PROCEEDINGS OF SCIENCE

AdS/CFT

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In this lectures we give a review of some basic, features of the gauge/gravity correspondence. We concentrate on the developements in the $1/2$ BPS sector were progress was accomplished through reduction to simple, solvable matrix models. We give a detailed discussion of the reduction both in supergravity and in Yang-Mills theory. We then discuss the physical picture and the relationship with the noncritical string.

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1. Introduction

A certain class of correlation functions of $\mathcal{N} = 4$ super Yang-Mills theory can be evaluated exactly. This follows from the fact that for this class of operators one has nonrenormalization theorems. A particularly simple subset is given by the multi-trace local operators belonging to the $1/2$-BPS sector. In this case, the evaluation of correlators reduces to a matrix model (of a complex matrix pair) which can be further mapped into the system of free nonrelativistic fermions [1, 2]. This, through its collective representation is described by a bosonic (droplet) theory in 1+1 dimensions. It is remarkable that an equivalent Fermi droplet picture can also be shown to arise in the AdS dual description. On the supergravity side [3] a nontrivial ansatz results in a class of (bubbling) geometries with the same amount of supersymmetry as the $1/2$- BPS states of the gauge theory. One is also able to demonstrate a precise agreement between the dynamics on both sides of the ADS/CFT correspondence. It is a misnomer to call the (matrix model) picture a 'toy model' as it represent an exact statement underlying the $AdS_5 \times S^5$ theory. It provides a deeper insight into the origin of holography through the same map that defined the relationship between the (old) $c = 1$ matrix model and the 2d noncritical string theory.

One of the basic features [4] of the AdS/CFT correspondence is the fact that CFT correlators are related to amplitudes of propagation of supergravity modes connecting AdS boundary to AdS boundary. This is reminiscent of a scattering picture where one has a projection to the on- shell surface. However a precise scattering interpretation of ADS/CFT was never fully implemented. Through Euclidean continuation of the LLM construction and the correspondence with the $c = 1$ theory we will be able to describe such an S-matrix interpretation.

Our review is in no sense complete. It concentrates on aspect of the correspondence which we consider most basic. For background one should consult some of the classic reviews on ADS/CFT [5],[6].

We will not follow the chronological development of the subject either. After a short introduction in sect.2 we describe first in detail the construction of LLM which provides a reduction of the theory in the supergravity sector. This reduction (from 10 to 2 dimensions) results in a dynamical system of a Fermi droplet associated with a matrix model construction. In section 4 we then give a detailed presentation of the collective fermion droplet theory itself. Here we discuss the scattering picture that provides the basis of the holographic interpretation. In the concluding section 5 we shortly comment on extensions and generalizations that are being contemplated both on the gauge theory and on the supergravity side. The reference list that we give is in no sense complete, we only give reference to works that directly concern the discussion in the text or material that we expect is of direct help to a student.

2. Yang-Mills reduction

In this section we set the notation and discuss briefly the origin of a reduced (matrix) model in the Yang-Mills picture. The goal is to quickly identify the model and proceed with its detailed study.

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The action of $\mathcal{N}=4$ SYM defined on $R\times S^3$ in $SU(4)$ notation, is given by

$$S=\frac{2}{g_{YM}^2}\int d^4x\,\mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}^2-\frac{1}{2}\left(D_\mu X_{AB}\right)^2\left(D^\mu X^{AB}\right)^2-\frac{1}{2}X_{AB}X^{AB}\right.$$
$$+\frac{1}{4}\left[X_{AB},X_{CD}\right]\left[X^{AB},X^{CD}\right]+\imath\overline{\lambda}_{+A}\not{D}\lambda_+^A$$
$$\left.+\overline{\lambda}_{+A}\left[X^{AB},\lambda_{-B}\right]+\overline{\lambda}_{-A}\left[X^{AB},\lambda_{+B}\right]\right)$$

The $SU(4)_R$ symmetry generators are

$$\delta\lambda_+^A=\imath T_B^A\lambda_+^A,\quad \delta\overline{\lambda}_-^A=-\imath T_B^A\overline{\lambda}_-^A,\quad \delta X^{AB}=\imath T_C^AX^{CB}+\imath T_C^BX^{AC}$$

giving the conserved charges

$$J_B^A=\frac{2}{g_{YM}^2}\int_{S^3}\mathrm{Tr}\left(-2\imath X^{AC}D_0X_{CB}-\overline{\lambda}_{+B}\gamma^0\lambda_+^A\right).$$

Early confirmation of the ADS/CFT correspondence came from comparison of correlation functions [7, 8] of selected operators in Yang-Mills theory with analogous observables of Supergravity. The most explicit results concern operators invariant under certain degree of supersymmetry[9]. Simplest is the set of operators preserving $1/2$ of Yang-Mills supersymmetries. In particular the chiral primary 1/2- BPS operators are characterized by $SO(4)$ symmetry and the conformal dimensions $\Delta=J$,

$$\mathcal{O}^J_{(J_1,J_2,\ldots,J_n)}(x)\equiv\mathrm{Tr}\left(Z(x)^{J_1}\right)\mathrm{Tr}\left(Z(x)^{J_2}\right)\cdots\mathrm{Tr}\left(Z(x)^{J_n}\right),\quad J=\sum_i J_i \tag{2.1}$$

where $Z=(\phi_5+i\phi_6)/\sqrt{2}$ is the complex scalar field with a unit R-charge $J=1$ with respect to the rotation in the 5-6 plane. Due to a non-renormalization property correlation functions of these operators with their conjugate set of operators constructed in terms of $\overline{Z}=(\phi_5-i\phi_6)/\sqrt{2}$ are given by the free-field results,

