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# String Vacua and Moduli Stabilization

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We give a very basic introduction to the classification of string vacua in the presence of fluxes as well as to the related moduli stabilization problem in the corresponding 4-dimensional effective theories.

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## 1. Introduction

The Standard Model has been very successful in describing elementary particles and their interactions and by now it provides a standard undergraduate textbook topic. From a purely theoretical point of view, this same model is not completely satisfactory because of the many dimensionless free parameters that do not have an explanation and that can only be fixed by the experiment. Moreover, there are hints coming from cosmological observations that there is new physics beyond the Standard Model.

String Theory, on the other hand, is a very strong candidate for a unified theory of elementary interactions and it has no dimensionless free parameters at all. The only input parameter is the fundamental string length. Consistent superstring theories require a 10-dimensional space-time, though. In order to extract meaningful phenomenology out of String Theory, one has to consider mechanisms that lead to effective theories living in 4 space–time dimensions and that resemble the Standard Model at sufficiently low energies.

The first (and most used) such mechanism is given by Kaluza–Klein compactification. With this procedure the 10-dimensional space-time is taken to be a product of our 4-dimensional world and a 6-dimensional internal compact manifold of small size. Field fluctuations on this internal space are then seen from the 4-dimensional point of view as ordinary masses and charges. This means that the effective physics depends on the geometrical properties of the chosen internal manifold. The modern incarnation of this original idea is given by the so-called Intersecting-Brane-Worlds models (see Fig. 1), which let us obtain phenomenologically viable models with chiral fermions in representations of gauge groups similar to $SU(3) \times SU(2) \times U(1)$ of the Standard Model. In addition to the ordinary compactification manifold, this approach involves the use of stacks of space-time filling D-branes, possibly wrapped on the cycles of the internal manifold, allowing for a richer structure of matter and gauge interactions.

This procedure clearly weakens the possibility of making definite predictions for string phenomenology as these heavily depend on the choice of the internal manifold. Moreover, the compactification procedure leads to effective theories where scalar fields appear, associated to the sizes and shapes of the internal manifold. These fields $\phi^i$ finally determine the parameters of the standard model by their vacuum expectation value. Unfortunately, in the basic scenario, given a manifold, it does not cost any energy to change its size and shape. This means that, at least at the classical level, these fields are free moduli whose value is completely arbitrary. The outcome is that one introduces a huge vacuum degeneracy with a consequent loss of most of the predictive power given by a theory which does not contain any dimensionless free parameter as string theory.

In conclusion, the low-energy properties of our world depend on high-energy choices like the selection of the internal manifold! Of course, this means that the more we know on these choices, the more restrictive they are, the more we can constrain the theory and understand what kind of phenomenological consequences can be derived. **In the following we address this problem by studying a framework in which moduli stabilization can be achieved.**

One natural way to resolve this degeneracy is obtained by considering quantum corrections, but it is often difficult to compute them explicitly, keep these calculations under control and then determine the value of the lifted moduli.

Flux compactifications are an alternative scenario which can be employed already at the clas-

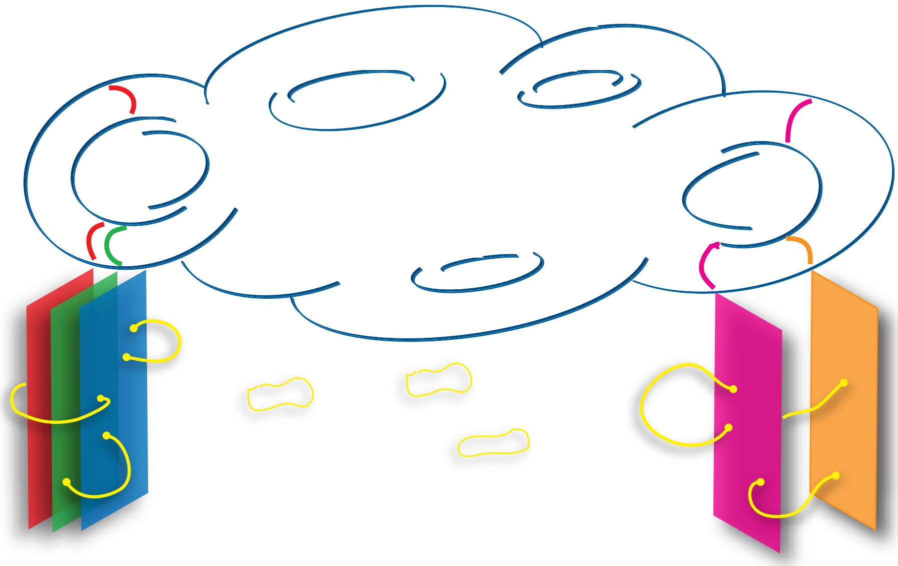

**Figure 1:** An Intersecting Brane World example. The 10-dimensional space-time is compactified on a generic manifold, whose topology determines the effective interactions. Gravity (closed strings) propagate on the full 10-dimensional space-time, while gauge interactions (open strings) are confined to 4 dimensions.

sical level, instead. The main idea is that one can consider compactifications where the Ramond–Ramond and Neveu–Schwarz fields appearing in the ten–dimensional theories acquire a non–trivial expectation value. These fields are $(p+1)$-form gauge potentials which couple to extended $p$-brane objects appearing in string theory, just like the electro-magnetic 1-form potential couples to the 0-dimensional electron. Giving a vacuum expectation value to their field-strength means that there are non-trivial fluxes in the theory. These fluxes further imply that additional energy has been introduced in the effective theory, deforming it. We will see that this deformation can be described in 4 dimensions by a scalar potential $V(\phi^i)$ for the moduli. These fields are then forced to roll down the potential and relax at an extremum (a vacuum). The choice of vacuum then follows by minimization of this potential $\partial_i V = 0$ and imposes relations on the moduli fields which eventually get fixed to values that can be explicitly determined. These values allow for explicit predictions on the parameters of the effective theory. Of course, the ideal situation would be to fix all of the moduli of the theory and generically also break supersymmetry. Practically, one studies this scenario as a promising (and quite simple) way to address this problem. Also, one should not be interested only to the (meta-)stable vacua of the potential, but to the full potential structure (the so-call *landscape of flux vacua*), because it may address further physical problems like inflation, or the explanation of the value of today's cosmological constant. It should be stressed once more that the power of this approach is that one can find explicit expressions for the potential $V(\phi^i)$ and hence for the effective couplings $< \phi^i >$.

In a world of justice, this analysis should lead to finding the precise vacuum describing our universe and therefore explaining the Standard Model couplings and masses. It is possible, however, that the values of the Standard Model parameters are just environmentally selected in a vast and complicated landscape of vacua and that there is no way to understand the vacuum selection process. Once the full landscape is known, however, one can perform a description of the vacua distribution through statistical methods and possibly obtain entropic explanations for the vacuum selection. At this stage any conclusion is premature, but there are already interesting results on the possible parameter distributions in the landscape of flux vacua.

The plan of these lectures is the following. First I will give a general introduction to the main properties of flux compactifications. Then I will focus on the geometry of the internal space and on the new techniques which can be employed in order to perform this analysis. Then I will pass to the effective theory point of view discussing the potentials and the moduli stabilization problem. Finally I will comment on some more recent approaches and progresses.

This proceeding provides mostly a transcription of the lectures held during the school and does not want to be an exhaustive review on the subject. For some good and quite comprehensive reviews the reader can look at [1, 2, 3, 4, 5, 6]. For the same reason I am not giving here a complete list of references on the subject, which can be found in the same papers just mentioned.

