ km) and reactor $\bar{\nu}_e$ (at $L \sim 180$ km), driven by nonzero ν-masses and ν-mixing, can be considered as practically established.

The neutrino oscillation data imply the existence of 3-neutrino mixing in vacuum. In the present lectures we review the theory of neutrino oscillations, the phenomenology of 3-ν mixing, and the current data on the 3-ν mixing parameters. We discuss also the open questions and the main goals of future research in the field of neutrino mixing and oscillations.

2. Neutrino Oscillations in Vacuum

We shall consider first the simplest possibility of two-neutrino oscillation in vacuum (see, e.g. [10, 13, 24]). Let us assume that the state vector of the electron neutrino, $|\nu_e\rangle$, produced in vacuum with momentum $\vec{p}$ in some weak interaction process, is a coherent superposition of the state vectors $|\nu_i\rangle$ of two neutrinos ν_i, i=1,2, having the same momentum $\vec{p}$ and definite but different masses in vacuum, $\mathrm{m_i}$, $\mathrm{m_1} \neq \mathrm{m_2}$, while the linear combination of $|\nu_1\rangle$ and $|\nu_2\rangle$, which is orthogonal to $|\nu_e\rangle$, represents the state vector $|\nu_x\rangle$ of another weak-eigenstate neutrino, $|\nu_x\rangle = |\nu_{\mu(\tau)}\rangle$ or $|\nu_s\rangle$, ν_s being a sterile neutrino:

$$\begin{aligned} |\nu_e\rangle &= |\nu_1\rangle \cos\theta + |\nu_2\rangle \sin\theta \;, \\ |\nu_\mu\rangle &= -|\nu_1\rangle \sin\theta + |\nu_2\rangle \cos\theta \;, \end{aligned} \tag{2.1}$$

where θ is the neutrino mixing angle in vacuum and we have chosen (for concreteness) $\nu_x \equiv \nu_\mu$. Obviously, $|\nu_{1,2}\rangle$ are eigenstates of the Hamiltonian of the ν-system in vacuum, H_0:

$$H_0\;|\nu_i\rangle \;= E_i\;|\nu_i\rangle, \quad E_i = \sqrt{\vec{p}^{\;2} + m_i^2},\; i = 1,2. \tag{2.2}$$

If ν_e is produced at time $t = 0$ in the state given by (2.1), after a time t the latter will evolve (in vacuum) into the state

$$|\nu_e(t)\rangle = e^{-iE_1 t}\;|\nu_1\rangle \cos\theta + e^{-iE_2 t}\;|\nu_2\rangle \sin\theta = A_{ee}(t)\;|\nu_e\rangle + A_{\mu e}(t)\;|\nu_\mu\rangle\;, \tag{2.3}$$

where we have ignored the overall space coordinate dependent factor $\exp(i\,\vec{p}\,\vec{r})$ in the right-hand side of (2.3) and used (2.1). Here

$$A_{ee} = e^{-iE_1 t} \cos^2\theta + e^{-iE_2 t} \sin^2\theta,\; A_{\mu e} = \frac{1}{2}\sin 2\theta (e^{-iE_2 t} - e^{-iE_1 t}) \tag{2.4}$$

are the probability amplitudes to find respectively ν_e and ν_μ at time t of the evolution of the ν-system if neutrino ν_e has been produced at time $t = 0$. Thus, if $m_1 \neq m_2$ and if neutrino mixing exists in vacuum, $\theta \neq n\pi/2$, $n = 0,1,2,...$, we have $|A_{\mu e}(t)|^2 \neq 0$ and transitions in flight between ν_e and ν_μ are possible. Assuming that ν_1 and ν_2 are stable and relativistic, we obtain from (2.4) the probabilities that a ν_e will not change into ν_μ, $P(\nu_e \to \nu_e)$, or will transform into ν_μ, $P(\nu_e \to \nu_\mu)$:

$$\begin{aligned} P(\nu_e \to \nu_e;t) = |A_{ee}(t)|^2 = 1 - \tfrac{1}{2}\sin^2 2\theta\;\left(1 - \cos 2\pi \tfrac{L}{L_\nu}\right)\;, \\ P(\nu_e \to \nu_\mu;t) = |A_{\mu e}(t)|^2 = \tfrac{1}{2}\sin^2 2\theta\;\left(1 - \cos 2\pi \tfrac{L}{L_\nu}\right)\;, \end{aligned} \tag{2.5}$$

where $\Delta m^2 = m_2^2 - m_1^2$, $L \cong t$ is the distance traveled by neutrinos and

$$L_\nu = 4\pi \frac{E}{\Delta m^2} \cong 2.48\text{ m}\;\frac{E[MeV]}{\Delta m^2[eV^2]} \tag{2.6}$$

is the oscillation length in vacuum. In obtaining (2.5) we have used the equality $E_2 - E_1 \cong E + \Delta m^2/(2E)$, $E \cong |\vec{p}|$, valid for relativistic neutrinos $\nu_{1,2}$. The quantities Δm^2 and $\sin^2 2\theta$ are typically considered as free parameters to be determined by the analysis of the neutrino oscillation data. A comprehensive theory of neutrino mixing should predict, or at least should be able to explain, the values of these parameters found from the data.

Our derivation of the expressions for the oscillation probabilities (2.5) was based on the assumption that the states $|\nu_1\rangle$ and $|\nu_2\rangle$ in the coherent superposition representing the state $|\nu_e\rangle$ are produced with the same momentum. It can be shown [25] that one arrives at the same result, eq. (2.5), if the states are produced with different momenta.

It should be clear from the above discussion that the neutrino oscillations are a purely quantum mechanical phenomenon. The requirements of coherence between the states $|\nu_1\rangle$ and $|\nu_2\rangle$ in the superposition (2.1) representing the ν_e (or $\nu_{\mu(\tau)}$) at the production point, and that the coherence be

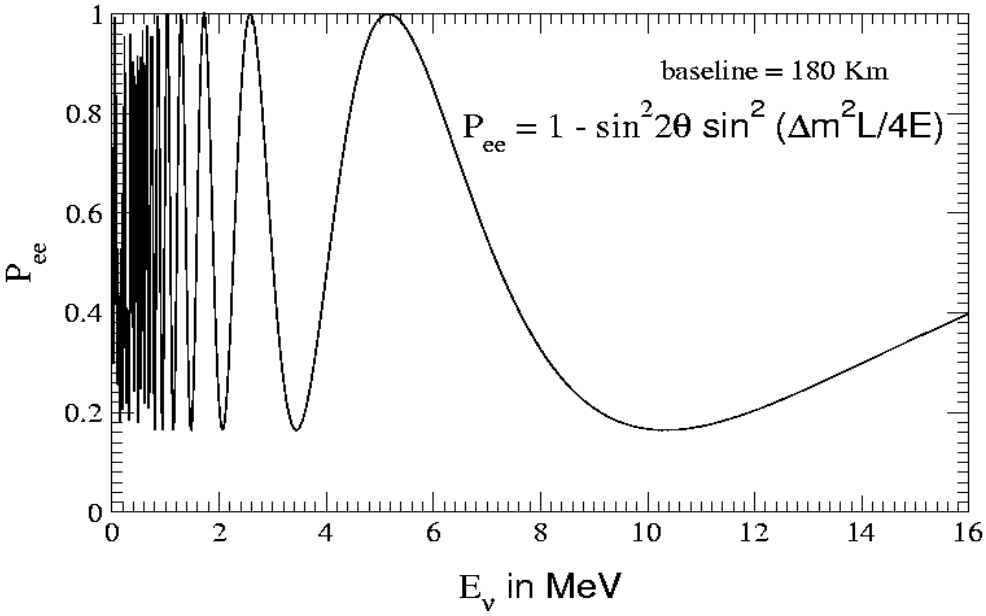

Figure 1: The probability of ν_e ($\bar{\nu}_e$) survival, $P(\nu_e \to \nu_e;t) = P(\bar{\nu}_e \to \bar{\nu}_e;t)$, as a function of the neutrino energy for $L = 180$ km and $\Delta m^2 = 8.0 \times 10^{-5}$ eV^2 (from [27]).

maintained during the evolution of the neutrino system up to the moment of neutrino detection, are crucial for the neutrino oscillations to occur. The subtleties and the implications of the coherence condition for neutrino oscillations continue to be discussed (see, e.g., [10, 26, 28]).