$$\langle\overline{\mathcal{O}}^J_{(J_1',J_2',\ldots,J_m')}(x')\,\mathcal{O}^J_{(J_1,J_2,\ldots,J_n)}(x)\rangle=f(\{(J'),(J)\},N)D_4(x,x')^J \tag{2.2}$$

where $D_4(x,x')\propto|x-x'|^{-2J}$ with $J=\sum_i J_i=\sum_i J_i'$ is the massless free-field propagator in 4 dimensions and the function $f(\{(J'),(J)\},N)$ is determined by the free-field contraction among the indices of scalar fields ϕ_i between $O^J(x)$ and $\overline{O}^J(x')$. As such the function f appearing in the numerator is completely independent of spacetime coordinates.

All the nontrivial information is contained in the function $f(\{(J'),(J)\},N)$. Concentrating on the numerator and ignoring the spacetime coordinates and the spacetime dependent factor $D_4(x,x')$ leads one to the matrix model picture. The matrix model is defined to reproduce the numerator appearing in the Yang-Mills correlator.

The degrees of freedom representing the matrix model can be deduced from a reduction of $\mathcal{N}=4$ Super Yang-Mills theory on $R\times S^3$. In particular the 6 Higgs fields become quantum mechanical matrix coordinates $\Phi_a(t)$, $a=1\ldots6$. For the study of $\frac{1}{2}$ BPS states (and corresponding giant gravitons) one can concentrate on the dynamics of a two matrix model. It is useful to perform the dimensional reduction on S^3 in some detail. Expanding the fields in spherical harmonics on S^3 we obtain the mass spectrum for the dimensionally reduced theory

$$M_{scalar} = l+1, \quad M_{fermion} = l+\frac{3}{2}, \quad M_{vector} = l+2$$

In particular for the scalars we write

$$X_{AB}(t,\Omega) = \frac{g_{YM}}{2}\sum_{l=0}^{\infty} x^l_{AB} Y^l(\Omega)$$

The free part of the Lagrangian for the scalars gives

$$L = \mathrm{Tr}\sum_{l=0}^{\infty}\left[\frac{1}{2}\dot{x}^l_m \dot{x}^{l\,m} - \frac{1}{2}(l+1)^2 x^l_m x^{l\,m}\right].$$

It is consistent to only keep

$$Z_l = \frac{2\sqrt{2}\pi}{g_{YM}} X^{12} = \frac{1}{2}\left(X^1_l + iX^4_l\right) \to e^{\imath\phi} Z$$
$$\bar{Z}_l = \frac{2\sqrt{2}\pi}{g_{YM}} X^{34} = \frac{1}{2}\left(X^1_l - iX^4_l\right) \to e^{-\imath\phi} \bar{Z}$$

After decomposing as

$$Z_l = \frac{1}{\sqrt{2(l+1)}}\left(A^\dagger_l + B_l\right)$$

the Hamiltonian reads

$$H = \mathrm{Tr}\sum_{l=0}^{\infty}(l+1)\left[B^\dagger_l B_l + A^\dagger_l A_l\right]$$

while the angular momentum is

$$J = \mathrm{Tr}\sum_{l=0}^{\infty}\left[A^\dagger_l a_l - B^\dagger_l B_l\right]$$

We see that $H = J$ means that we only keep the conjugate pair $\left(A^\dagger_0, A_0\right)$

$$H = \mathrm{Tr} A^\dagger_0 A_0$$

The eigenstates of this model are labeled by Young diagrams R of irreducible representations

$$\chi_R\left(A^\dagger_0\right) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\,\mathrm{Tr}\left(\sigma A^\dagger_0\right)$$

where

$$\mathrm{Tr}\left(\sigma A^\dagger_0\right) = \sum_{i_1,i_2,\ldots,i_n}\left(A^\dagger_0\right)^{i_1}_{i_{\sigma(1)}}\left(A^\dagger_0\right)^{i_2}_{i_{\sigma(2)}}\cdots\left(A^\dagger_0\right)^{i_n}_{i_{\sigma(n)}}$$

Characteristic examples are the *AdS* giant graviton

$$R = (L,0,\ldots,0), \quad L \approx N$$

and the multiply wound *AdS* giant graviton

$$R = (L, L, \ldots, 0), \quad L \approx N$$

In summary one can think of the reduction of 4d Yang-Mills theory in two stages. First one has the reduction to the zero modes and in the $SU(2)$ sector this results in the complex matrix model. Through study of $1/2$ BPS correlators it was realized that a further reduction occurs[1]. This second reduction is analogous to the case of Hall effect where a system of 2d fermions is further reduced to 1d. In terms of the canonical decomposition one has

$$Z = \frac{1}{\sqrt{2}}(A^\dagger + B), \quad \bar{Z} = \frac{1}{\sqrt{2}}(A + B^\dagger)$$

Then the creation-annihilation operator pair $A, A^\dagger$ determines the dynamics the 1/2 BPS sector. In parallel with the lowest Landau level condition the additional canonical pair $\left(B, B^\dagger\right)$ is eliminated.