## 2. General properties of flux compactifications

The first approach to the problem of obtaining 4-dimensional effective theories out of string theory consists in compactifying the 10-dimensional theories on a compact internal manifold $Y_6$, preserving 4-dimensional Poincarè invariance. The simplest way to do so is by setting all the fields to zero, but the metric. This latter is then taken to be a product of a 4-dimensional Minkowski space and a metric to be determined on $Y_6$. It is then easy to see that the equations of motion imply that this product is direct and give some conditions on the geometry of $Y_6$.

We can however consider further constraints to our problem that simplify the solution. String Theory and its low energy effective theory given by supergravity are 10-dimensional theories which contain supersymmetry. Unfortunately, the observed 4-dimensional world is not supersymmetric and therefore we need that the final vacuum breaks it explicitly. In global supersymmetry, breaking one supersymmetry, breaks them all. On the other hand, when this symmetry is local as in supergravity, it is possible to introduce additional intermediate scales by breaking supersymmetry only partially. For this reason we can setup the problem in a way that first considers the compactification to supersymmetry preserving spaces and then analyze further effects that may break it completely. As we will see, supersymmetry is a very powerful tool that allows full control on the effects on the 4-dimensional physics.

In order to obtain a supersymmetric background we therefore have to verify both the supersymmetry transformations as well as the equations of motion on the background. For a bosonic background, the supersymmetry transformations of the bosonic fields are always trivially satisfied as they always contain some fermions. If only the metric is not vanishing, most of the supersymmetry transformations of the fermions are identically satisfied, too, with the exception of the gravitino one (at least when the theory does not contain higher-derivative terms). The latter becomes an equation imposing constraints on the geometry of the solution. Let us see this.



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The supersymmetry requirement coming from the gravitino transformation rule imposes that there exist a spinor $\varepsilon$ which is parallel with respect to the Levi–Civita connection

$$\delta\psi_m = \nabla_m\varepsilon = \left(\partial_m + \frac{1}{4}\omega_m{}^{ab}\gamma_{ab}\right)\varepsilon = 0\,. \tag{2.1}$$

This equation specifies $\varepsilon$, but it is also clear that it admits solutions only for special choices of the connection $\omega_m{}^{ab}$ and therefore of the internal geometry. This latter can be specified by looking at the integrability condition deriving from (2.1):

$$[\nabla_m, \nabla_n]\,\varepsilon = -\frac{1}{4}R_{mn}{}^{pq}\gamma_{pq}\,\varepsilon = 0\,. \tag{2.2}$$

This integrability condition can be interpreted as the fact that certain combinations of the tangent space generators

$$T_{mn} \equiv \frac{1}{4}R_{mn}{}^{pq}\gamma_{pq} \tag{2.3}$$

annihilate $\varepsilon$ as well as the fact that the curvature is constrained. The first fact implies that the holonomy of the space is reduced. For what concerns the second comment we can see explicitly what happens by further contracting (2.2) with one gamma matrix:

$$\gamma^n\gamma^{pq}R_{mnpq}\,\varepsilon = \gamma^{npq}R_{m[npq]}\,\eta - 2R_{mn}\gamma^n\varepsilon = 0\,. \tag{2.4}$$

In the end, by using that the first term ($R_{m[npq]} = 0$) vanishes by construction for the Levi-Civita connection, one obtains that the solution must be Ricci-flat: $R_{mn} = 0$.

Summing up, the possible internal manifolds $Y_6$ must have special-holonomy and must be Ricci-flat. These spaces have been classified by Berger [7], and we can see from his analysis how to obtain minimal supersymmetry in four dimensions. For instance, we can compactify M-theory on $G_2$-manifolds, whose holonomy is contained in the group $G_2 \subset SO(7)$, or the Heterotic string theory on a Calabi–Yau manifold that preserves $SU(3) \subset SO(6)$. These same manifolds can be used in compactifications of type II string theories to obtain $\mathcal{N} = 2$ effective theories. Berger's classification applies more generally to all types of solutions that can be obtained for purely geometric compactifications of any (ungauged) supergravity theory preserving some supersymmetry. The smaller the holonomy group the bigger the number of supersymmetries preserved.

Adding fluxes obviously changes this situation. We have seen that these are vacuum expectation values (vev) for the $(p+1)$-forms field-strengths. In order to preserve Poincarè invariance, these are chosen to be non-vanishing only when all the form indices are on the internal manifold $< F_{mnp\ldots} > \neq 0$, or proportional to the 4-dimensional volume form $< F > \sim \mathrm{Vol}_4 \wedge f_{int}$. A simple consequence of this addition is given by the backreaction of these fluxes onto the internal geometry. The Einstein equation will now read

$$R_{mn} = F_m{}^{c_1\ldots c_{p-1}}F_{nc_1\ldots c_{p-1}} + \ldots\,, \tag{2.5}$$

where $F_{m_1\ldots m_p}$ is some $p$-form whose vacuum expectation value (vev) is assumed to be different from zero. On the internal sector this generically implies that the space is no longer Ricci-flat. But we can say more. If we again look for supersymmetric configurations, the gravitino supersymmetry law tells us that there must exist a non-trivial spinor $\eta$ which is covariantly constant with respect to a certain connection $\mathscr{D}$, which now contains information from the fluxes. Schematically

$$\delta\psi_m = \mathscr{D}_m\eta \equiv \nabla_m\eta + H_m\eta\,, \tag{2.6}$$

where we called $H_m$ the flux contribution, which for instance may be that of a 3-form flux in the form $H_m = H_m{}^{np}\gamma_{np}$. Integrability of (2.6) implies that the internal space has again reduced (generalized) holonomy, but also that it is not Ricci flat anymore, not even with respect to the generalized connection $\mathscr{D}$. Explicitly, the new integrability condition reads

$$[\mathscr{D}_m, \mathscr{D}_n]\,\eta = -\frac{1}{4}\widehat{R}_{mn}{}^{pq}\gamma_{pq}\,\eta = 0\,, \tag{2.7}$$

but now we cannot follow the same line of reasoning as before because the connection which defines the generalized Riemann tensor contains torsion terms generated by the fluxes and therefore $\widehat{R}_{m[npq]} \neq 0$.

The outcome of this simple analysis is that we need some new tools to classify these geometries, and this is given by the *group structures* (Introduced first in this context in [8, 9]). Before discussing this tool in detail, let us finish this part with some more comments on the general properties of this kind of compactifications.

Unfortunately, it is not so straightforward to obtain solutions of the above type. It is known since the eighties that, under very simple assumptions, solutions like the ones presented above are inconsistent. This no-go theorem, which was first formulated in [10] and recently perfected in [11], starts from assuming that the theory we analyze has a standard action, which means that there are no higher derivative terms like higher order terms in the curvature: $R^2$ or $R^4$ and so on. The second assumption is that all massless fields have positive kinetic energy. Then we further ask that the starting theory has a potential which is non-positive definite $V_D \leq 0$, as in the case of 10-dimensional supergravities. Finally, we look for smooth (some type of singularities may be allowed, cfr. [11]) solutions of the form

$$ds^2 = e^{2A(y)}\left(dx^\mu dx^\nu \eta_{\mu\nu} + ds_6^2(y)\right)\,, \tag{2.8}$$

where the 4-dimensional Poincaré invariance is preserved also by the vevs of the other fields. Under all these assumptions one can easily argue from the analysis of the Einstein equation that for a *compact internal space* all the vevs of the various fields with the exception of the metric must be vanishing. Since type II supergravity theories satisfy these requirements we obtain that compactifications of type II supergravities to 4-dimensional Minkowski space-time in the presence of fluxes are not possible.