It follows from CPT-invariance, which we will assume to hold, that

$$P(\nu_e \to \nu_e;t) = P(\bar{\nu}_e \to \bar{\nu}_e;t)\ ,\ \ P(\nu_e \to \nu_\mu;t) = P(\bar{\nu}_\mu \to \bar{\nu}_e;t)\ . \tag{2.7}$$

Combined with the probability conservation, $P(\nu_e \to \nu_e;t) + P(\nu_e \to \nu_\mu;t) = 1$, $P(\bar{\nu}_e \to \bar{\nu}_e;t) + P(\bar{\nu}_e \to \bar{\nu}_\mu;t) = 1$, eq. (2.7) implies that in the simple case of two-neutrino oscillations we are considering one has

$$P(\nu_e \to \nu_\mu;t) = P(\bar{\nu}_e \to \bar{\nu}_\mu;t) = P(\nu_\mu \to \nu_e;t) = P(\bar{\nu}_\mu \to \bar{\nu}_e;t)\ . \tag{2.8}$$

As it follows from (2.5), $P(\nu_e \to \nu_\mu;t)$ depends on two factors: on $(1 - \cos 2\pi L/L_\nu)$, which exhibits oscillatory dependence on the distance L and on the ν energy E (hence the name "neutrino oscillations"), and on $\sin^2 2\theta$ which determines the amplitude of the oscillations. In order to have $P(\nu_e \to \nu_\mu;t) \cong 1$, two conditions have to be fulfilled: the neutrino mixing in vacuum must be large, $\sin^2 2\theta \cong 1$, and the oscillation length in vacuum L_ν has to be of the order of or smaller than the distance traveled by the neutrinos, $L_\nu \lesssim 2\pi L$. If $L_\nu \gg 2\pi L$, the oscillations do not have enough time to develop on the way to the neutrino detector and one has $P(\nu_e \to \nu_\mu;t) \cong 0$. This is illustrated in Fig. 1 showing the dependence of the probability $P(\nu_e \to \nu_e;t) = P(\bar{\nu}_e \to \bar{\nu}_e;t)$ on the neutrino energy.

A given experiment searching for neutrino oscillations, is specified, in particular, by the average energy of the neutrinos being studied, $\bar{E}$, and by the distance traveled by the neutrinos to the

Source	Type of ν	$\bar{E}$[MeV]	L[km]	$\min(\Delta m^2)$[eV^2]
Reactor	$\bar{\nu}_e$	~ 1	1	$\sim 10^{-3}$
Reactor	$\bar{\nu}_e$	~ 1	100	$\sim 10^{-5}$
Accelerator	$\nu_\mu, \bar{\nu}_\mu$	$\sim 10^3$	1	~ 1
Accelerator	$\nu_\mu, \bar{\nu}_\mu$	$\sim 10^3$	1000	$\sim 10^{-3}$
Atmospheric ν's	$\nu_{\mu,e}, \bar{\nu}_{\mu,e}$	$\sim 10^3$	10^4	$\sim 10^{-4}$
Sun	ν_e	~ 1	1.5×10^8	$\sim 10^{-11}$

detector L. The requirement $L_\nu \lesssim 2\pi L$ determines the minimal value of Δm^2 to which the experiment is sensitive (figure of merit of the experiment): $\min(\Delta m^2) \sim 2\bar{E}/L$. Because of the interference nature of neutrino oscillations, the ν- oscillation experiments can probe, in general, rather small values of Δm^2 (see, e.g., [10, 13]). Values of $\min(\Delta m^2)$, characterizing qualitatively the sensitivity of different experiments are given in Table 1. They correspond to the reactor experiments CHOOZ ($L \sim 1$ km) and KamLAND ($L \sim 100$ km), to accelerator experiments - past ($L \sim 1$ km), recent, current and future (K2K, MINOS, OPERA, T2K, NOνA)), to Super-Kamiokande experiment studying atmospheric and solar neutrino oscillations, and to the solar neutrino experiments. Due to the large Sun - Earth distance the relatively low energies of the solar ν_e, the experiments with solar neutrinos have a remarkable sensitivity to Δm^2.

In certain cases the dimensions of the neutrino source, ΔR, are not negligible in comparison with the oscillation length. Similarly, when analyzing neutrino oscillation data one has to include the energy resolution of the detector, ΔE, etc. in the analysis. As can be shown [13], if $2\pi \Delta R/(L_\nu) \gg 1$, and/or $L\Delta m^2 \Delta E/(E^2) \gg 1$, the oscillating term in the neutrino oscillation probability will be strongly suppressed. In this case the effects of ν-oscillations will be effectively determined by the average probabilities:

$$\bar{P}(\nu_e \to \nu_e) \cong 1 - \frac{1}{2}\sin^2 2\theta \ , \quad \bar{P}(\nu_e \to \nu_\mu) \cong \frac{1}{2}\sin^2 2\theta \ . \tag{2.9}$$

As we have seen, if (2.1) is realized and $\Delta m^2 L/(2E) \gtrsim 1$ for reactor $\bar{\nu}_e$, for instance, they can take part in vacuum oscillations on the way to the detector (see eqs. (2.8) and (2.7)). In this case the flavour content of the $\bar{\nu}_e$ state vector will change periodically on the way to the detector due to the different time evolution of the vector's massive neutrino components. The amplitude of these oscillations is determined by the value of $\sin^2 2\theta$. If $\sin^2 2\theta$ is sufficiently large, the neutrinos that are being detected at distance L will be in states representing, in general, certain superpositions of the states of [1] $\bar{\nu}_e$ and $\bar{\nu}_\mu$. The reactor $\bar{\nu}_e$ have energies $E \lesssim 12$ MeV and are detected through the charged current (CC) reaction $\bar{\nu}_e + p \to e^+ + n$. Obviously, the $\bar{\nu}_\mu$ component of the state being detected will not give a contribution to the signal in the detector. As a result, the measured signal in the reactor $\bar{\nu}_e$ oscillation experiment should be noticeably smaller than the predicted one in the absence of oscillations. This is what is observed in the KamLAND experiment [5, 21], which has

[1]Obviously, if ν_e mixes with ν_μ and/or ν_τ, these states will be superpositions of the states of $\bar{\nu}_\mu$ and/or $\bar{\nu}_\tau$.

a baseline roughly of [2] $L \sim 180$ km. Knowing the initial $\bar{\nu}_e$ flux and comparing it with the flux measured at the detector, one can get information about the neutrino oscillation parameters. From the data accumulated in the KamLAND experiment, the following values of the two parameters were obtained [21] (see also [29] and Section 4):

$$|\Delta m^2_{21}| \sim 8 \times 10^{-5}\ \mathrm{eV}^2\,, \quad \sin^2 2\theta_{12} \sim 0.84\,. \tag{2.10}$$

Similar considerations apply to the case of mixing and oscillations between ν_μ ($\bar{\nu}_\mu$) and ν_τ ($\bar{\nu}_\tau$), which is relevant for the interpretation of the Super-Kamiokande experimental results on atmospheric neutrinos [4, 16, 17]. The data is described perfectly well in terms of two-neutrino $\nu_\mu \to \nu_\tau$, $\bar{\nu}_\mu \to \bar{\nu}_\tau$ oscillations with parameters:

$$|\Delta m^2_{31}| \cong 2.2 \times 10^{-3}\ \mathrm{eV}^2\,, \quad \sin^2 2\theta_{23} \cong 1.0\,. \tag{2.11}$$

Finally, in the CHOOZ reactor neutrino experiment with a baseline $L \cong 1$ km, no disappearance of reactor $\bar{\nu}_e$ was observed. For the energies of the reactor $\bar{\nu}_e$, the oscillations due to $|\Delta m^2_{21}| \cong 8 \times 10^{-5}$ eV^2 cannot develop on the distance of 1 km: we have for, e.g $E = 4$ MeV, $2\pi L/L_\nu \cong 0.063 \ll 1$, $\cos 2\pi L/L_\nu \cong 1$, and correspondingly $P(\bar{\nu}_e \to \bar{\nu}_e) \cong 1$. In the range of values of $|\Delta m^2_{31}|$, determined from the atmospheric neutrino oscillation data, $|\Delta m^2_{31}| \sim 2.5 \times 10^{-3}$ eV^2, the following limit on the relevant mixing angle was obtained [31]:

$$\sin^2\theta_{13} < 0.06\,. \tag{2.12}$$

We postpone to Section 4 a more detailed discussion of the ranges of values of neutrino oscillation parameters determined by the current global neutrino oscillation data.

3. Matter Effects in Neutrino Oscillations

The presence of matter can drastically change the pattern of neutrino oscillations: neutrinos can interact with the particles forming the matter. Accordingly, the Hamiltonian of the neutrino system in matter differs from the Hamiltonian of the neutrino system in vacuum H_0,

$$H_m = H_0 + H_{int}\ , \tag{3.1}$$

where H_{int} describes the interaction of neutrinos with the particles of matter. When, e.g., ν_e propagate in matter, they can scatter (due to the H_{int}) on the electrons (e$^-$), protons (p) and neutrons (n) present in matter. The incoherent elastic and the quasi-elastic scattering, in which the states of the initial particles change in the process (destroying the coherence between the neutrino states), are not of interest - they have a negligible effect on the solar neutrino propagation in the Sun and on the solar, atmospheric and reactor neutrino propagation in the Earth [3] : even in the center of the Sun, where the matter density is relatively high (~ 150 g/cm^3), a ν_e with energy of 1 MeV has a mean free path with respect to the indicated scattering processes, which exceeds 10^{10} km. We

[2]The KamLAND detector, which is situated in the Kamioka mine in Japan, actually receives $\bar{\nu}_e$ flux principally from 16 reactors in Japan, located at different distances from the Kamioka mine. The baseline of 180 km we quote represents a mean distance to the reactors contributing to the signal in the KamLAND detector (see [5, 21]).