The single creation-annihilation operator pair $A, A^\dagger$ can be used to define a Hermitian matrix model with a harmonic oscillator potential

$$L = \frac{1}{2}\mathrm{Tr}\left(\dot{M}^2 - M^2\right)$$

After diagonalizing

$$M_{ij} = diag\left(\lambda_1, \ldots, \lambda_N\right)$$

this results in the even simpler system of N free fermions. The dynamics of excitations above (and below) the Fermi surface is nontrivial. It was used in [1] to introduce the particle-hole picture of giant gravitons in AdS_5 and in S^5. The nonlinear dynamics underlying the theory is given by the bosonized (collective) field theory. The collective boson is

$$\phi\left(t, x\right) = \sum_{i=1}^{N} \delta\left(x - \lambda_i\right)$$

with the collective Hamiltonian is given by

$$H_{coll} = \int dx \left[\frac{1}{2}\Pi_{,x}\phi\Pi_{,x} + \frac{\pi^2}{6}\phi^3 + x^2\phi\right]$$

with the fields satisfying the standard Poisson brackets

$$\left\{\phi\left(x\right), \Pi\left(x'\right)\right\} = \delta\left(x - x'\right)$$

After the canonical transformation

$$y_\pm = \pm\pi\phi + \Pi_{,x}$$

the collective Hamiltonian takes the form

$$H_{coll} = \frac{1}{2}\int \frac{dx}{2\pi}\left[\frac{1}{3}\left(y_+^3 - y_-^3\right) + x^2\left(y_+ - y_-\right)\right]$$

with the Poisson brackets

$$\left\{y_\pm\left(x\right), y_\pm\left(y\right)\right\} = \pm 2\pi\partial_x\delta\left(x - y\right)$$

In this parametrization one one has the closed curve$(x, y(x))$ which represents the boundary of the fermionic droplet. Its dynamics is governed by the cubic collective Hamiltonian.

If we parametrize the boundary in polar coordinates which reflect the action-angle coordinates for the fermions on the boundary

$$\vec{r}(\phi,t) = \rho(\phi,t)\cos(\phi)\,\hat{x} + \rho(\phi,t)\sin(\phi)\,\hat{y}$$

and in this case we have

$$\partial_t \vec{r} \times \partial_\phi \vec{r} = \frac{1}{2}\partial_t \rho^2(\phi,t)$$
$$\partial_t \rho^2 = \partial_\phi \rho^2.$$

The above theory, but with the inverted harmonic oscillator potential was the basis for $c = 1$ / 2d string correspondence [10]. The collective provided an explicit mapping between the two sides giving the first example of an 'emerging' extra spatial dimension. The cubic interacting Hamiltonian was seen to correctly reproduce elements of scattering of 2d tachyons both at tree and loop level. For a review of these results the reader is referred to[11]. We will discuss their relevance for the ADS/CFT correspondence in Sect.4.

3. LLM Ansatz in ADS Supergravity

The simple dynamical model found on the gauge theory side can also demonstrated in 10d Supergravity.This was accomplished by Lin, Lunin and Maldacena who succeeded in identifying in $AdS_5 \times S^5$ Supergravity the degrees of freedom with $1/2$ of supersymmetry. What emerges is a reduction of the 10 dimensional theory to two dimensional 'bubbling' configurations whose dynamics is identical with the collective fermion droplet dynamics of the reduced matrix model.

It is useful to present the LLM reduction in some generality. Analogous 1/2 BPS reductions are possible for a larger class of SUGRA theories. The theory in question,type II B Supergravity in 10 dimensions includes in its field content the spacetime metric g_{MN} and the self-dual five form field strength $F_{M_1M_2M_3M_4M_5}$ which play the central role in the reduction. The Ansatz given by

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu + e^{H+G}d\hat{\Omega}_m^2 + e^{H-G}d\tilde{\Omega}_n^2 \\ F &= \hat{F}_{\mu_1\cdots\mu_{5-m}}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{5-m}}\wedge d\hat{\Omega}_m + \tilde{F}_{\mu_1\cdots\mu_{5-n}}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{5-n}}\wedge d\tilde{\Omega}_n. \end{aligned} \tag{3.1}$$

corresponds to a reduction on an $S^m \times S^n$ sphere with arbitrary (space-time) depending radii. One then has a metric on a resulting $10-n-m$ dimensional space-time described by the Greek indices $\mu, \nu = 1\ldots 10-m-n$. For the case of $\frac{1}{2}$ BPS configurations on $AdS_5 \times S^5$ one sets $m = 3$ and $n = 3$.

We register the useful identities

$$\sqrt{-G} = \sqrt{g_{10-m-n}}\,e^{\left(\frac{m+n}{2}\right)H}e^{\left(\frac{m-n}{2}\right)G}$$

which expresses the determinant of the ten dimensional metric G_{MN} in terms of the determinant of the metric of the base space $g_{\mu\nu}$ and the two scalars H and G. We also have for the spin connection

$$\omega_{\hat{\mu}m\hat{\nu}} = -\frac{1}{2}e^{\frac{1}{2}(H+G)}\hat{e}_{\hat{\mu}\hat{n}}e^\gamma_m\partial_\gamma(H+G)$$
$$\omega_{\tilde{\mu}m\tilde{\nu}} = -\frac{1}{2}e^{\frac{1}{2}(H-G)}\tilde{e}_{\tilde{\mu}\tilde{n}}e^\gamma_m\partial_\gamma(H-G)$$

where the English indices m,n are local Lorentz indices, the indices with a hat represent coordinates on $\hat{S}_m$ and the indices with a tilde note indices on $\tilde{S}_n$. The self-duality condition of the five form $F_5 = \star_{10} F_5$ gives the constrain

$$\hat{F}_{5-m} = \frac{(5-m)!}{(5-n)!} e^{\left(\frac{m+n}{2}\right)G} e^{\left(\frac{m-n}{2}\right)H} \star_{10-m-n} \tilde{F}_{5-n}.$$