How can we avoid such no go theorem then? Well, there are various possibilities given by relaxing some of the above requirements, though we will mainly focus on the two that are more natural to adopt in ordinary string theory. One could indeed start from exotic theories like type II$^*$ theories, which have fields with negative kinetic energy or one could try to use non–compact solutions for which the interactions could be confined to a 4-dimensional effective theory, but we will not follow these ideas. One of the assumptions of the no-go theorem is that there are no singularities. An easy way to avoid this is by introducing sources, which imply that the solution is not smooth everywhere and also that one has to solve the equations of motion of the coupled system and not just of the gravity theory as it was done to derive the theorem. In string theory this is a natural consequence of the existence of D-branes. However, since fluxes give positive contributions to the Einstein equation they can really be compensated only by negative tension objects, like orientifold planes, which also act as sinks for the fluxes.

Another possibility is to add higher-derivative terms to the action. This is also natural in string theory as they appear when looking at $\alpha'$ corrections to their low-energy effective action. These are especially required by Heterotic and type I theories where consistency due to the cancellation of anomalies imposes that the Bianchi identity of the 3-form be modified by $\alpha'$ contributions $dH = \alpha' \left[\mathrm{tr}\, R^2 - \mathrm{tr}\, F^2\right]$. By supersymmetry, this further imposes $\alpha'$ correction terms to the action $S = S_0 + \alpha' \int R^2$, thus violating the assumptions of the no-go theorem.

So far we discussed the gross features of flux compactifications. Now we move to a more detailed description of some aspects of the two main problems one faces when dealing with flux compactifications. First, we will address the problem of describing the vacua configurations in the presence of fluxes. By this we mean understanding the geometrical properties of the solutions for a given flux $H \neq 0$. We have already seen that a non-trivial flux generates a backreaction on the 10-dimensional metric and in particular it makes the internal space $Y_6$ non Ricci-flat, but now we want to be more precise. If we want to specify what kind of spaces $Y_6$ can be used to obtain consistent compactifications in the presence of fluxes, we have to find an effective way to describe the consistency conditions in terms of geometrical quantities. As we have seen above, holonomy is not a good guide anymore, but it can be replaced by the tool of group structures on the tangent bundle.

The second point we will address is the possibility of having a collective description of these vacua. Since the net effect of fluxes on the effective theory is that of generating a scalar potential, we would like to see how one can determine such potential in terms of the fluxes and finally address the moduli stabilization problem.

## 3. Geometry of Flux Compactifications

We are now ready to discuss in more detail the tools that we can use in order to construct and classify the supersymmetric solutions describing flux compactifications. Let us start by exploring the geometric consequences of equation (2.1) further.

The Levi–Civita connection appearing in (2.1) takes values in the tangent space group $\mathrm{Spin}(1,d-1)$ and actually, since it preserves the metric $\nabla_M g_{NP} = 0$, in $\mathrm{SO}(1,d-1)$. This is the structure group of the tangent bundle for a generic Riemannian manifold, i.e. the group required to patch the tangent bundle over the manifold. For the Levi–Civita connection it coincides with the holonomy group.

When solving the supersymmmetry conditions one reduces this group, as we have seen before, because the Killing spinors, solutions of (2.1), are annihilated by some of the generators of this group. This means that in order to patch together the tangent bundle over the manifold, only a subgroup $G \subset \mathrm{SO}(1,d-1)$ is needed. This fact is equivalent to the Killing spinor being a singlet of $G$: it does not transform under an action of its generators. Clearly, in order to have a reduction of the structure group over the whole manifold, this invariant must be globally defined. This is granted for the solutions of (2.1). The Killing spinors are parallel with respect to the Levi–Civita connection and therefore any solution of (2.1) can be transported using this connection to any other point of the manifold (at least if this is simply connected). This means that once the supersymmmetry conditions are solved in one patch, the solution can be extended globally over the manifold. Moreover, since the metric is preserved by this connection, it is clear that the norm of all the invariants is preserved and the invariants whose norm is never vanishing are globally defined.

Following this discussion, any reduced group structure, and therefore any reduced holonomy group, implies the existence of a set of singlet tensor fields (or spinors) with respect to the structure group. For instance, for the compactifications of string theory to four dimensions, we have seen that the holonomy group of the internal manifold is reduced to $SU(3)$ and therefore they must be Calabi–Yau manifolds. These are Kähler manifolds with vanishing first Chern class. From a differential point of view they can be described by a closed Kähler form $dJ = 0$ and a closed holomorphic 3-form $d\Omega = 0$. Obviously, these conditions must be equivalent to the supersymmmetry condition on the spinor $\nabla\eta = 0$, which we have seen specifying the same constraint

$$dJ = 0 = d\Omega \qquad \Longleftrightarrow \qquad \nabla\eta = 0.$$

This is indeed the case, as we will see in a while from the fact that the invariant tensors $J$ and $\Omega$ can be constructed as bilinears from the invariant spinor $\eta$.

The group structure reduction from $SO(6) \simeq SU(4)$ to $SU(3)$ can be characterized by looking at the decomposition of the $SU(4)$ irrepses under $SU(3)$:

$$\begin{aligned} \mathbf{4} &\to \mathbf{1}+\mathbf{3}, \\ \mathbf{6} &\to \mathbf{3}+\overline{\mathbf{3}}, \\ \mathbf{10} &\to \mathbf{1}+\mathbf{3}+\mathbf{6}, \\ \mathbf{15} &\to \mathbf{1}+\mathbf{3}+\overline{\mathbf{3}}+\mathbf{8}. \end{aligned} \tag{3.1}$$

From this decomposition we deduce that there is one complex globally defined invariant spinor on the manifold, as there is only one singlet in the decomposition of the spinor representation of SO(6) in terms of SU(3) representations. In the same way we can understand that beside the metric tensor (that is a singlet of the general SO$(1,d)$ structure group of a Riemannian manifold), there are a 2-form and a complex 3-form field that are invariant under SU(3): the symplectic form $J$ and the holomorphic form $\Omega$. It is useful to notice that this tensors can be obtained by contractions of the invariant spinor with the 6-dimensional gamma matrices:

$$J_{mn} = -i\eta^\dagger \gamma_{mn}\eta\,, \qquad \Omega_{mnp} = -i\,\eta^T \gamma_{mnp}\eta. \tag{3.2}$$

This also implies that $J$ and $\Omega$ are parallel with respect to the Levi-Civita connection $\nabla J = 0 = \nabla\Omega$, by applying (2.1) and finally, by antisymmetrizing, we get the Calabi–Yau conditions:

$$dJ = 0, \quad d\Omega = 0. \tag{3.3}$$

We can actually infer more from simple group theory. Since the decomposition of the $SO(6)$ vector under SU(3) gives no singlets (and the same is true for the dual 5-forms) and there is only one 0-form, the following compatibility and completeness relations must hold:

$$\begin{aligned} J\wedge\Omega &= 0 \\ J\wedge J\wedge J &= \frac{3}{4} i\Omega\wedge\overline{\Omega}. \end{aligned} \tag{3.4}$$

Summmarizing, the manifold is completely specified by the supersymmetry conditions!