[3]These processes are important, however, for the supernova neutrinos (see, e.g., [30]).

recall that the solar radius is much smaller: $R_\odot = 6.96 \times 10^5$ km. The oscillating ν_e and ν_μ can scatter also elastically in the forward direction on the e^-, p and n, with the momenta and the spin states of the particles remaining unchanged. In such a process the coherence of the neutrino states is being preserved.

The ν_e and ν_μ coherent elastic scattering on the particles of matter generates nontrivial indices of refraction of the ν_e and ν_μ in matter [11]: $\kappa(\nu_e) \neq 1$, $\kappa(\nu_\mu) \neq 1$. Most importantly, we have $\kappa(\nu_e) \neq \kappa(\nu_\mu)$. The difference $\kappa(\nu_e) - \kappa(\nu_\mu)$ is determined essentially by the difference of the real parts of the forward $\nu_e - e^-$ and $\nu_\mu - e^-$ elastic scattering amplitudes [11] [4] and can be calculated in the Standard Theory. One finds [11, 33, 34]:

$$\kappa(\nu_e) - \kappa(\nu_\mu) = -\frac{1}{p}\sqrt{2} G_F N_e \,, \tag{3.2}$$

where G_F is the Fermi constant and N_e is the e^- number density in matter. Knowing $\kappa(\nu_e) - \kappa(\nu_\mu)$, it is possible to write the system of evolution equations which describes the $\nu_e \leftrightarrow \nu_\mu$ oscillations in matter [11]:

$$i\frac{d}{dt}\begin{pmatrix} A_e(t,t_0) \\ A_\mu(t,t_0) \end{pmatrix} = \begin{pmatrix} -\varepsilon(t) & \varepsilon' \\ \varepsilon' & \varepsilon(t) \end{pmatrix}\begin{pmatrix} A_e(t,t_0) \\ A_\mu(t,t_0) \end{pmatrix} \tag{3.3}$$

where $A_e(t,t_0)$ ($A_\mu(t,t_0)$) is the amplitude of the probability to find neutrino ν_e (ν_μ) at time t of the evolution of the neutrino system if at time t_0 the neutrino ν_e or ν_μ has been produced, $t \geq t_0$, and

$$\varepsilon(t) = \frac{1}{2}\,[\frac{\Delta m^2}{2E}\cos 2\theta - \sqrt{2}G_F N_e(t)],\ \varepsilon' = \frac{\Delta m^2}{4E}\sin 2\theta. \tag{3.4}$$

The term $\sqrt{2}G_F N_e(t)$ in the parameter $\varepsilon(t)$ accounts for the effects of matter on neutrino oscillations. The system of evolution equations describing the oscillations of antineutrinos $\bar{\nu}_e \leftrightarrow \bar{\nu}_\mu$ in matter has exactly the same form except for the matter term in $\varepsilon(t)$ which changes sign.

Consider first the case of $\nu_e \leftrightarrow \nu_\mu$ oscillations in matter with constant density: $N_e(t) = N_e = \ const$. Due to the interaction term H_{int} in H_m, the eigenstates of the Hamiltonian of the neutrino system in vacuum, $|\nu_1\rangle$ and $|\nu_2\rangle$, are not eigenstates of H_m. It proves convenient to find the states $|\nu^m_{1,2}\rangle$, which diagonalize the evolution matrix in the r.h.s. of the system (3.3) or equivalently, the Hamiltonian H_m. We have:

$$\begin{aligned} |\nu_e\rangle &= \ \ |\nu^m_1\rangle \cos\theta_m + |\nu^m_2\rangle \sin\theta_m \,, \\ |\nu_\mu\rangle &= -|\nu^m_1\rangle \sin\theta_m + |\nu^m_2\rangle \cos\theta_m \,. \end{aligned} \tag{3.5}$$

Here θ_m is the neutrino mixing angle in matter [11],

$$\sin 2\theta_m = \frac{\varepsilon'}{\sqrt{\varepsilon^2 + \varepsilon'^2}} = \frac{\tan 2\theta}{\sqrt{(1 - \frac{N_e}{N_e^{res}})^2 + \tan^2 2\theta}}, \tag{3.6}$$

where the quantity

[4] We standardly assume that the weak interaction of the flavour neutrinos ν_e, ν_μ and ν_τ and antineutrinos $\bar{\nu}_e$, $\bar{\nu}_\mu$ and $\bar{\nu}_\tau$ is described by the standard (Glashow-Salam-Weinberg) theory of electroweak interaction (for an alternative possibility see, e.g., [32]). Let us add that the imaginary parts of the forward scattering amplitudes (responsible, in particular, for decoherence effects) are proportional to the corresponding total scattering cross-sections and in the case of interest are negligible in comparison with the real parts.

$$N_e^{res} = \frac{\Delta m^2 \cos 2\theta}{2E\sqrt{2}G_F} \tag{3.7}$$

is called “resonance density” [33]. The matter-eigenstates $|\nu^m_{1,2}\rangle$ (which are also called “adiabatic”) have energies $E^m_{1,2}$ whose difference is given by

$$E_2^m - E_1^m = 2\sqrt{\varepsilon^2 + \varepsilon'^2} = \frac{\Delta m^2}{2E}\ \left((1 - \frac{N_e}{N_e^{res}})^2 \cos^2 2\theta + \sin^2 2\theta\right)^{\frac{1}{2}}. \tag{3.8}$$

It should be clear from (3.5) and (3.8) that the probability of $\nu_e \rightarrow \nu_\mu$ transition in matter with $N_e = const.$ is given by [12]

$$P_m(\nu_e \rightarrow \nu_\mu; t) = |A_\mu(t)|^2 = \frac{1}{2}\sin^2 2\theta_m\ [1 - \cos 2\pi\frac{L}{L_m}\ ]\ , \tag{3.9}$$

where $L_m = (E_2^m - E_1^m)/(2\pi)$ is the oscillation length in matter. As (3.6) indicates, the dependence of the amplitude of $\nu_e \leftrightarrow \nu_\mu$ oscillations in matter, $\sin^2 2\theta_m$, on N_e has a resonance character [12]. Indeed, if $\Delta m^2 \cos^2 2\theta > 0$, for any $\sin^2 2\theta \neq 0$ there exists a value of N_e equal to N_e^{res}, such that

$$\sin^2 2\theta_m = 1\,, \quad for\ N_e = N_e^{res}\ , \tag{3.10}$$

even if the mixing angle in vacuum is small, i.e., if $\sin^2 2\theta \ll 1$. This implies that the presence of matter can lead to a strong enhancement of the oscillation probability $P_m(\nu_e \rightarrow \nu_\mu; t)$ even when the $\nu_e \leftrightarrow \nu_\mu$ oscillations in vacuum are strongly suppressed due to a small value of $\sin^2 2\theta$. For obvious reasons the condition

$$N_e = N_e^{res} = \frac{\Delta m^2 \cos 2\theta}{2E\sqrt{2}G_F}\,, \tag{3.11}$$

is called “resonance condition”, while the energy at which (3.11) holds for given N_e, Δm^2 and $\cos 2\theta$, is referred to as “resonance energy”, E^{res}.

The oscillation length at resonance is given by [12] $L_m^{res} = L_\nu / \sin 2\theta$, while the width in N_e of the resonance (i.e., the “distance” in N_e between the points at which $\sin^2 2\theta_m = 1/2$) reads $\Delta N_e^{res} = 2N_e^{res} \tan 2\theta$. Thus, if the mixing angle in vacuum is small the resonance is narrow, $\Delta N_e^{res} \ll N_e^{res}$, and L_m at resonance is relatively large, $L_m^{res} \gg L_\nu$. As it follows from (3.8), the energy difference $E_2^m - E_1^m$ has a minimum at the resonance: $(E_2^m - E_1^m)^{res} = \min\,(E_2^m - E_1^m) = (\Delta m^2/(2E))\sin 2\theta$.

It is instructive to consider two limiting case. If $N_e \ll N_e^{res}$, as it follows from (3.6) and (3.8), $\theta_m \cong \theta$, $\mathrm{L_m} \cong \mathrm{L_v}$ and the neutrinos oscillate practically as in vacuum. In the opposite limit, $N_e \gg N_e^{res}$, $N_e^{res}\tan^2 2\theta$, $\theta_m \cong \pi/2$ ($\cos 2\theta_m \cong -1$) and the presence of matter suppresses the $\nu_e \leftrightarrow \nu_\mu$ oscillations. In this case we get from (3.5) and (3.6): $|\nu_e\rangle \cong |\nu_2^m\rangle$, $|\nu_\mu\rangle = -|\nu_1^m\rangle$, i.e., ν_e practically coincides with the heavier of the two matter-eigenstate ν_2^m, while the ν_μ coincides with the lighter one ν_1^m.