For the case of $\frac{1}{2}$ BPS solutions on $AdS_5 \times S^5$ that we will follow one has $m=3$ and $n=3$. The equations of motion of the ten dimensional theory are given by

$$\begin{aligned} R_{MN} &= \frac{1}{2\,(4)!}\left[\left(F^2\right)_{MN} - \frac{1}{10} g_{MN} F^2\right] \\ dF &= 0 \\ \star F &= F. \end{aligned}$$

In terms of the 4 dimensional fields the reduced equations of motion are found to read

$$\begin{aligned} R_{\mu\nu} &= \frac{3}{2}\left(\partial_\mu H \partial_\nu H + \partial_\mu G \partial_\nu G\right) + 3\nabla_\mu \nabla_\nu H \\ &\quad + \frac{1}{2} e^{-3(H+G)} \left(F^2_{\mu\nu} - \frac{1}{4} g_{\mu\nu} F^2\right) + \frac{1}{2} e^{-3(H-G)} \left(\tilde{F}^2_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \tilde{F}^2\right) \\ \Box H + 3\partial^\mu H \partial_\mu H &= 4 e^{-H} \cosh G \\ \Box G + 3\partial^\mu G \partial_\mu G &= -\frac{1}{4} e^{-3(H+G)} F^2 + \frac{1}{4} e^{-3(H-G)} \tilde{F}^2 - 4 e^{-H} \sinh G. \end{aligned}$$

These reduced 4-dimensional dynamics can be associated with the Lagrangian

$$\mathscr{L}_4 = e \cdot e^{3H}\left[R + \frac{15}{2}\partial H^2 - \frac{3}{2}\partial G^2 - \frac{1}{4} e^{-3(H+G)} F^2_{\mu\nu} - \frac{1}{4} e^{-3(H-G)} \tilde{F}^2_{\mu\nu} + 12 e^{-H} \cosh G\right]$$

where one needs to impose the duality constrain

$$\tilde{F} = -\star_4 e^{3G} F$$

inherited from the ten dimensional duality constraints on the five-form. The second stage of reduction comes from the imposition of a 1/2 supersymmetry requirement. This follows from a detailed discussion of the spinorial equations which we shortly describe next.

3.1 Spinorial reduction

For the case of interest $m=3$ and $n=3$ one decomposes the gamma matrices Γ_M in the following way

$$\begin{aligned} \Gamma_\mu &= \gamma_\mu \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \hat{\sigma}_1 \\ \Gamma_\alpha &= \gamma_5 \otimes \sigma_\alpha \otimes \mathbb{I}_2 \otimes \hat{\sigma}_2 \\ \Gamma_{\tilde{\alpha}} &= \gamma_5 \otimes \mathbb{I}_2 \otimes \tilde{\sigma}_\alpha \otimes \hat{\sigma}_1 \end{aligned}$$

where $\gamma_5 = \imath\gamma_1 \ldots \gamma_4$, $\Gamma_{11} = \Gamma_1 \ldots \Gamma_{10} = \gamma_5 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \hat{\sigma}_3$ and the chirality condition for the IIB spinors gives

$$\Gamma_{11}\eta = \eta \Rightarrow \hat{\sigma}_3 \eta = \eta.$$

For the spinors we give the decomposition

$$\eta = \varepsilon \otimes \chi_\alpha \otimes \tilde{\chi}_b \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where the two spinors χ_α and $\tilde{\chi}_b$ are two Killing spinors on the unit sphere and they satisfy the Killing spinor equation

$$\begin{aligned} \hat{\nabla}_{\hat{a}} \chi_\alpha &= \frac{i\alpha}{2} \sigma_{\hat{a}} \chi_\alpha, \qquad a = \pm 1 \\ \tilde{\nabla}_{\tilde{\alpha}} \tilde{\chi}_b &= \frac{ib}{2} \sigma_{\tilde{\alpha}} \tilde{\chi}_b, \qquad b = \pm 1 \end{aligned}$$

The ten dimensional Killing spinor equation reads

$$\nabla_M \eta + \frac{\imath}{480} \Gamma^{M_1 \ldots M_5} F_{M_1 \ldots M_5} \Gamma_M \eta = 0. \tag{3.2}$$

The reduced equation now gives one differential and two algebraic constrains for the Killing spinor

$$\nabla_\mu \varepsilon - \imath N \gamma_\mu \varepsilon = 0 \tag{3.3}$$

$$\left[\frac{\imath\alpha}{2} e^{-\frac{1}{2}(H+G)} - \frac{i}{4} \gamma^\lambda \partial_\lambda (H+G) + N \right] \varepsilon = 0 \tag{3.4}$$

$$\left[\frac{\imath b}{2} e^{-\frac{1}{2}(H-G)} + \frac{1}{4} \gamma_5 \gamma^\lambda \partial_\lambda (H-G) - \imath \gamma_5 N \right] \varepsilon = 0 \tag{3.5}$$

where

$$N = -\frac{1}{4} \not{\hat{F}} e^{-\frac{3}{2}(G+H)}.$$

At this point we list a set of interesting spinor bilinears constructed from the Killing spinor ε

$$f_1 = \bar{\varepsilon} \gamma_5 \varepsilon \tag{3.6}$$

$$f_2 = \imath \bar{\varepsilon} \varepsilon \tag{3.7}$$

$$K_\mu = \bar{\varepsilon} \gamma_\mu \varepsilon \tag{3.8}$$

$$L_\mu = \bar{\varepsilon} \gamma_\mu \gamma_5 \varepsilon \tag{3.9}$$

$$Y_{\mu\lambda} = \imath \bar{\varepsilon} \gamma_{\mu\nu} \gamma_5 \varepsilon \tag{3.10}$$