*Let us then see what happens when fluxes are turned on.* We remind that the supersymmetry condition coming from the gravitino transformation rule has changed:

$$\delta\psi_m = \mathscr{D}_m\eta \equiv \nabla_m\eta + H_m\eta = 0\,. \tag{3.5}$$

However, we can still characterize the solutions in terms of the structure group of the tangent bundle, by using the properties of the new connection defined by $\mathscr{D}$. A solution to (3.5) defines a spinor $\eta$ which reduces the group structure. This same spinor defines once more the invariant tensors $J$ and $\Omega$, but now $dJ \neq 0$ and $d\Omega \neq 0$. This means that knowing $dJ$ and $d\Omega$ we can specify the geometry of the internal manifold $Y_6$. There is a difference with the previous case, though. The new connection is not simply the Levi–Civita connection and therefore it is not straightforward to prove that once (3.5) is solved in one patch, we can infer a global solution. For a generic value of the fluxes, this connection does not lie in $\mathrm{Spin}(1,d-1)$, as not all the terms in the gravitino supersymmetry rule can be rewritten in terms of Levi–Civita-plus-torsion terms. Actually, it does not generically preserve the metric, defining the reduction of the structure group to $\mathrm{SO}(1,d-1)$,

$$\mathscr{D}_M g_{NP} = Q_{MNP} \neq 0, \tag{3.6}$$

and the generic decomposition of the connection will contain an explicit dependence on these terms $\mathscr{D}_M = \nabla_M + \tau_M{}^{NP}\gamma_{NP} + \widetilde{Q}_M$, where $\tau$ is the contorsion tensor. As a generic consequence the spinors that solve the supersymmetry equations are no longer globally defined. This *does not* imply that the solution does not preserve supersymmetry anymore. In order to preserve supersymmetry one just needs to solve the supersymmetry preserving conditions on every patch of the manifold, but the solutions need not be globally non-vanishing.

A similar phenomenon appears when looking for solutions of the Killing vector equations on a manifold. Consider for instance $S^2 \equiv \frac{SO(3)}{SO(2)}$. This manifold has a local $\mathrm{SO}(3)$ symmetry group. This implies that in every patch one can define 3 non-vanishing vector fields that generate $\mathrm{SO}(3)$. At the same time, parallel transport of these fields changes their norm, as they are not parallel to the Levi–Civita connection and can therefore vanish at some point, as they actually do. Nonetheless, the group of isometries is $\mathrm{SO}(3)$ at each point on the manifold. The same phenomenon takes place for the supersymmetry equations and the Killing spinors, solving the supersymmetry conditions. In this case $N$ spinor fields $\eta$ satisfying $\mathscr{D}\eta = 0$ define an $N$-supersymmetric background, even if some of the $\eta$ vanish at some point. However, for the special cases where $Q = 0$, the connection $\mathscr{D}$ lies in $\mathrm{Spin}(1,d-1)$, and solutions to $\mathscr{D}\eta = 0$ can be parallel-transported using this connection and therefore become globally defined. In the first case, the structure group is reduced only locally. In the second case, the structure group is globally reduced and the intrinsic torsion completely specifies the supersymmetric solutions.

In any case, one can use the group structure tool locally, to obtain geometrical constraints on the solutions and then analyze the topological consequences when trying to extend this solutions globally.

Let us then see more in detail how we can use the group structures in order to classify the solutions (In the following we are going to assume that $Q = 0$). When the supersymmetry parameter is not preserved by the Levi–Civita connection $\nabla\eta \neq 0$, also the G-invariant structures are not preserved $\nabla_m J_n{}^p = \tau_{mn}{}^p \neq 0$ and the $\tau$ tensor measures the departure from the Calabi–Yau condition and, following the arguments above, specifies completely the manifold. If we want to classify $Y_6$ we can then use the torsion $\tau$ and its modules, as it was done first for this purpose in [12, 13]. This can be read from the supersymmetry equations as the torsion piece that appears in the new covariant derivative $\mathscr{D}$ preserving the supersymmetry parameter $\eta$. All the different $Y_6$ are classified by the irreducible $G$-modules under which $\tau$ can be decomposed. The intrinsic torsion tensor $\tau$ is a 1-form valued in SO(6): $\tau \in \Lambda^1 \otimes SO(6)$. Its irreducible modules can be determined by decomposing the so(6) algebra in the Lie algebra of the group structure $g$ and its complement $g^\perp$. Obviously, since $\eta$ is a G-singlet, the action of the elements in $g$ on it is trivial $g\eta = 0$ and the G-modules can be obtained by decomposing the remainder. For instance, different $SU(3)$ structures are classified by the decomposition of the torsion $\tau$ into five complex modules

$$
\begin{aligned}
\tau \to \left(\mathbf{3}+\overline{\mathbf{3}}\right) \times \left(\overbrace{\mathbf{1}+\mathbf{3}+\overline{\mathbf{3}}}^{g^\perp} + \overbrace{\not{\mathbf{8}}}^{g}\right) &= (\mathbf{1}+\mathbf{1}) + (\mathbf{8}+\mathbf{8}) + (\mathbf{6}+\overline{\mathbf{6}}) + (\mathbf{3}+\overline{\mathbf{3}}) + (\mathbf{3}+\overline{\mathbf{3}}) \\
&= \quad \mathscr{W}_1 \quad + \quad \mathscr{W}_2 \quad + \quad \mathscr{W}_3 \quad + \quad \mathscr{W}_4 \quad + \quad \mathscr{W}_5 \, .
\end{aligned}
\tag{3.7}
$$

The interesting part of this analysis is that such modules are completely determined by simply computing the action of the exterior differential on the invariant tensors defining the group structure. For the case at hand, they are determined by $dJ$ and $d\Omega$ in the following way:

$$
dJ = \frac{3}{4} i \left(\mathscr{W}_1 \overline{\Omega} - \overline{\mathscr{W}}_1 \Omega\right) + \mathscr{W}_3 + J \wedge \mathscr{W}_4 \, , \tag{3.8}
$$

$$
d\Omega = \mathscr{W}_1 J \wedge J + J \wedge \mathscr{W}_2 + \Omega \wedge \mathscr{W}_5 \, , \tag{3.9}
$$

where $J \wedge \mathscr{W}_3 = J \wedge J \wedge \mathscr{W}_2 = 0$ and $\Omega \wedge \mathscr{W}_3 = 0$. Since $J_m{}^n$ is globally defined we can always introduce hodge type projectors $P_m^{\pm n} = \frac{1}{2}\left(\delta_m{}^n \pm i J_m{}^n\right)$ and therefore $J$ is always of type (1,1) with respect to this decomposition and $\Omega$ is of type (3,0). The fact that the $(2,2)$ piece of $d\Omega$ defines the same class as the $(0,3)$ piece of $dJ$ is a consequence of the first relation in (3.4).

Let us now give some names to manifolds having different $SU(3)$ structures. Since exterior differentiation preserves the Hodge type for complex manifolds, we can immediately recognize complex versus non-complex manifolds by evaluating $[dJ]^{(3,0)}$ and $[d\Omega]^{(2,2)}$. For a complex manifold these must vanish and therefore so must $\mathscr{W}_1$ and $\mathscr{W}_2$, too. Provided $Y_6$ satisfies this requirement we can list some of these manifolds and their allowed structures.

| Complex manifolds | | $\mathscr{W}_1 = \mathscr{W}_2 = 0$ |
|---|---|---|
| Calabi Yau | $dJ = d\Omega = 0$ | $\tau = 0$ |
| Kähler | $dJ = 0$ | $\tau \in \mathscr{W}_5$ |
| Balanced | $J \wedge dJ = 0$ | $\tau \in \mathscr{W}_3 \oplus \mathscr{W}_5$ |
| Special Hermitean | $J \wedge dJ = 0$<br>$d\Omega = 0$ | $\tau \in \mathscr{W}_3$ |

Table I. Complex manifolds and their group structures.