Since the neutral current weak interaction of neutrinos in the Standard Theory is flavour symmetric, the formulae and results we have obtained are valid for the case of $\nu_e - \nu_\tau$ mixing and $\nu_e \leftrightarrow \nu_\tau$ oscillations in matter as well. The case of $\nu_\mu - \nu_\tau$ mixing, however, is different. It is possible to show that to a relatively good precision we have for the ν_μ and ν_τ indexes of refraction

$\kappa(\nu_\mu) \cong \kappa(\nu_\tau)$. As a consequence, the $\nu_\mu \leftrightarrow \nu_\tau$ oscillations in matter (e.g., in the Earth) proceed as in vacuum [5].

The analogs of eqs. (3.6) - (3.9) for oscillations of antineutrinos, $\bar{\nu}_e \leftrightarrow \bar{\nu}_\mu$, in matter can formally be obtained by replacing N_e with $(-N_e)$ in the indicated equations. It should be clear that depending on the sign of $\Delta m^2 \cos 2\theta$, the presence of matter can lead to resonance enhancement either of the $\nu_e \leftrightarrow \nu_\mu$ or of the $\bar{\nu}_e \leftrightarrow \bar{\nu}_\mu$ oscillations, but not of the both types of oscillations. This is a consequence of the fact that the matter in the Sun or in the Earth we are interested in, is not charge-symmetric (it contains e^-, p and n, but does not contain their antiparticles) and therefore the oscillations in matter are neither CP- nor CPT- invariant [35] [6].

The formalism we have developed can be applied, e.g., to the study of the matter effects in the $\nu_e \leftrightarrow \nu_{\mu(\tau)}$ ($\nu_{\mu(\tau)} \leftrightarrow \nu_e$) oscillations of neutrinos which traverse the Earth mantle (but do not traverse the Earth core). Indeed, the Earth density distribution in the existing Earth models [38] is assumed to be spherically symmetric and there are two major density structures - the core and the mantle, and a certain number of substructures (shells or layers). The Earth radius is 6371 km; the Earth core has a radius of 3486 km, so the Earth mantle depth is 2885 km. The mean electron number densities in the mantle and in the core read [38]: $\bar{N}_e^{man} \cong 2.2\ N_A cm^{-3}$, $\bar{N}_e^c \cong 5.4\ N_A cm^{-3}$, m_N and N_A being the nucleon mass and Avogadro number [7] The electron number density N_e changes relatively little around the mean values of $\bar{N}_e \cong 2.3\ cm^{-3}\ N_A$ and $\bar{N}_e^c \cong 5.4\ N_A cm^{-3}$, along the trajectories of neutrinos which cross a substantial part of the Earth mantle, or the mantle and the core, and the $N_e = const.$ approximation was shown to be remarkably accurate in what concerns the calculation of ν-oscillation probabilities [36, 39]. This is related to the fact that the changes of density along the path of the neutrinos in the mantle (or in the core) take place over path lengths which are typically considerably smaller than the corresponding oscillation length in matter. If, for example, $\Delta m^2 = 10^{-3}$ eV^2, $E = 1$ GeV and $\sin^2 2\theta \cong 0.5$, we have: $N_e^{res} \cong 4.6\ cm^{-3}\ N_A$, $\sin^2 2\theta_m \cong 0.8$ and the oscillation length in matter, $L_m \cong 3 \times 10^3$ km, is of the order of the depth of the Earth mantle.

In the case of neutrinos crossing the Earth core, new resonance-like effects become apparent. For $\sin^2\theta < 0.05$ and $\Delta m^2 > 0$, we can have $P_m^{2\nu}(\nu_e \to \nu_\mu) = P_m^{2\nu}(\nu_\mu \to \nu_e) \equiv P_m^{2\nu}(\Delta m^2, \theta) \cong 1$ *only due to the effect of maximal constructive interference between the amplitudes of the* $\nu_e \to \nu_\mu$ *transitions in the Earth mantle and in the Earth core* [39, 40]. The effect differs from the MSW one [39] and the enhancement happens in the case of interest at a value of the energy between the resonance energies corresponding to the density in the mantle and that of the core. The *mantle-core enhancement effect* is caused by the existence (for a given ν-trajectory through the Earth core) of *points of resonance-like total neutrino conversion*, $P_m^{2\nu}(\Delta m^2, \theta) = 1$ in the corresponding space of ν-oscillation parameters [40]. The points where $P_m^{2\nu}(\Delta m^2, \theta) = 1$ are determined by the

[5]In what concerns the possibility of mixing and oscillations between the ν_e and a sterile neutrino ν_s, $\nu_e \leftrightarrow \nu_s$, the relevant formulae can be obtained from the formulae derived for the case of $\nu_e \leftrightarrow \nu_{\mu(\tau)}$ oscillations by [35] replacing N_e with $(N_e - 1/2N_n)$, where N_n is the number density of neutrons in matter.

[6]The matter effects in the $\nu_e \leftrightarrow \nu_\mu$ ($\bar{\nu}_e \leftrightarrow \bar{\nu}_\mu$) oscillations will be invariant with respect to the operation of time reversal if the N_e distribution along the neutrino path is symmetric with respect to this operation. The latter condition is fulfilled for the N_e distribution along a path of a neutrino crossing the Earth [36, 37].

[7]The change of N_e from the mantle to the core can be well approximated by a step function [38].

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conditions [40]:

$$\tan\phi' \pm \sqrt{\frac{-\cos 2\theta''_m}{\cos(2\theta''_m - 4\theta'_m)}},\ \tan\phi'' = \pm\frac{\cos 2\theta'_m}{\sqrt{-\cos 2\theta''_m \cos(2\theta''_m - 4\theta'_m)}} \tag{3.12}$$

where the signs are correlated and $\cos 2\theta''_m \cos(2\theta''_m - 4\theta'_m) \leq 0$. In eq. (3.12) $2\phi' = (E^{m,m}_2 - E^{m,m}_1)L^{man}$ and $2\phi'' = (E^{m,c}_2 - E^{m,c}_1)L^{core}$ are the oscillation phases (phase differences) accumulated by the (two) neutrino states after crossing respectively the first mantle layer and the core, $E^{m,m}_{1,2}$ ($E^{m,c}_{1,2}$) and L^{man} (L^{core}) being the energies of the two states and the neutrino path length in the mantle layer (core), and θ'_m and θ''_m are the ν-mixing angles in the mantle and in the core. For $\Delta m^2 < 0$ the mantle-core enhancement can take place for the antineutrino transitions, $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ and $\bar{\nu}_e \rightarrow \bar{\nu}_\mu$. A rather complete set of values of $\Delta m^2/E > 0$ and $\sin^2 2\theta$ for which both conditions in eq. (3.12) hold and $P^{2\nu}_m(\Delta m^2, \theta) = 1$ was found in [40]. The location of these points determines the regions where $P^{2\nu}_m(\Delta m^2, \theta)$ is large, $P^{2\nu}_m(\Delta m^2, \theta) \gtrsim 0.5$. For $\sin^2\theta < 0.05$, there are two sets of values of Δm^2 and $\sin^2\theta$ for which eq. (3.12) is fulfilled and $P^{2\nu}_m(\Delta m^2, \theta) = 1$. These two solutions of eq. (3.12) occur for, e.g., values of the Nadir angle $\theta_n = 0;\ 13^0; 23^0$, at 1) $\sin^2 2\theta = 0.034;\ 0.039;\ 0.051$, and at 2) $\sin^2 2\theta = 0.15;\ 0.17;\ 0.22$ (see Table 2 in the last article quoted in [40]). For $\Delta m^2 = 2.0\ (3.0) \times 10^{-3}$ eV^2, for instance, $P^{2\nu}_m(\Delta m^2, \theta) = 1$ occurs in the case of the first solution [8] at $E \cong (2.8 - 3.1)$ GeV ($E \cong (4.2 - 4.7)$ GeV).

The effects of the mantle-core enhancement of $P^{2\nu}_m(\nu_e \rightarrow \nu_\mu) = P^{2\nu}_m(\nu_\mu \rightarrow \nu_e) \equiv P^{2\nu}_m(\Delta m^2, \theta)$ are relevant, in particular, for the searches of subdominant $\nu_{e(\mu)} \rightarrow \nu_{\mu(e)}$ oscillations of atmospheric neutrinos (see, e.g., [39, 41, 42]). In the case of three neutrino mixing, for which we have compelling experimental evidences (see Section 4), and energies of the atmospheric neutrinos crossing the Earth core $E \gtrsim 2$ GeV, the $\nu_{e(\mu)} \rightarrow \nu_{\mu(e)}$ transition probabilities of interest, $P^{3\nu}_m(\nu_e \rightarrow \nu_\mu) = P^{3\nu}_m(\nu_\mu \rightarrow \nu_e)$, are simply related to the two-neutrino transition probabilities discussed above [43] (see also [41]): $P^{3\nu}_m(\nu_e \rightarrow \nu_\mu) = P^{3\nu}_m(\nu_\mu \rightarrow \nu_e) \cong \sin^2\theta_{23} P^{2\nu}_m(\Delta m^2_{31}, \theta_{13})$, where θ_{23} and Δm^2_{31} are the atmospheric neutrino mixing angle and neutrino mass squared difference, responsible for the dominant $\nu_\mu \rightarrow \nu_\tau$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_\tau$ oscillations of atmospheric neutrinos, and θ_{13} is the CHOOZ angle (see eqs. (2.11) and (2.12)).