$$V_{\mu\nu} = \bar{\varepsilon} \gamma_{\mu\nu} \varepsilon \tag{3.11}$$

With the aid of the Killing spinor conditions and the Fierz identities in four dimensions one can find the constraints imposed on the geometry. For our case, using the differential equation (3.3), one can show that the vector (3.8) is a Killing vector

$$\nabla_{(\mu} K_{\nu)} = 0 \tag{3.12}$$

and that the vector (3.9) gives a closed form

$$\nabla_{[\mu} L_{\nu]} = 0. \tag{3.13}$$

The Fierz identities constrain the two vectors to be orthogonal and the Killing vector to be timelike

$$L^2 = -K^2 = f_1^2 + f_2^2 \tag{3.14}$$

$$L \cdot K = 0. \tag{3.15}$$

Partial fixing of our gauge is done through the identification of the coordinate y such that

$$L_\mu = \gamma dy, \quad \gamma^2 = 1. \tag{3.16}$$

and also use the Killing vector to identify the time coordinate. Using the Killing spinor conditions one may also show for the scalars appearing in our ansatz that

$$f_1 = e^{\frac{1}{2}(H-G)} \tag{3.17}$$

$$f_2 = e^{\frac{1}{2}(H+G)} \tag{3.18}$$

$$e^H = \gamma y \tag{3.19}$$

At this point the ten dimensional metric takes the form

$$ds^2 = -\frac{1}{h^2}(dt+A)^2 + h^2 dy^2 + h^2 h_{mn} dx^m dx^n + \gamma y e^G d\hat{\Omega}_3^2 + \gamma y e^{-G} d\tilde{\Omega}_3^2, \qquad m = 1,\ldots,4. \tag{3.20}$$

At this point we can take advantage of the fact that the two dimensional space spanned by x_m, $m = 1,2$ and is equipped with the $y-$dependent metric

$$ds_2^2 = h_{mn}(x_m, y)\, dx^m dx^n \tag{3.21}$$

In order to draw conclusions about this two dimensional metric we consider the following bilinears

$$\omega_\mu = \varepsilon^t C \gamma_\mu \varepsilon$$

where C is the charge conjugation matrix

$$\gamma_\mu^T = -C\gamma_\mu C^{-1}.$$

Using the differential equation (3.3) from the Killing spinor conditions we see that

$$d\omega = 0. \tag{3.22}$$

At this point we will specify the form of the spinor ε in order to make use of (3.22). In order to do this we consider the linear combination of the projectors (3.4) and (3.5) in which we substitute the known expressions to write it as

$$\left(\gamma^3\left(a^2+b^2\right)^{\frac{1}{2}} - ae^{-G} + \imath\gamma_5 b\right)\varepsilon = 0$$

which is easily seen to be solved by

$$\varepsilon = e^{i\alpha\gamma_5\gamma_3}\varepsilon_1$$

where

$$\sinh\alpha = \frac{b}{a}e^G.$$

The Fierz identity also implies that the Killing spinor has to satisfy the projection

$$\gamma^0\gamma^5\gamma^3\varepsilon = -a\varepsilon$$

which gives us

$$\gamma^3\varepsilon_1 = \varepsilon_1.$$

Choosing in particular the chiral representation for the gamma matrices we can see that

$$\varepsilon_1 = \begin{bmatrix} \varepsilon_0 \\ 0 \\ -\imath\varepsilon_0 \\ 0 \end{bmatrix}.$$

Finally from the norm the bilinears f_1 and f_2 we have that

$$|\varepsilon_0|^2 = \frac{1}{2}e^{\frac{1}{2}(H+G)}\sinh(2\alpha).$$

Using the above form in equation (3.22) we see that the conformal factor is independent of y and that the two dimensional metric (3.21) is flat. The differential equations that we can construct for the scalars (3.6) and (3.7) determines the two form

$$F = -de^{2(G+H)}\wedge(dt+V) - h^2e^{3G}\star_3 de^{(H-G)}$$

where the hodge duality is meant with respect to the flat three dimensional space spanned by x_1, x_2 and y. Finally, using the Killing vector K_μ one can show that

$$\begin{aligned} \frac{1}{2}h^{-2}dA &= -h^2e^H\star_3 dG \\ dA &= -\frac{1}{2}y^{-1}\star_3 d\tanh G. \\ &= y^{-1}\star_3 dz \end{aligned}$$

The integrability condition coming from the last equation gives us a second order equations for the scalar z

$$\begin{aligned} d\left(y^{-1}\star_3 dz\right) = 0 \Rightarrow \\ \left(\partial_1^2 + \partial_2^2 + y\partial_y\frac{1}{y}\partial_y\right)z = 0. \end{aligned}$$

At this point we see that all the fields involved in our ansatz can be solved for as soon as the scalar z has been determined. In order to summarize we give the relevant equations

$$\begin{aligned} ds^2 &= -h^{-2}(dt+A)^2 + h^2\left(dy^2 + dx_m dx_m\right) + ye^G d\hat{\Omega}_3^2 + ye^{-G}d\tilde{\Omega}_3^2 \\ h^{-2} &= 2y\cosh G \\ y\partial_y V_m &= \varepsilon_{mn}\partial_n z, \qquad y(\partial_m V_n - \partial_n V_m) = \varepsilon_{mn}\partial_y z \\ z &= \frac{1}{2}\tanh G \\ F &= -de^{2(G+H)}\wedge(dt+V) - h^2e^{3G}\star_3 de^{(H-G)} \end{aligned} \tag{3.23}$$

We will see next how the regularity of the solution impose strong constraints on the function z.