Non-complex manifolds can have torsion in $\mathscr{W}_1$ and $\mathscr{W}_2$, instead.

| | | |
|---|---|---|
| Nearly Kähler | $dJ \sim \Omega$<br>$d\Omega \sim J \wedge J$ | $\tau \in \mathscr{W}_1$ |
| Almost Kähler | $dJ = 0$<br>$d\Omega \sim J \wedge A$ | $\tau \in \mathscr{W}_2$ |

Table II. Non-complex manifolds.

An interesting type of manifolds which may be both complex or not is given by half-flat manifolds. They have $\tau \in \mathscr{W}_1^- \oplus \mathscr{W}_2^- \oplus \mathscr{W}_3$. The interest in these manifolds lies in the fact that a real fibration over them gives a $G_2$-holonomy 7-manifold [14].

We finish this section by giving one example: the common sector of string theory [13, 15]. This sector contains the metric $g_{mn}$, the dilaton $\phi$, the 2-form $B_{mn}$, the gravitino $\psi_m$ and a dilatino $\lambda$. $\mathscr{N} = 1$ supersymmetric compactifications of heterotic/type II superstring theory can be obtained by using SU(3) structure manifolds as internal solutions $Y_6$. This can be realized by analizing the supersymmetry transformations of this sector:

$$\delta\psi_m = \nabla_m \varepsilon - \frac{1}{4} H_{mnp}\gamma^{np}\varepsilon = 0, \tag{3.10}$$

$$\delta\lambda = \partial\!\!\!/ \varepsilon + \frac{1}{24} H_{mnp}\gamma^{mnp}\varepsilon = 0. \tag{3.11}$$

The information on $Y_6$ coming from these equations can be extracted by decomposing the 10-dimensional Majorana-Weyl spinors as

$$\varepsilon_{10} = \varepsilon_4 \otimes \eta_6 + \varepsilon_4^* \otimes \eta_6^* \tag{3.12}$$

and projecting the above equations on $\eta^\dagger$, $\eta^\dagger\gamma^m$ and so on... As an example let us look at the projection $\eta^\dagger\delta\lambda = 0$. This projection reads

$$H_{mnp}\eta^T\gamma^{mnp}\eta = H_{mnp}\Omega^{mnp} = 0. \tag{3.13}$$

This implies that the (0,3) component (and by reality the (3,0) component as well) of the 3-form flux $H$ is vanishing. With the same technique we can also obtain the differential conditions specifying the structure by looking at the gravitino equation. The result is that

$$\begin{aligned} dJ &= -4J \wedge d\phi + 2\star H, \\ d\Omega &= 8\Omega \wedge d\phi. \end{aligned} \tag{3.14}$$

Let us finally remind once more that these conditions specify the internal manifolds that are compatible with the supersymmetry conditions, but that the Bianchi identities and the equations of motion still need to be imposed in order to obtain real vacua of the theory.

## 4. Potentials from flux compactifications and moduli stabilization

As we mentioned above, the second main problem in flux compactifications is the derivation and properties of the scalar potential induced by the fluxes on the effective 4-dimensional theory.



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So far, we have seen how the backgrounds preserving supersymmetry can be classified and constructed, using the tool of the group structure of the tangent bundle. Of course, we are as well interested in the effective theories coming from compactifications that do not satisfy the 10- or 11-dimensional equations of motion, but still give some supercharges in the effective theory that may be spontaneously broken. This requirement is related again to the existence of some globally defined spinors on the internal manifold (therefore implying a reduction of the group structure).

The idea is that, like the supersymmetry parameter of the previous section, all the spinor fields are reduced to effective 4-dimensional fields using these globally defined fields [16, 17]. For instance, the transverse part of the M-theory gravitino $\Psi_\mu$ can be split in the 4-dimensional part $\psi_\mu$ and the internal globally defined spinors $\eta$ as $\Psi_\mu = \psi_\mu \otimes \eta + \psi_\mu^* \otimes \eta^*$. For an $N = 1$ compactification of M-theory, from the supersymmetry transformation of the 11-dimensional gravitino

$$\delta\Psi_A = \left\{D_A[\omega] + \frac{1}{144} G_{BCDE} \left(\Gamma^{BCDE}{}_A - 8\Gamma^{CDE}\eta^B{}_A\right)\right\}\varepsilon_{11}, \tag{4.1}$$

we can extract the supersymmetry transformation of the 4-dimensional field

$$\delta\psi_\mu = D_\mu \varepsilon_4 + \ldots + ie^{K/2} W \gamma_\mu \varepsilon_4^c, \tag{4.2}$$

by comparison of the various terms in the reduction after integration over the internal space. In (4.2) $\varepsilon_4^c$ denotes the charge-conjugate spinor and we have emphasized only the superpotential term, neglecting in the dots the various terms with the vector fields.

The superpotential term is then written as an integral over the internal space of the fluxes appearing in (4.1) and the non-vanishing contractions of the gamma matrices between the globally defined spinors. These contractions, as we saw in the previous section, describe the structure group of the internal manifold, and they are represented by globally defined forms.

When trying to extract an effective 4-dimensional theory, the first question one should ask is how to identify the light modes in the compactification (Here we follow a discussion in [12]). Since fluxes introduce a potential, not all fields are massless anymore. Moreover, in any dimensional reduction there is an infinite tower of Kaluza–Klein states and thus we need a criterion for determining which modes to keep in the effective action. When fluxes are zero the metric is a direct product of Minkowski spacetime with a Calabi–Yau 3-fold. If fluxes are added the geometry can radically change. Still, at least in the IIB case, we can think of the flux as a small perturbation of the Calabi–Yau geometry. A heuristic argument is the following. Making a product expansion in $H$, at the linear order in $H$, the flux appears only in its equations of motion, whereas at the quadratic order it appears in both the Einstein and dilaton equations of motion

$$d * H = \ldots \tag{4.3}$$

$$R_{mn} = H_m{}^{ij} H_{nij} + \ldots \tag{4.4}$$

The backreaction will be small provided $H$ is small compared to the curvature of the compactification, set by the inverse size of the Calabi–Yau manifold $1/t$. Recall, however, that in string theory the flux is quantized in units of $\alpha'$

$$\frac{1}{(2\pi)^2\alpha'} \int_{C^3} H = N. \tag{4.5}$$

Consequently $H \sim \alpha'/t^3$ and so for a small backreaction we require $H$ to be small compared to the curvature $1/t$:

$$H \sim \frac{\alpha'}{t^3} \ll \frac{1}{t}\,. \tag{4.6}$$

In other words, we must be in the large volume limit where the Calabi–Yau manifold is much larger than the string length, which anyway is the region where supergravity is applicable. The Kaluza–Klein masses will be of order $1/t$. The mass correction due to $H$ is proportional to $\alpha'/t^3$ and so it is comparatively small in the large volume limit. Thus in the dimensional reduction it is consistent to keep only the zero-modes on $Y_6$, which get small masses of order $\alpha'/t^3$, and to drop all the higher Kaluza–Klein modes with masses of order $1/t$.