4. Oscillations of Solar Neutrinos

Consider next the oscillations of solar ν_e while they propagate from the central part, where they are produced [44], to the surface of the Sun. For details concerning the production, spectrum, magnitude and particularities of the solar neutrino flux, the methods of detection of solar neutrinos, description of solar neutrino experiments and of the data they provided we refer the reader to [44, 14, 24]. The electron number density N_e changes considerably along the neutrino path in the Sun: it decreases monotonically from the value of ~ 100 cm^{-3} N_A in the center of the Sun to 0 at the surface of the Sun. According to the contemporary solar models (see, e.g., [44, 45]), N_e decreases approximately exponentially in the radial direction towards the surface of the Sun:

$$N_e(t) = N_e(t_0) \exp\left\{-\frac{t - t_0}{r_0}\right\}, \tag{4.1}$$

[8] The first solution corresponds to $\cos 2\phi' \cong -1$, $\cos 2\phi'' \cong -1$ and $\sin^2(2\theta''_m - 4\theta'_m) = 1$. The enhancement effect in this case was called "neutrino oscillation length resonance" (NOLR) in [39].

where $(t-t_0) \cong d$ is the distance traveled by the neutrino in the Sun, $N_e(t_0)$ is the electron number density in the point of ν_e production in the Sun, r_0 is the scale-height of the change of $N_e(t)$ and one has [45] $r_0 \sim 0.1R_\odot$.

The system of evolution equations (3.3) does not admit, in general, exact solutions. However, there are few notable exceptions in which the evolution equations can be solved exactly (see, e.g., [46, 47]). Remarkably, these include the case of exponentially varying N_e [48, 49], eq. (4.1), relevant for the description of the solar neutrino oscillations in the Sun. Perhaps even more remarkable is the fact that [50] the system of evolution equations (3.3), with N_e given by eq. (4.1), describing the solar neutrino oscillations in the Sun, is equivalent to a second order differential equation - the confluent hypergeometric equation [51], which coincides in form with the Schrödinger (energy eigenvalue) equation obeyed by the radial part of the non-relativistic wave function of the hydrogen atom [52]. On the basis of the corresponding exact solutions expressed in terms of confluent hypergeometric functions, using the asymptotic series expansions of the latter [51], a simple expression for the solar neutrino survival probability, $P_\odot(\nu_e \to \nu_e)$, containing only elementary functions, has been derived [48, 53] (see also [54]). It was also demonstrated that the expression for $P_\odot(\nu_e \to \nu_e)$ thus found provides a very precise (and actually, the most precise) analytic description of the MSW oscillations and transitions of the solar neutrinos in the Sun [55, 56, 57]. The expression of interest for $P_\odot(\nu_e \to \nu_e)$ has the form [48, 53]:

$$P_\odot(\nu_e \to \nu_e) = \bar{P_\odot} + P_1^{osc}, \tag{4.2}$$

where $\bar{P_\odot}$ is the average probability of solar ν_e survival,

$$\bar{P_\odot} = \frac{1}{2} + \left(\frac{1}{2} - \mathrm{P}_c\right)\cos 2\theta_m^0 \cos 2\theta, \tag{4.3}$$

and P_1^{osc} is an oscillating term

$$P_1^{osc} = -\sqrt{P_c(1-P_c)}\cos 2\theta_m^0 \sin 2\theta \cos(\Phi_{21} - \Phi_{22})\ . \tag{4.4}$$

In eqs. (4.3) and (4.4)

$$P_c = \frac{\exp\left[-2\pi \mathrm{r}_0 \frac{\Delta m^2}{2E}\sin^2\theta\right] - \exp\left[-2\pi \mathrm{r}_0 \frac{\Delta m^2}{2E}\right]}{1 - \exp\left[-2\pi \mathrm{r}_0 \frac{\Delta m^2}{2E}\right]} \tag{4.5}$$

is [48] the "jump" or "level-crossing" probability for exponentially varying electron number density N_e [9], and θ_m^0 is the neutrino mixing angle in matter [11] in the point of ν_e production in the Sun. The phases Φ_{21} and Φ_{22} in the oscillating term, eq. (4.4), have a simple physical interpretation [53, 50]. In the exponential density approximation one finds [53]:

$$\begin{aligned}\Phi_{21} - \Phi_{22} \ &= \ -2\arg\Gamma(1-c) - \arg\Gamma(a-1) + \arg\Gamma(a-c) \\ &\quad -r_0\frac{\Delta m^2}{2E}\ln[r_0\sqrt{2}G_F N_e(x_0)] + \frac{\Delta m^2}{2E}(L - x_0)\end{aligned} \tag{4.6}$$

[9] An expression for the "jump" probability corresponding to the case of density (N_e) varying linearly along the neutrino path was derived a long time ago by Landau and Zener [58]. An analytic description of the average probability of solar neutrino transitions based on the linear approximation for the change of N_e in the Sun and on the Landau-Zener result was proposed in [59]. The drawbacks of this description, which in certain cases (e.g., non-adiabatic transitions with relatively large $\sin^2 2\theta$) is considerably less accurate [55] than the description based on the results obtained in the exponential density approximation, were discussed in [46, 48, 55].

where $a = 1 + ir_0 \Delta m^2/(2E)\sin^2\theta$, $c = 1 + ir_0\Delta m^2/(2E)$, $\Gamma(y)$ is the Gamma function and $L = 1$ A.U. The part of the phase $(\Phi_{21} - \Phi_{22})$ given by $\Delta m^2(L - R_\odot)/(2E)$, is accumulated on the path of neutrinos in vacuum from the solar surface to the surface of the Earth; the rest is generated in the Sun. Numerical studies have shown that $(\Phi_{21} - \Phi_{22})$ does not depend on the value of $N_e(x_0)$, i.e., on the point of ν_e production in the Sun [57].

Few comments are in order. Both eqs. (4.5) and (4.6) are valid for any value of Δm^2 (or $\Delta m^2/(2E)$) and for any θ, including $\theta \geq \pi/4$ [53]. The solar neutrino transitions are called "adiabatic" [12] if $P_c \cong 0$; otherwise they are called "non-adiabatic" [10]. As was shown in [54], the oscillating term P_1 can be relevant in the solar neutrino transitions, i.e., can give a non-negligible contribution in $P_\odot(\nu_e \to \nu_e)$, only for $\Delta m^2/(2E) \lesssim 10^{-8}$ eV^2/MeV: at $\Delta m^2/(2E) \gtrsim 5\times 10^{-8}$ eV^2/MeV we have effectively $P_\odot(\nu_e \to \nu_e) \cong \bar{P}_\odot$. In the latter case one speaks about solar neutrino transitions. At $\Delta m^2/(2E) \lesssim 10^{-8}$ eV^2/MeV a very precise and easy to use expression for the phase $(\Phi_{21} - \Phi_{22})$ was found in [57]:

$$\Phi_{21} - \Phi_{22} \cong 0.130\,(\frac{\Delta m^2}{2E}R_\odot) + 1.67\times 10^{-3}(\frac{\Delta m^2}{2E}R_\odot)^2\cos 2\theta + \frac{\Delta m^2}{2E}(L - R_\odot). \tag{4.7}$$

The effects of solar matter in the $\nu_e \to \nu_{\mu(\tau)}$ oscillations or transitions of solar neutrinos become negligible at sufficiently large [12] and sufficiently small [48, 53, 54] Δm^2. For solar neutrinos we have at $\Delta m^2 \gtrsim 6\times 10^{-4}$ eV^2: $P_c \cong 0$, $P_1 \cong 0$, $\cos 2\theta_m^0 \cong \cos 2\theta$, and $P_\odot(\nu_e \to \nu_e) \cong 1 - 1/2\ \sin^2 2\theta$, which coincides with the average probability of survival of ν_e when the oscillations take place in vacuum. At $\Delta m^2 \lesssim 5\times 10^{-10}$ eV^2 one finds [48, 53, 54] $P_c \cong \cos^2\theta$, $\cos 2\theta_m^0 \cong -1$, $(\Phi_{21} - \Phi_{22}) \cong \Delta m^2(L - x_0)/(2E)$, and correspondingly $P_\odot(\nu_e \to \nu_e) \cong 1 - 1/2\ \sin^2 2\theta\ [1 - \cos(\Delta m^2(L - x_0)/(2E))]$, i.e., the solar neutrinos oscillate as in vacuum. For 5×10^{-10} eV^2 $\lesssim \Delta m^2 \lesssim 2\times 10^{-8}$ eV^2 the solar matter effects are still not negligible and solar neutrinos take part is the so-called "quasi-vacuum oscillations (QVO)". Finally, for $\sin^2\theta \sim 0.3$ of interest for the description of the solar neutrino data (see further), we have $P_c \cong 0$, $P_1 \cong 0$, $\cos 2\theta_m^0 \cong -1$ and correspondingly $P_\odot(\nu_e \to \nu_e) \cong \sin^2\theta$, approximately for $\Delta m^2/(2E) \sim (10^{-6} - 5\times 10^{-8})$ eV^2/MeV. The analytic expression for $P_\odot(\nu_e \to \nu_e)$ given by eqs. (4.2) - (4.6) and (4.7) provides a very precise analytic description of the solar ν_e oscillations/transitions [55, 57].