3.2 Regularity of the solution and the ground state

As one may see from the form of (3.23) when y goes close to zero, with G remaining finite, the spacetime is in danger of a conical singularity. In order to have a regular space time we find that the appropriate boundary conditions for z at $y=0$ is $z(x_1,x_2,y=0)=\pm\frac{1}{2}$. We consider the case where $z\sim\frac{1}{2}-y^2 f(x_1,x_2)$ so that $e^{-G}\sim yc(x_1,x_2)$. We see that close to $y=0$ the metric in the y direction and the second three sphere combines to form the regular piece

$$h^2dy^2+ye^{-G}d\tilde{\Omega}_3^2\sim c(x)\left(dy^2+y^2d\tilde{\Omega}_3^2\right)$$

while the radius of the first sphere remaining finite. From the above considerations one can have an explicit picture of a fermion droplet in type IIB supergravity by identifying $z(x_1,x_2,y=0)$ with the fermion density.

Using the above set of boundary conditions for z we have its solution being given by

$$z(x_1,x_2,y)=\frac{y^2}{\pi}\int_{\mathscr{D}}\frac{z(x_1',x_2',y)\,dx_1'dx_2'}{\left[(\mathbf{x}-\mathbf{x}')^2+y^2\right]^2}$$

and

$$V_m(x_1,x_2,y)=\frac{\varepsilon_{mn}}{\pi}\int_{\mathscr{D}}\frac{z(x_1',x_2',y)\,(x_n-x_n')\,dx_1'dx_2'}{\left[(\mathbf{x}-\mathbf{x}')^2+y^2\right]^2}.$$

As an example we give the configuration that will give us $AdS_5\times S^5$ which is given by a circular configuration of fermions in the $y=0$ hyperplane

$$z(r,\phi,y=0)=\begin{cases}-1/2, r<l\\ \ \ 1/2, r>l\end{cases}$$
$$x_1=r\cos\phi$$
$$x_2=r\sin\phi.$$

The various fields involved in the solution are given by

$$z(r,\phi,y)=\frac{r^2+y^2-l^2}{2\sqrt{(r^2+y^2-l^2)^2+4y^2l^2}}$$
$$V(r,\phi,y)=\frac{r^2+y^2+l^2}{2\sqrt{(r^2+y^2-l^2)^2+4y^2l^2}}d\phi.$$

After making the change of coordinates

$$y=l\sinh\rho\,\sin\theta$$
$$r=l\cosh\rho\,\cos\theta$$

the fields are written as

$$
\begin{aligned}
z(\rho,\phi,\theta) &= \frac{1}{2}\frac{\sinh^2\rho - \sin^2\theta}{\sinh^2\rho + \sin^2\theta} \\
e^G &= \frac{\sinh\rho}{\sin\theta} \\
V(\rho,\phi,\theta) &= -\frac{1}{2}\frac{\cosh^2\rho + \cos^2\theta}{\cosh^2\rho - \cos^2\theta}d\phi .
\end{aligned}
$$

Using the above expressions in the solution (3.23) we have after a little algebra

$$
ds^2 = l\left(-\cosh^2\rho\left(dt - \frac{1}{2}d\phi\right)^2 + d\rho^2 + \sinh^2\rho d\hat{\Omega}_3^2 + \cos^2\theta\left(dt + \frac{1}{2}d\phi\right)^2 + d\theta^2 + \sin^2\theta d\tilde{\Omega}_3^2\right)
$$

where we observe the mixing between the coordinates ϕ of the sphere and the global AdS_5 time t. Defining

$$
\tilde{\phi} = \phi - \tau, \tag{3.24}
$$

the LLM metric above reduces to the standard $\mathrm{AdS}_5\times\mathrm{S}^5$ metric expressed in terms of the global coordinate,

$$
ds^2 = r_0\left[-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2 + d\theta^2 + \cos^2\theta d\tilde{\phi}^2 + \sin^2\theta d\tilde{\Omega}_3^2\right]. \tag{3.25}
$$

3.3 Euclidean continuation

It is useful at this point to describe how by double Wick rotation the above configuration of a circular droplet transforms into a hyperbolic one. The circular configuration represents the ground state of a harmonic oscillator the hyperbolic one is related to the ground state of an inverted oscillator. Through this one will have a connection with the c=1 theory. The relevance of an Euclidean picture in ADS was pointed out in [12, 13] in connection with the question of 'holography'. Consider then the transformation $\tau \to -i\tau, \phi \to -i\psi$ ($\tilde{\psi} \equiv \psi - \tau \to -i\tilde{\psi}$) under which both the AdS metric and the RR-fields are transformed 'covariantly' into the Euclideanized AdS (EAdS $_5\times$ $\mathrm{S}^{4,1}$) background with the metric,

$$
ds^2 = r_0\left[\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2 + d\theta^2 - \cos^2\theta d\tilde{\psi}^2 + \sin^2\theta d\tilde{\Omega}_3^2\right]. \tag{3.26}
$$

Since the signature of this metric is still $9+1$ in 10-dimensional sense, supersymmetries can be preserved by a suitable renaming of spinor variables. The two-dimensional coordinates (x_1,x_2) are then transformed as

$$
x_1 \to x_1 = r\cosh\psi, \quad x_2 \to ix_2 = ir\sinh\psi. \tag{3.27}
$$

This exercise implies that for discussing generic Euclideanized LLM ansatz, it is sufficient to make the double Wick rotations $x_2 \to ix_2$ and $\tau \to -i\tau$. The vector field V_i must also be rotated covariantly as $V_1 \to -iV_1$, $V_2 \to V_2$.