Of course, when doing this kind of reduction one has to be aware of the approximation used and that therefore all the results which can be derived from this effective theory should be compatible with it. Moreover, not all flux vacua can be described in this way. The deformations can be so drastic that the light fields $\sigma$-model is unrelated or only partially related to those describing the fluxless compactifications. In any case, *it is not generically true that the obtained 4-dimensional theory is a consistent truncation of the original 10-dimensional one*. It would actually be surprising if this was the case. As a positive point, however, it should be emphasized that the effective theory may describe more than just the possible $Y_6$ vacua, but may yield also interesting information on the vacuum dynamics.

What type of models describe these effective theories then? We expect these theories to allow for a potential. Hence we can argue that either they are described by $N = 1$ supergravity, in which case the potential is also related to a superpotential, or they must be gauged supergravity theories. Actually, when the number of preserved supersymmetries (at the level of the lagrangian) is bigger than one, then the only consistent supergravity theories containing a potential are gauged supergravities. Let us then briefly review the concepts underlying these theories.

### 4.1 Gauged supergravities as effective theories

Let us specify what gauged supergravity means. The scalar $\sigma$–model interactions in any standard supergravity theory are described by a scalar manifold whose coordinates are the scalar fields themselves. A subgroup of the isometries of this manifold is realized as a global symmetry group of the full theory, or at least of the equations of motion. The gauging procedure, by which we name the theory, consists in a deformation which makes these global symmetries (including $R$–symmetry) local. The standard procedure is to introduce new connections for the charged objects, for instance substituting simple derivatives in front of the charged scalars by covariant ones $\partial_\mu \to \partial_\mu + gA_\mu$. This process obviously modifies the Lagrangean breaking its supersymmetry invariance. In order to restore it we have to modify also the supersymmetry rules: a mass term for the fermions must be introduced at $O(g)$, as well as shifts to the supersymmetry rules, whereas at $O(g^2)$ a scalar potential appears. It is remarkable that such a process, for consistent gaugings, does not introduce $O(g^3)$ terms.

Of course, there are many possible deformations of standard supergravity theory leading to gauged supergravities and only for a little set the stringy origin has been understood. Anyway, these theories must be the relevant ones in flux compactifications because only such theories allow for a scalar potential (with the exception of $\mathcal{N} = 1$ theories in 4 dimensions).

Let us now see explicitly how this works in 4 dimensions, with a schematic general construction which is valid for any number of supersymmetries [18]. The generic field content of 4 dimensional theories is given by a graviton, $\mathscr{N}$ gravitini and a number of vector, spin 1/2 and scalar fields $\{g_{\mu\nu}, \psi^i_\mu, A^I_\mu, \lambda^A, \phi^a\}$. These latter parameterize a scalar manifold $\mathscr{M}$ and we are going to consider the isometries of $\mathscr{M}$

$$\delta\phi^a = \varepsilon^\alpha k^a_\alpha(\phi)\,, \tag{4.7}$$

which are realized as symmetries of the full theory.

The gauging is performed by introducing the appropriate vector fields

$$D_\mu\phi^a = \partial_\mu\phi^a + g A^I_\mu k^a_I(\phi)\,. \tag{4.8}$$

As explained above, we preserve supersymmetry at the level of the Lagrangean by modifying the Fermi supersymmetry rules by shifts $S_{ij}$, $N_i^A$ as follows:

$$\delta\psi^i_\mu = D_\mu\varepsilon^i + h_I(\phi)\, F^{I\,\nu\rho}\gamma_{\mu\nu\rho}\varepsilon^i + g\,\gamma_\mu S^{ij}\varepsilon_j\,, \tag{4.9}$$

$$\delta\lambda^A = e^A_{ai}(\phi)\, \not{D}\phi^a\,\varepsilon^i + f^A_I(\phi)\,\gamma^{\mu\nu}F^I_{\mu\nu}\varepsilon^i + g N^A_i\varepsilon^i\,. \tag{4.10}$$

These shifts have the remarkable property of satisfying some general gradient flow relations

$$D_a S_{ij} = \mathscr{N}_i^A e^a_{Aj} + k^a_I f^I_{ij} \tag{4.11}$$

and more importantly to completely determine the scalar potential. It can be proven that for any gauged supergravity the scalar potential follows by a generalized Ward identity as the square of the shifts of the fermi fields supersymmetry rules

$$\mathscr{V} = \mathscr{N}_A^i g^A{}_B N_i^B - \mathrm{tr}\, S^2\,. \tag{4.12}$$

This quick review also outlines the strategy to be used in the following to determine the scalar potential in flux compactifications.

If we can determine which isometry is gauged, the potential follows directly by the strict construction of consistent supersymmetric theories. This means that in general there is no need to perform a full compactification, but it is enough to look for the right couplings of the scalars with the gauge vectors as we will now show.

being clear that potentials arise only in gauged supergravities, why fluxes should give rise to gaugings? This can be seen in one simple example: the reduction of the kinetic term of a 3-form $H$.

The kinetic term of the Neveu–Schwarz 3-form common to any string theory is given by

$$\int H\wedge\star H. \tag{4.13}$$

If we assign a vev to H, $<H>\neq 0$, and try to expand around this background using the various 4-dimensional fluctuations, we can see that $H$ contains derivatives of a tensor field in 4 dimensions $B_{\mu\nu}$, some vector fields $B_{\mu a}$ and some scalar fields $B_{ab}$. In the same way the metric contains the 4-dimensional metric $g_{\mu\nu}$, some vector fields $g_{\mu a}$ and some scalar fields $g_{ab}$. By inspecting the kinetic term (4.13) we can see that among the various 4-dimensional terms there are a kinetic term for the 4-dimensional vectors coming from the 2-form $B$, as well as a (generically non-abelian) coupling between the vectors coming from the 2-form and those coming from the metric:

$$\int H\wedge\star H = \int d^4x\sqrt{-g_4}\left(\partial_\mu B_\nu{}^a\partial^\mu B^{\nu b}g_{ab} + \boxed{\partial_\mu B_\nu{}^a g^{\mu b}g^{\nu c}H_{abc}} + \ldots\right) \tag{4.14}$$

The coloured term emphasizes the coupling due to the gauging. It is therefore clear that the flux $H_{abc}$ becomes the structure constants (including the couplings) of the gauge group of the effective theory and that the scalar potential will arise (following also the previous discussion on gauged supergravity) from the $H^2$ terms. For an explicit discussion of this see [19, 20, 21, 22].

### 4.2 An example of an $\mathcal{N}=1$ superpotential and of full moduli stabilization

As we have discussed above, in order to obtain consistent compactifications of string theory with fluxes, we need also sinks for the fluxes to satisfy the charge conservation conditions. These that are usually provided by orientifold planes. Generic O-plane configurations explicitly break supersymmetry. For instance, Calabi–Yau compactifications of type IIB String Theory leads to $\mathcal{N}=2$ supergravity in 4 dimensions (we have only one complex spinor on the internal manifold), but the consistent addition of fluxes requires also O-planes leading to an effective $\mathcal{N}=1$ theory. Let us then see how we can describe these vacua by means of an effective superpotential.