Let us note that the solar neutrino energies relevant for the interpretation of the results of the solar neutrino experiments lie in the interval $E \cong (0.233 - 14.4)$ MeV. As we shall see, the neutrino mass squared difference responsible for the solar neutrino oscillations is constrained by the data to be in the range $\Delta m^2_\odot = \Delta m^2_{21} \cong (7.0 - 9.0)\times 10^{-5}$ eV^2. Under these conditions we have $P_1 \cong 0$ and $P_c \cong 0$. The SNO experiment is sensitive to solar ν_e neutrinos with energies $E \gtrsim 6.5$ MeV. Thus, for $E \gtrsim 10$ MeV, the solar neutrino survival probability relevant for the interpretation of the data from SNO experiment is given by $P_\odot(\nu_e \to \nu_e) \cong \sin^2\theta$. This allows, in particular, a direct determination of the solar neutrino mixing parameter $\sin^2\theta$ from the SNO data.

5. Determining the Neutrino Mixing Parameters

The formalism of neutrino oscillations in vacuum and in matter we have developed is used in the analyses of the neutrino oscillation data provided by the solar, atmospheric and reactor neutrino experiments as well as by the experiments with accelerator neutrinos.

[10] For a more rigorous definition of the adiabatic and non-adiabatic neutrino transitions see [60, 55, 57].

The Super-Kamiokande atmospheric neutrino data and the K2K and MINOS data are well described in terms of (dominant) 2-neutrino $\nu_\mu \to \nu_\tau$ ($\bar{\nu}_\mu \to \bar{\nu}_\tau$) vacuum oscillations (see [16, 18,

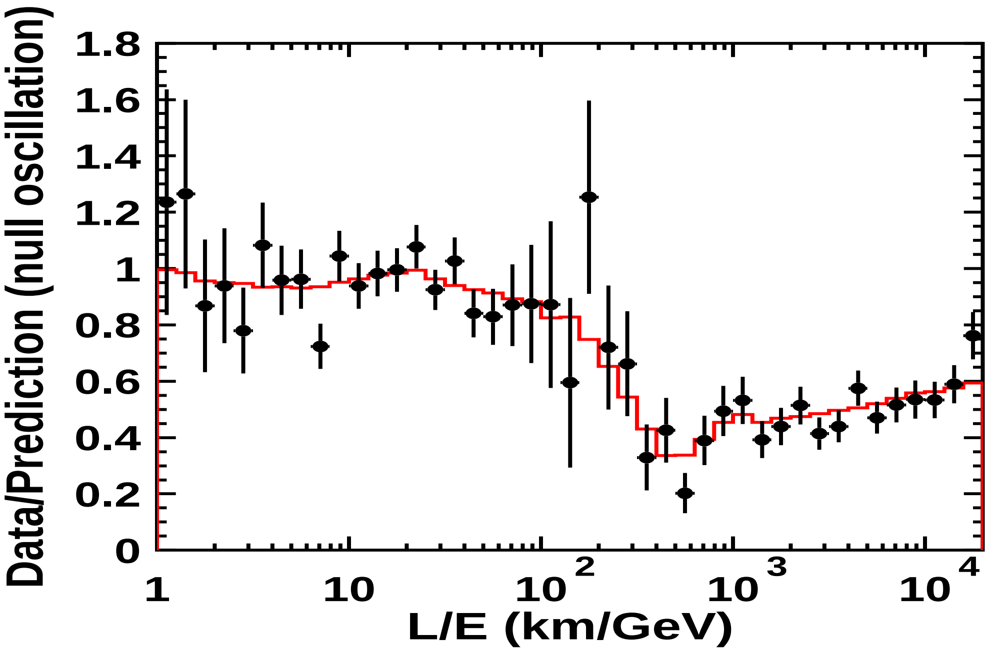

Figure 2: The L/E dependence of the μ-like atmospheric neutrino event rate observed in the Super-Kamiokande experiment [16].

19]). The corresponding $\nu_\mu \to \nu_\tau$ oscillation probability is given by:

$$P(\nu_\mu \to \nu_\mu) \cong 1 - \sin^2 2\theta_{\rm A} \sin^2 \frac{\Delta m^2_{\rm A} {\rm L}}{4{\rm E}},$$
$$P(\nu_\mu \to \nu_\mu) = 1 - P(\nu_\mu \to \nu_\tau) = P(\bar{\nu}_\mu \to \bar{\nu}_\mu) = 1 - P(\bar{\nu}_\mu \to \bar{\nu}_\tau)\ . \tag{5.1}$$

The best fit values and the 99.73% C.L. allowed ranges of the atmospheric neutrino ($\nu_{\rm A}$-) oscillation parameters read [61]:

$$|\Delta m^2_{\rm A}| = 2.5 \times 10^{-3}\ {\rm eV}^2\ ,\quad \sin^2 2\theta_{\rm A} = 1.0\ ,$$
$$|\Delta m^2_{\rm A}| = (1.9 - 3.2) \times 10^{-3}\ {\rm eV}^2\ ,\quad \sin^2 2\theta_{\rm A} \geq 0.87\ . \tag{5.2}$$

The sign of $\Delta m^2_{\rm A}$ and of $\cos 2\theta_{\rm A}$, if $\sin^2 2\theta_{\rm A} \neq 1.0$, cannot be determined using the existing data. The latter implies that when, e.g., $\sin^2 2\theta_{\rm A} = 0.92$, one has $\sin^2 \theta_{\rm A} \cong 0.64$ or 0.36.

In ref. [16] SK collaboration presented the first evidence for an "oscillation dip" in the $L/E-$dependence, L and E being the distance traveled by neutrinos and the neutrino energy, of a particularly selected sample of (essentially milti-GeV) $\mu-$like events [11]. Such a dip is predicted due to the oscillatory dependence of the $\nu_\mu \to \nu_\tau$ ($\bar{\nu}_\mu \to \bar{\nu}_\tau$) oscillation probability on Ł$/E$: the $\nu_\mu \to \nu_\tau$ ($\bar{\nu}_\mu \to \bar{\nu}_\tau$) transitions of atmospheric neutrinos are predominantly two-neutrino transitions governed by vacuum oscillation probability. The dip in the observed L/E distribution corresponds

[11]These are $\mu-$like events for which the relative uncertainty in the experimental determination of the L/E ratio does not exceed 70%.

to the first oscillation minimum of the ν_μ ($\bar{\nu}_\mu$) survival probability, $P(\nu_\mu \to \nu_\mu)$ ($P(\bar{\nu}_\mu \to \bar{\nu}_\mu)$), as L/E increases starting from values for which $|\Delta m^2_{\rm A}|L/(2E) \ll 1$ and $P(\nu_\mu \to \nu_\mu) \cong 1$. This beautiful result represents the first ever observation of a direct effect of the oscillatory dependence on L and E of the probability of neutrino oscillations in vacuum.

The combined neutrino oscillation analysis of the solar neutrino and the KamLAND data shows [21, 29, 62] that the $\nu_\odot$-oscillation parameters lie in the so-called "low-LMA" region. The best fit values and the 99.73% C.L. allowed ranges of values of $\Delta m^2_\odot$ and $\sin^2\theta_\odot$ read:

$$\begin{aligned} \Delta m^2_\odot &= 8.0 \times 10^{-5}~{\rm eV}^2~, \quad \sin^2\theta_\odot = 0.30~, \\ \Delta m^2_\odot &= (7.1 - 8.9) \times 10^{-5}~{\rm eV}^2~, \quad \sin^2\theta_\odot = (0.24 - 0.40)~. \end{aligned} \tag{5.3}$$

The value of $\Delta m^2_\odot$ is determined with a remarkably high precision. Maximal $\nu_\odot$-mixing is ruled out at $\sim 6\sigma$ [29, 61]; at 95% C.L., $\cos 2\theta_\odot \geq 0.28$. One also has: $\Delta m^2_\odot / |\Delta m^2_{\rm A}| \sim 0.03 \ll 1$.