After the rotation the Laplace equation turns into a hyperbolic wave equation

$$
(\partial_1^2 - \partial_2^2)z + y\partial_y(\frac{\partial_y z}{y}) = 0. \tag{3.28}
$$

One can repeat and analyze the (nonsingular) solutions of this equation. One sees that for the EAdS solution solution is obtained by replacing the circular disk (with value $-1/2$) of the Lorentzian theory by an infinite domain ($0 < r < r_0$) in the wedge region bounded by a hyperbola at $r = r_0$. It can be checked by explicitly evaluating the above integral that the expression of z for the EAdS takes the same form as in the Lorentzian case

$$\begin{aligned}
&-\frac{y^2}{\pi}\mathrm{Im}\Big[\int_{\mathrm{D}} dx_1' dx_2' \frac{z(x_1',x_2',0)}{[(x_1-x_1')^2-(x_2-x_2')^2+y^2-i\varepsilon]^2}\Big] \\
&= 1/2 + \frac{y^2}{\pi}\mathrm{Im}\Big[\int_{\mathrm{H}} dx_1' dx_2' \frac{1}{[(x_1-x_1')^2-(x_2-x_2')^2+y^2-i\varepsilon]^2}\Big] \\
&= \frac{1}{2}\frac{r^2-r_0^2+y^2}{\sqrt{(r^2+r_0^2+y^2)^2-4r^2r_0^2}} \to \begin{cases} 1/2 & \text{if } r>r_0\,, y=0 \\ -1/2 & \text{if } r<r_0\,, y=0 \end{cases}
\end{aligned} \tag{3.29}$$

Thus one can define Euclideanized bubbling geometries by setting hyperbolical droplet at $y = 0$ in a similar way as in the Lorentzian case. In comparison with the circular droplet which is located close to the center of the geometries the hyperbolic droplet has an infinite range bounded by a hyperbola. Consequently there naturally emerges a possibility for scattering picture.

3.4 Energy and Symplectic Form

We now turn to the important topic of specifying the dynamics of the droplet. The energy of general 1/2 BPS configuration will be given by an integral over the droplets domain D. This was deduced in the original work of [3]. In addition it was also seen that the dynamical degree of freedom is contained in the (closed) curve, representing the boundary of the droplet. For coincidence with the fermion picture it is relevant to establish the symplectic structure (namely the Poisson brackets) of the dynamical curve. This was accomplished in [14] following a method formulated by Crnkovic and Witten [15].

For a theory with a Lagrangian density $\mathscr{L}\left(\phi^I, \partial\phi^I\right)$ one defines the (CWZ) symplectic current by

$$J^\mu = \delta\left(\frac{\partial L}{\partial\partial_\mu\phi^I}\right)\wedge\delta\phi^I$$

After specifying a Cauchy surface (e.g. $\mu = t$) the symplectic form is obtained from

$$\omega = \int dx^d J^t$$

For LLM geometries we have the Lagrangian

$$S = \int d^{10}x\sqrt{-g}\left(R - \frac{1}{2\cdot 5!}F_{(5)}^2\right)$$

The fields under study are $\phi^I = \left\{g_{mn}, A_{(4)}\right\}$ and the first order form of the action is

$$S = \int d^{10}x\sqrt{-g}\left(g^{ik}\Gamma^m_{il}\Gamma^l_{km} - g^{ik}\Gamma^l_{ik}\Gamma^m_{lm} - \frac{1}{2\cdot 5!}F_{(5)}^2\right)$$

The symplectic form density $J^l_{bulk} = J^l_G + J^l_F$ is then given by

$$J^l_G = -\delta\Gamma^l_{mn} \wedge \delta\left[\sqrt{-g}g^{mn}\right] + \delta\Gamma^n_{mn} \wedge \delta\left[\sqrt{-g}g^{lm}\right] \tag{3.30}$$

$$J^l_F = -\frac{1}{3}\delta\left(\sqrt{-g}F^{lk_1\ldots k_4}\right) \wedge \delta A_{k_1\ldots k_4} \tag{3.31}$$

It can be shown that the symplectic form is invariant under regular gauge transformations

$$\delta g_{mn} \to \delta g_{mn} + \nabla_{(m}\xi_{n)}$$
$$\delta A_{k_1\ldots k_4} \to \delta A_{k_1\ldots k_4} + \partial_{[k_1}\Lambda_{k_1\ldots k_3]}$$

For the case of the five form we have that under a gauge transformation the symplectic form transforms according to

$$\delta_\Lambda \omega_f = \int dx^d \delta_\Lambda J^0_F = -\frac{1}{3}\int dx^d \partial_k \left[\delta\left(\sqrt{-g}F^{0kk_1\ldots k_3}\right) \wedge \delta\Lambda_{k_1\ldots k_3}\right] \tag{3.32}$$

which of course is a total derivative and one should be extremely cautious dropping it.