Fluxes deform the internal manifold, but we expect that from the effective 4-dimensional theory all their effects can be described by a potential for the moduli $V(\phi^i)$. We stress once more that these two descriptions can be concealed only by carefully choosing the approximation in which we work. In the case of type IIB on a Calabi–Yau manifold $Y_6$, we can add 3-form fluxes $<G>=<F-\tau H>\neq 0$, giving an expectation value to the Ramond–Ramond and Neveu–Schwarz forms on the 3-cycles of $Y_6$. After these fluxes are introduced, the $Y_6$ manifold is deformed, but for some simple choices the only deformation is a conformal factor in front of the original Calabi–Yau metric:

$$ds^2_{10} = \mathrm{e}^{2A(y)}ds^2_4(x) + \mathrm{e}^{-2A(y)}ds^2_{CY}(y). \tag{4.15}$$

The equations of motion relate the warp-factor $A(y)$ to the fluxes and therefore we can now specify the previous comments on the flux "smallness". Small fluxes imply that the warp factor remains close to unity

$$\mathrm{e}^A \sim 1\,. \tag{4.16}$$

This further implies that the Calabi–Yau moduli can still be described as the lightest fields in the effective 4-dimensional description. Obviously, these fields now get masses, once more related to the warp-factor deformations. For instance, looking at the expansion of the 2-form $B$ on the internal space we obtain some scalar fields by looking at the sector with all indices on the internal manifold:

$$B_{ab}(x,y) = B_I(x)Y^I_{ab}(y), \tag{4.17}$$

where $Y^I$ are 2-form harmonics on the Calabi–Yau. When expanding the 10-dimensional equations of motion for this field around the vacuum, we can now see that these fields get massive:

$$\Box_{10}B_{ab}(x,y) \sim \Box_4 B_I(x)Y^I_{ab}(y) + B_I(x)\Box_6 Y^I_{ab}(y). \tag{4.18}$$



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Clearly $\Box_6 Y^I_{ab}(y) = 0$ on the Calabi–Yau, but the introduction of a warping implies that $\Box_6 Y^I_{ab}(y) \sim m^2 Y^I_{ab}(y) + ...$, where $m^2$ comes from the action of $\Box_6$ on the warp factor by which the harmonic forms get rescaled. The effective description is then given by the same moduli space as before, but now with a non-trivial potential. In the case of $\mathcal{N} = 1$ supergravity the potential is determined as

$$V = \mathrm{e}^K \left( g^{i\bar{\jmath}} D_i W \overline{D}_{\bar{\jmath}} \overline{W} - 3 W \overline{W} \right) + D^2, \tag{4.19}$$

with $g_{i\bar{\jmath}} = \partial_i \overline{\partial}_{\bar{\jmath}} K$ the metric of the Kähler scalar manifold. In the case at hand and for a Calabi–Yau with only one volume modulus $\rho$ the Kähler potential reads (this discussion follows the seminal papers [23, 24])

$$K = -\log\left(-i(\tau - \bar{\tau})\right) - 3\log\left(-i(\rho - \bar{\rho})\right) - \log\left(-i \int \Omega \wedge \overline{\Omega}\right), \tag{4.20}$$

and $\Omega(\phi^\alpha)$ gives the dependence on the complex structure mooduli. The potential then follows from the kinetic term of the 3-form

$$V = -\frac{1}{2} \int_{Y_6} d^6 y \sqrt{g} \frac{G_{mnp} \overline{G}^{mnp}}{Im\tau} = -\frac{1}{2} \int_{Y_6} \frac{G \wedge \star \overline{G}}{Im\tau}. \tag{4.21}$$

In order to compute this integral (or better, to rewrite it in the (4.19) form) it is useful to define the (anti) selfdual combinations:

$$G^\pm \equiv \frac{1}{2}(G \pm i \star G)$$

and to remember that $\star_6^2 = -1$ and therefore $\star G^\pm = \mp i G^\pm$. The kinetic term then gives two terms,

$$V = \underbrace{-\frac{1}{2}\frac{1}{Im\tau} \int G^+ \wedge \star \overline{G}^+}_{\text{potential}} + \underbrace{\frac{i}{Im\tau} \int G \wedge \overline{G}}_{\text{topological term}} \tag{4.22}$$

Supergravity disregards the topological term, though it is needed in string theory to cancel tadpoles, and we are therefore left with the real potential. In the small flux approximation ($\mathrm{e}^A \sim 1$) we can expand $G^+$ in harmonics on the Calabi–Yau as

$$dG^+ = 0 = d \star G^+. \tag{4.23}$$

A basis of harmonic 3-forms is given by

$$\Omega, \qquad \overline{\Omega}, \qquad D_\alpha \Omega, \qquad \overline{D_\alpha \Omega} \equiv \chi_{\bar{\alpha}},$$

where $D_\alpha = \partial_\alpha + \partial_\alpha K$ and $\Omega \wedge \chi_\alpha = 0$. Moreover, it is easy to prove the self-duality properties

$$\begin{aligned} \star\Omega &= -i\Omega, & \star\chi_\alpha &= i\chi_\alpha, \\ \star\overline{\Omega} &= i\overline{\Omega}, & \star\overline{\chi}_\alpha &= -i\overline{\chi}_\alpha. \end{aligned} \tag{4.24}$$

Hence, the expansion of the 3-form becomes

$$G^+ \simeq A\Omega + B^{\bar{\alpha}} \chi_{\bar{\alpha}}. \tag{4.25}$$

The $A$ and $B^{\bar{\alpha}}$ coefficients can be fixed by integrating (4.25) as

$$\int G^+ \wedge \overline{\Omega} = A \int \Omega \wedge \overline{\Omega}, \tag{4.26}$$

$$\int G^+ \wedge \overline{\chi}_\alpha = B^{\bar{\beta}} \int \chi_\alpha \wedge \chi_{\bar{\beta}}, \tag{4.27}$$

and by using the fact that $G^- \wedge \overline{\Omega} = G^- \wedge \overline{\chi}_\alpha = 0$, by selfduality. This implies that

$$A = \frac{\int G \wedge \overline{\Omega}}{\int \Omega \wedge \overline{\Omega}}, \tag{4.28}$$

and

$$B^{\bar{\beta}} = g^{\bar{\beta}\alpha} \int G \wedge \chi_\alpha, \tag{4.29}$$

where we introduced $g_{\alpha\bar{\beta}} \equiv \frac{\int \chi_\alpha \wedge \chi_{\bar{\beta}}}{\int \Omega \wedge \overline{\Omega}}$. In conclusion the potential reads

$$V = \frac{1}{2Im\tau} \frac{i \int G \wedge \overline{\Omega} \int \overline{G} \wedge \Omega + g^{\alpha\bar{\beta}} \int G \wedge \chi_\alpha \int \overline{G} \wedge \chi_\beta}{\int \Omega \wedge \overline{\Omega}} \tag{4.30}$$

that compares with the $\mathcal{N} = 1$ potential (4.19) by using

$$W = \int G \wedge \Omega, \tag{4.31}$$

and $i = \tau, \rho, \alpha$. This matching can be seen explicitly by computing

$$\begin{aligned} D_\alpha W &= \int G \wedge D_\alpha \Omega = \int G \wedge \chi_\alpha \\ D_\tau W &= \frac{1}{\overline{\tau} - \tau} \int \overline{G} \wedge \Omega = \int \left[ -H - \frac{1}{\overline{\tau} - \tau} (F - \tau H) \wedge \Omega \right] \\ D_\rho W &= \overbrace{\partial_\rho W}^{0} + \partial_\rho K W = -\frac{3}{\rho - \bar{\rho}} W, \end{aligned} \tag{4.32}$$

It should also be noted that since the superpotential does not depend on the volume modulus $\rho$, the potential becomes positive definite, hence forcing supersymmetric vacua to be flat. Supersymmetric critical points are obtained for $D_i W = 0$,but since $\partial_\rho W = 0$, we can equate $D_\rho W = 0$ with $W = 0$:

$$\begin{aligned} W &= \int G \wedge \Omega &= 0 \Rightarrow G^{(0,3)} &= 0, \\ D_\alpha W &= \int G \wedge \chi_\alpha &= 0 \Rightarrow G^{(1,2)} &= 0, \\ D_\tau W &= \int \overline{G} \wedge \Omega &= 0 \Rightarrow G^{(3,0)} &= 0, \end{aligned} \tag{4.33}$$

It is remarkable that the same conditions can be obtained from the 10-dimensional supersymmetry conditions as a consistency check of the validity of the approximation used.