The interpretation of the solar and atmospheric neutrino, K2K, KamLAND and MINOS data in terms of ν-oscillations requires the existence of 3-ν mixing in the weak charged lepton current:

$$\nu_{l{\rm L}} = \sum_{j=1}^{3} U_{lj}\, \nu_{j{\rm L}}, \quad l = e, \mu, \tau, \tag{5.4}$$

where ν_{lL} are the flavour neutrino fields, $\nu_{j{\rm L}}$ is the left-handed field of neutrino ν_j having a mass m_j and U is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) ν-mixing matrix [6, 7]. All existing ν-oscillation data, except the data of LSND experiment [12] [20], can be described assuming 3-ν mixing in vacuum and we will consider only this possibility. The minimal 4-ν mixing scheme which could incorporate the LSND indications for ν-oscillations is strongly disfavored by the data [64]. The ν-oscillation explanation of the LSND results is possible assuming 5-ν mixing [65].

The PMNS matrix can be parametrized by 3 angles and, depending on whether the massive neutrinos ν_j are Dirac or Majorana particles, by 1 or 3 CP-violation (*CPV*) phases [66, 67]. In the standard parameterization [86]

$$U_{\rm PMNS} = V(\theta_{12}, \theta_{13}, \theta_{23}, \delta)\ {\rm diag}(1, e^{i\alpha}, e^{i\beta}),$$

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13}e^{i\delta} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}e^{i\delta} \end{pmatrix}, \tag{5.5}$$

where $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$, the angles $\theta_{ij} = [0, \pi/2]$, $\delta = [0, 2\pi]$ is the Dirac *CPV* phase and α, β are two Majorana *CPV* phases [66, 67]. One can identify $\Delta m^2_\odot = \Delta m^2_{21} > 0$. In this case $|\Delta m^2_{\rm A}|$=$|\Delta m^2_{31}| \cong |\Delta m^2_{32}|$, $\theta_{12} = \theta_\odot$, $\theta_{23} = \theta_{\rm A}$. The angle θ_{13} is limited by the data from the CHOOZ experiment [31]. The existing $\nu_{\rm A}$-data is essentially insensitive to θ_{13} obeying the CHOOZ limit [17]. The probabilities of survival of reactor $\bar{\nu}_e$ and solar ν_e, relevant for the interpretation of the

[12] In the LSND experiment indications for oscillations $\bar{\nu}_\mu \to \bar{\nu}_e$ with $(\Delta m^2)_{\rm LSND} \simeq 1~{\rm eV}^2$ were obtained. The LSND results are being tested in the MiniBooNE experiment [63].

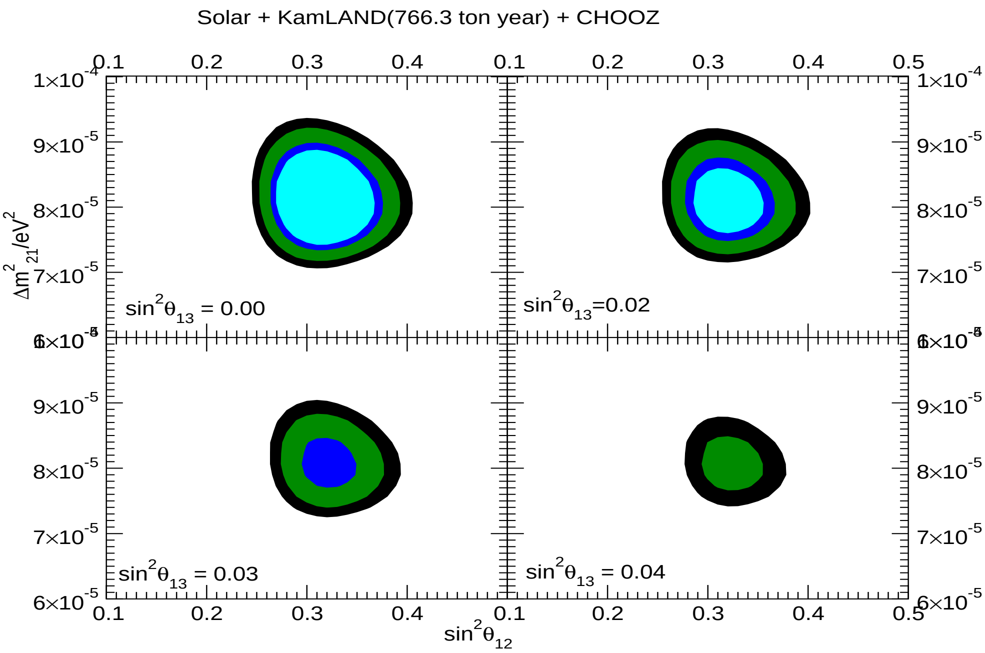

Figure 3: The 90%, 95%, 99% and 99.73% C.L. allowed regions in the Δm^2_{21} -$\sin^2\theta_{12}$ plane, obtained in a three-neutrino oscillation analysis of the solar neutrino, KamLAND and CHOOZ data [29].

KL, CHOOZ and $\nu_{\odot}$- data, depend on θ_{13} (see, e.g., [69]):

$$P^{3\nu}_{\rm KL} \cong \sin^4\theta_{13} + \cos^4\theta_{13}\left[1 - \sin^2 2\theta_{12}\sin^2\frac{\Delta m^2_{21}{\rm L}}{4E}\right],$$
$$P^{3\nu}_{\rm CHOOZ} \cong 1 - \sin^2 2\theta_{13}\sin^2\frac{\Delta m^2_{31}{\rm L}}{4{\rm E}}, \qquad (5.6)$$
$$P^{3\nu}_{\odot} \cong \sin^4\theta_{13} + \cos^4\theta_{13}\, P^{2\nu}_{\odot}(\Delta m^2_{21}, \theta_{12}; \theta_{13}),$$

where $P^{2\nu}_{\odot}$ is the 2-ν mixing solar ν_e survival probability, eq. (4.2), in the case of transitions driven by Δm^2_{21} and θ_{12}, in which (the solar e^- number density) N_e is replaced by $N_e\cos^2\theta_{13}$ [43], $P^{2\nu}_{\odot} = \bar{P}_{\odot} + P^{osc}_{1}$ (see eqs. (4.3) and (4.4)). In the LMA solution region one has [54] $P^{2\nu}_{\odot\ {\rm osc}} \cong 0$. Using the existing atmospheric and solar neutrino, CHOOZ and KamLAND data, one finds [29, 61]:

$$\sin^2\theta_{13} < 0.041, \quad 99.73\%\ {\rm C.L.} \qquad (5.7)$$

In Fig. 3 we show the allowed regions in the $\Delta m^2_{21} - \sin^2\theta_{12}$ plane for few fixed values of $\sin^2\theta_{13}$.

Thus, the fundamental parameters characterizing the 3-neutrino mixing are: i) the 3 angles θ_{12}, θ_{23}, θ_{13}, ii) depending on the nature of ν_j - 1 Dirac (δ), or 1 Dirac + 2 Majorana (δ, α, β), *CPV* phases, and iii) the 3 neutrino masses, m_1, m_2, m_3. It is convenient to express the two larger masses in terms of the third mass and the measured $\Delta m^2_{\odot} = \Delta m^2_{21} > 0$ and $\Delta m^2_{\rm A}$. In the convention we are using, the two possible signs of $\Delta m^2_{\rm A}$ correspond to two types of ν-mass spectrum:

- with normal hierarchy, $m_1 < m_2 < m_3$,
  $\Delta m^2_{\rm A} = \Delta m^2_{31} > 0$, $m_{2(3)} = (m_1^2 + \Delta m^2_{21(31)})^{\frac{1}{2}}$, and
- with inverted hierarchy, $m_3 < m_1 < m_2$,
  $\Delta m^2_{\rm A} = \Delta m^2_{32} < 0$, $m_2 = (m_3^2 - \Delta m^2_{32})^{\frac{1}{2}}$, etc.

The spectrum can also be

• *normal hierarchical (NH)*: $m_1 \ll m_2 \ll m_3$,
$m_2 \cong (\Delta m^2_{\odot})^{\frac{1}{2}} \sim$0.009 eV, $m_3 \cong |\Delta m^2_{\rm A}|^{\frac{1}{2}} \sim$0.05; or
• *inverted hierarchical (IH)*: $m_3 \ll m_1 < m_2$,
with $m_{1,2} \cong |\Delta m^2_{\rm A}|^{\frac{1}{2}} \sim$0.05 eV; or
• *quasi-degenerate (QD)*: $m_1 \cong m_2 \cong m_3 \cong m_0$, $m_j^2 \gg |\Delta m^2_{\rm A}|$. In this case one has $m_0 \gtrsim 0.10$ eV.