One interesting point of this analysis is that we can only fix the dilaton $\tau$ and the complex structure moduli $\phi^\alpha$, but nothing can be said on the value of the volume modulus, which can be fixed at any arbitrary value. We see from this example that fluxes help in stabilizing the moduli, but that it is quite common to obtain flat directions in the potential. Moreover, we learned that there is a clear difference between supersymmetric Minkowski and Anti-de Sitter vacua. AdS vacua do not allow for flat directions in the potential. A flat direction would imply $\partial_i W = 0$ and supersymmetry $D_i W = \partial_i W + K_i W = 0$, that means that the vacuum must be Minkowski.

At this point the question moves to the possible improvements of the situation in other theories, for instance in the IIA one. For the case at hand, one should not expect much improvements as the fluxless Calabi–Yau backgrounds exhibit a duality called mirror symmetry that exchanges the dual backgrounds and effective theories. If type IIB fluxes fix the complex structure moduli, we expect that the IIA one fixes only the Kähler ones. It should also be clear however that the deformations due to the fluxes can give different result, which deform away from the calabi–Yau condition by size deformations different than just a conformal factor. Indeed in type IIB supergravity the 3-form flux couples to $\Omega_3$ (The 5-form flux would couple to the harmonic $X_1$, but there is no such form on a Calabi–Yau), in type IIA the even-form fluxes $F_{(0,2,4,6)}$ couple to $J$, instead. As expected, the superpotential should therefore contain terms like

$$\int F_6 + \int J \wedge F_4 + \int J \wedge J \wedge F_2 + \int J \wedge J \wedge J F_0 \tag{4.34}$$

and stabilize the size moduli. The type IIA theory however contains also a 3-form flux in the common sector and therefore the superpotential may further contain terms of the form[1]

$$W = \int (H + i dJ) \wedge \Omega, \tag{4.35}$$

where $dJ \wedge \Omega$ is allowed by the fact that in type IIA the $Y_6$ manifolds specifying the background can be more general than conformal Calabi–Yaus. There is therefore the chance to stabilize everything and there are indeed examples of IIA AdS vacua! The first model of complete moduli stabilization has indeed been obtained in [27]. Of course at this point two kinds of questions arise:

- What are the IIB duals of backgrounds with all moduli stabilized?
- How do we study the effective theories on more complicated $Y_6$? (not conformally Calabi–Yau)

We are not going to discuss these two questions in detail here, but it should suffice to know that the answers are related to the so called twisted tori, for when $Y_6$ is given by one of these manifolds (or their orbifolds) we can indeed address both questions.

A simple heuristic explanation can be given in the following way. An ordinary straight 3-torus has a flat metric $ds^2 = dx^2 + dy^2 + dz^2$, which can be obtained by 3 vielbeins $e^1 = dx$, $e^2 = dy$ and $e^3 = dz$. In order to have a compact manifold we have to identify the coordinates as $x \sim x+1$, $y \sim y+1$, $z \sim z+1$. After this identification only a U(1)$^3$ symmetry generated by $\partial_x$, $\partial_y$ and $\partial_z$ survives. A twisting is a deformation of one of the circles defining the torus (or more than one), for

[1] This kind of contributions to the superpotential was first argued in [25, 26] for the Heterotic theory.

instance $e^3 = dz + gxdy$, so that the identifications get changed ($x \sim x+1$ must be accompanied by $z \sim z - gy$). This further implies that the new symmetries generated by $Z_3 = \partial_x$, $Z_1 = \partial_x + gy\partial_z$ and $Z_2 = \partial_y - gx\partial_z$ acquire a non-abelian structure:

$$[Z_1, Z_2] = -2gZ_3. \tag{4.36}$$

By this deformation we can expect to obtain once more gauged supergravities as effective theories and therefore non-trivial potentials for the moduli fields [28, 29]. But which are the moduli fields now?

When $g$ is small we can still use the moduli of the original torus as the light fields and consider the $g$-terms as deformations[2]. This type of backgrounds is not flat anymore, as there is a non-trivial connection $de^i = \omega^{ij}e^j \neq 0$ and this implies that $dJ \neq 0$ and/or $d\Omega \neq 0$ This further implies that more terms may appear in the superpotentials as we argued above for a similar case in type IIA.

We are now in a position that allows us to compare the generic potentials obtained in the two cases. If we call $T^i$ the Kähler moduli, $U^i$ the complex structure ones and $S$ the axio/dilaton field, we can roughly obtain the effective superpotentials from the above formulae (4.31), (4.34) and (4.35), by expanding the structure forms in terms of the harmonic ones $\Omega = U^i\alpha_i$, $J = T^i\beta_i$, with $\alpha_i \in H^2$ and $\beta_i \in H^3$. The resulting superpotentials have the following structure:

$$W_{IIB} = (c_i^1 + Sc_i^2)U^i, \tag{4.37}$$

$$W_{IIA} = c_0 + c_iT^i + d_{ij}T^iT^j + f_{ijk}T^iT^jT^k + h_iU^i + m_{ij}T^iU^j. \tag{4.38}$$

If there is any extension of the mirror symmetry duality when fluxes are present the two potentials should look the same by exchanging $T \leftrightarrow U$. It is on the other hand clear that this is not the case. This means that further investigations on the possible effective theories with fluxes should lead to more general possibilities that may encompass such dualities. Recent progresses in this study show that non-geometric compactifications, using T-folds, precisely lead to the extra terms one could argue in this way [31].

As a final brief comment I would like to recall that on top of this scenario, we can always add further non-perturbative effects (D3-instantons, gaugino condensation), as seen also in the lectures by Burgess at this school. These effects give further contributions to the superpotential of the form

$$W \sim \sum A_i \mathrm{e}^{i\alpha_i T^i} \tag{4.39}$$

and may lead also to stable de Sitter vacua [32], solving other long-standing problems in string theory. When looking at these results however, it should be clear that there is very little control on the actual values of the parameters $A_i$ and $\alpha_i$ and therefore on the argued result. The only firm constraint that can be established is whether such coefficients admit non vanishing values [33].

There are many more interesting results that have been obtained in the context of flux compactifications and that we could not even touch in these lectures. However, we hope that these notes may be useful for the interested reader as a basic introduction to the fundaments (with simple and hopefully clear examples) of the physics and mathematics involved in this sector of string theory.

[2] There are clear subtleties on the allowed values of $g$ and therefore on the validity of such approximation. The interested reader can find this problem addressed in [30].

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