As is well-known, neutrino oscillations are not sensitive to the absolute scale of neutrino masses. Information on the absolute neutrino mass scale can be derived in ^{3}H β-decay experiments [70, 71, 72] and from cosmological and astrophysical data. The most stringent upper bounds on the $\bar{\nu}_e$ mass were obtained in the Troitzk [71] and Mainz [72] experiments:

$$m_{\bar{\nu}_e} < 2.3\text{eV} \quad \text{at 95\% C.L.} \tag{5.8}$$

We have $m_{\bar{\nu}_e} \cong m_{1,2,3}$ in the case of the QD ν-mass spectrum. The KATRIN experiment [72] is planned to reach a sensitivity of $m_{\bar{\nu}_e} \sim 0.20$ eV, i.e. it will probe the region of the QD spectrum. The CMB data of the WMAP experiment [73], combined with data from large scale structure surveys (2dFGRS, SDSS), lead to the following upper limit on the sum of neutrino masses (see, e.g. [74]):

$$\sum_j m_j \equiv \Sigma < (0.4\text{–}1.7)\ \text{eV} \quad \text{at 95\% C.L.} \tag{5.9}$$

Data on weak lensing of galaxies, combined with data from the WMAP and PLANCK experiments, may allow Σ to be determined with an uncertainty of $\xi \sim 0.04$ eV [74, 75].

The type of neutrino mass spectrum, i.e. sgn$(\Delta m^2_{\rm A})$, can be determined by studying oscillations of neutrinos and antineutrinos, say, $\nu_\mu \leftrightarrow \nu_e$ and $\bar{\nu}_\mu \leftrightarrow \bar{\nu}_e$, in which matter effects are sufficiently large. This can be done in long base-line ν-oscillation experiments [76]. If $\sin^2 2\theta_{13} \gtrsim 0.05$ and $\sin^2\theta_{23} \gtrsim 0.50$, information on sgn$(\Delta m^2_{\rm A})$ might be obtained in atmospheric neutrino experiments by investigating the effects of the subdominant transitions $\nu_{\mu(e)} \rightarrow \nu_{e(\mu)}$ and $\bar{\nu}_{\mu(e)} \rightarrow \bar{\nu}_{e(\mu)}$ of atmospheric neutrinos which traverse the Earth [41, 42]. For $\nu_{\mu(e)}$ (or $\bar{\nu}_{\mu(e)}$) crossing the Earth core, the corresponding $\nu_{\mu(e)}$ (or $\bar{\nu}_{\mu(e)}$) transition probabilities can be maximal [40] due to the mantle-core enhancement effect (neutrino oscillation length resonance) [39], discussed in Section 3. For $\Delta m^2_{\rm A} > 0$, the neutrino transitions $\nu_{\mu(e)} \rightarrow \nu_{e(\mu)}$ are enhanced, while for $\Delta m^2_{\rm A} < 0$, the enhancement of antineutrino transitions $\bar{\nu}_{\mu(e)} \rightarrow \bar{\nu}_{e(\mu)}$ takes place, which might allow to determine sgn$(\Delta m^2_{\rm A})$.

If $\sin^2\theta_{13}$ is sufficiently large, information about sgn$(\Delta m^2_{\rm A})$ can be obtained by studying the oscillations of reactor $\bar{\nu}_e$ on distances of $\sim (30-50)$ km [77]. An experiment with reactor $\bar{\nu}_e$, which might have the capability to determine sgn$(\Delta m^2_{\rm A})$, was proposed recently in [78].

6. Outlook

After the spectacular experimental progress made in the studies of neutrino oscillations, further understanding of the structure of neutrino masses and neutrino mixing, of their origins and of the status of CP-symmetry in the lepton sector requires an extensive and challenging program of research in neutrino physics. The main goals of this research program should include [69]:
• High precision measurement of the solar and atmospheric neutrino oscillations parameters, Δm^2_{21}, θ_{21}, and Δm^2_{31}, θ_{23}.
• Measurement of, or improving by at least a factor of (5 - 10) the existing upper limit on, θ_{13} - the only small mixing angle in $U_{\rm PMNS}$. Together with the Dirac CP-violating phase, the angle θ_{13}

determines the magnitude of CP-violation effects in neutrino oscillations.
• Determination of the sign of $\Delta m^2_{\rm A}$ (Δm^2_{31}) and of the type of ν-mass spectrum (NH, IH, QD, etc.).
• Determining or obtaining significant constraints on the absolute scale of ν-masses, or on $min(m_j)$.
• Determining the nature of massive neutrinos ν_j which can be Dirac fermions possessing distinct antiparticles, or Majorana fermions, i.e. spin 1/2 particles that are identical with their antiparticles. This is of fundamental importance for making progress in our understanding of the origin of neutrino masses and mixing and of the symmetries governing the lepton sector of particle interactions. The presence of massive Dirac neutrinos is associated with the existence of a conserved additive lepton charge, which can be, e.g. the total lepton charge $L = L_e + L_\mu + L_\tau$. If no lepton charge is conserved by the particle interactions, the massive neutrinos ν_j will be Majorana fermions (see, e.g., [13, 69, 79]).
• Establishing whether the CP-symmetry is violated in the lepton sector a) due to the Dirac phase δ, and/or b) due to the Majorana phases α and β if ν_j are Majorana particles.
• Searching with increased sensitivity for possible manifestations, other than flavour neutrino oscillations, of the non-conservation of the individual lepton charges L_l, $l = e, \mu, \tau$, such as $\mu \rightarrow e + \gamma$, $\tau \rightarrow \mu + \gamma$, etc. decays.
• Understanding at fundamental level the mechanism giving rise to neutrino masses and mixing and to L_l-non-conservation, i.e., finding the *Theory of neutrino mixing*. This includes understanding the origin of the patterns of ν-mixing and ν-masses suggested by the data. Are the observed patterns of ν-mixing and of $\Delta m^2_{21,31}$ related to the existence of new fundamental symmetry of particle interactions? Is there any relations between quark mixing and neutrino mixing, e.g., does the relation $\theta_{12} + \theta_c = \pi/4$, where θ_c is the Cabibbo angle, hold? Is $\theta_{23} = \pi/4$, or $\theta_{23} > \pi/4$ or else $\theta_{23} < \pi/4$? What is the physical origin of *CPV* phases in $U_{\rm PMNS}$? Is there any relation (correlation) between the (values of) *CPV* phases and mixing angles in $U_{\rm PMNS}$? Progress in the theory of ν-mixing might also lead, in particular, to a better understanding of the mechanism of generation of baryon asymmetry of the Universe [80].

Obviously, the successful realization of the experimental part of this research program would be a formidable task and would require many years. A number of experiments, which are expected to make important contributions to the future studies of neutrino mixing – T2K, Double CHOOZ, Daya Bay, CUORE, GERDA, etc., see [76, 81, 82, 83], are already under preparation.

The mixing angles, θ_{21}, θ_{23} and θ_{13}, Dirac *CPV* phase δ and Δm^2_{21} and Δm^2_{31} can, in principle, be measured with a sufficiently high precision in a variety of ν-oscillation experiments (see, e.g. [69]). The Dirac CP-violating phase δ is a source of CP-violation in ν-oscillations (see, e.g. [36, 84]). The magnitude of the CP-violation effects in ν-oscillations is controlled by $\sin\theta_{13}\sin\delta$.

The neutrino oscillation experiments, however, cannot provide information on the absolute scale of ν- masses and on the nature of massive neutrinos ν_j. The flavour neutrino oscillations are insensitive to the Majorana CP-violating phases α and β [66, 35]. Establishing whether ν_j have distinct antiparticles (Dirac fermions) or not (Majorana fermions) is of fundamental importance for understanding the underlying symmetries of particle interactions [13] and the origin of ν-masses. The only feasible experiments having the potential of establishing the Majorana nature of massive neutrinos at present are the $(\beta\beta)_{0\nu}$-decay experiments searching for the process $(A,Z) \rightarrow (A,Z+2) + e^- + e^-$ (for reviews see, e.g. [13, 83, 85]). The observation of $(\beta\beta)_{0\nu}$-decay and the measurement of the corresponding half-life with sufficient accuracy, would not only

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be a proof that the total lepton charge is not conserved, but might provide also a unique information on the i) type of neutrino mass spectrum [87] (see also [88]), ii) absolute scale of neutrino masses (see, e.g. [88]), and iii) Majorana *CP*-violating (CPV) phases [89, 86]. If ν_j are Majorana fermions, getting experimental information about the Majorana CP-violating phases in $U_{\rm PMNS}$ would be a remarkably challenging problem [90, 91, 92]. The phases α and β can affect significantly the predictions for the rates of the (LFV) decays $\mu \to e + \gamma$, $\tau \to \mu + \gamma$, etc. in a large class of supersymmetric theories with see-saw mechanism of neutrino mass generation [93]. The Majorana CPV phase(s) in the PMNS matrix can play the role of the CP-violating parameter(s) necessary for the generation of baryon asymmetry of the Universe (see [94] and the references quoted therein).

The compelling experimental evidences obtained during the last several years for existence of neutrino oscillations, caused by nonzero neutrino masses and neutrino mixing, opened a new exciting field of research in elementary particle physics and astrophysics. There is no doubt that progress in the studies of neutrino mixing and oscillations will lead to more profound understanding of the fundamental forces governing particle interactions and of the Universe we are living in.

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