It was realized [14] that the variation of the gauge field $\delta A_{k_1\ldots k_4}$, being in axial gauge, is singular at $y = 0$. For this reason one needs to perform a gauge transformation of the form

$$\delta\Lambda_{(4)} = -d\lambda \wedge d\Omega - d\tilde{\lambda} \wedge d\tilde{\Omega}$$

to regularize the variation everywhere. This transformation will give a finite contribution to the symplectic form by the addition of a boundary term at $y = 0$ as we may see from (3.32). The two scalar functions where determined close to $y = 0$ [14] by the regularity requirement

$$\delta B^{\text{reg}}_i\big|_{y=0} = 0$$
$$\delta \tilde{B}^{\text{reg}}_i\big|_{y=0} = 0 \quad i = 1,2$$

where the components δB^{reg}_i, $\delta\tilde{B}^{\text{reg}}_i$ appear in the variation of the 4-form field

$$\delta A^{\text{reg}}_{(4)} = \delta B^{\text{reg}}_i \wedge d\Omega + \delta\tilde{B}^{\text{reg}}_i \wedge d\tilde{\Omega}.$$

The bulk contribution, after summing (3.30) and (3.31), may be expressed as a total derivative

$$J^t_{bulk} = \partial_y I^y + \partial_i I^i, \quad i = 1,2$$

where

$$I^y = -\frac{y^4 z\left(\frac{1}{4}+z^2\right)}{\left(\frac{1}{4}-z^2\right)^2}\delta V_i \wedge \delta V_j - \left[\frac{y^4\varepsilon_{ij}V_j}{\frac{1}{4}-z^2} + \frac{2z^2y^4\varepsilon_{ij}V_j}{\left(\frac{1}{4}-z^2\right)^3}\right]\delta V_i \wedge \delta z$$
$$I^i = \frac{2y^3 z}{\frac{1}{4}-z^2}\delta V_i \wedge \delta z$$

After substitutions the symplectic form reads [14]

$$\begin{aligned}
\omega &= \int d^2xdy\, J^t_{bulk} + \int_{y=0} d^2x\, j_{bndr} \\
&= \int d^2xdy \left(\partial_y I^y + \partial_{x^i} I^{x^i}\right) + \int_{y=0} d^2x\, j_{bndr} \\
&= -\int_{y=0} d^2x I^y + \int_{y=0} d^2x\, j_{bndr} \\
&= \int_{y=0} d^2x \left[-\varepsilon_{ij}\delta V_i \wedge \delta\left(aV_j\right) - \varepsilon_{ij} a_i \wedge b_j + 8\left(\lambda \wedge \delta\tilde{F}_{12} - \tilde{\lambda}\,\delta F_{12}\right)\right].
\end{aligned}$$

where

$$\begin{aligned}
a_i &= \delta\left[\frac{y^2 V_i}{2\left(\frac{1}{4}-z^2\right)} + U_i\right] \\
b_i &= \delta\left[\frac{y^2 z V_i}{\frac{1}{4}-z^2}\right] \\
U_i &= \frac{\varepsilon_{ij}}{y^2} x_j \left(x^2+y^2\right) z \\
a &= -\frac{y^4 z\left(\frac{1}{4}+z^2\right)}{\left(\frac{1}{4}-z^2\right)^2}
\end{aligned}$$

The last expression may be further simplified to exactly match the collective field theory symplectic form

$$\omega = \frac{1}{32\pi\hbar}\oint\oint d\phi\, d\tilde{\phi}\, Sign\left(\phi-\tilde{\phi}\right)\delta\left[r^2\left(\phi\right)\right]\wedge\delta\left[r^2\left(\tilde{\phi}\right)\right] \tag{3.33}$$

In order to give the expression for the energy of these configurations (or the angular momentum since they saturate the BPS bound) we will follow [3] and at the geometry asymptotically. This will allow us to compare with the reduction of the type IIB theory to $SO(6)$ gauged five dimensional supergravity [16] and identify the electric charge of one of the $U(1)$ gauge fields which is for the LLM class of solutions remains non-trivial. At this point we will just give the asymptotic form without any of the details of the calculation

$$\begin{aligned}
ds^2 &= \left[1 + \frac{(3u^2-2Q)(Q^2-2W)+6(Q-u^2)\tilde{W}\cos 2\phi}{6Q^2v^2}\right. \\
&\quad \left. + \frac{((2Q-3u^2)(Q^2-2W)-6(Q-u^2)\tilde{W}\cos 2\phi)^2}{48Q^4v^4} - \frac{2g_2}{v^4}\right] \\
&\quad \times \left\{-\left(v^2+Q+\frac{Q^2-2W}{3v^2}\right)dt^2 + Q\frac{dv^2}{v^2}\left(1-\frac{Q}{v^2}\right) + v^2 d\tilde{\Omega}_3^2\right\} \\
&\quad + g_{uu}du^2 + 2g_{u\phi}duD\phi + g_{\phi\phi}D\phi^2 + g_{\Omega\Omega}d\Omega_3^2 \\
D\phi &= d\phi + dt - \frac{2W-Q^2}{Qv^2}dt \qquad (3.34)\\
Q^2 &= \frac{1}{\pi}\int_D d^2x' \qquad (3.35)\\
W &= Q^{-2}\frac{1}{\pi}\int_D d^2x' \left|x'\right|^2. \qquad (3.36)
\end{aligned}$$

Written in this form we have grouped in the first terms the *AdS* coordinates and in the second piece the S^5 part with the mixing caused by the $U(1)$ gauge field that appears in the form (3.34). From the above expression for the asymptotic form of the gauge field we can read off its charge or, since this is a BPS solution, the energy. It is given by a surface integral over the domain D:

$$\begin{aligned} H &= \frac{1}{2}\int_{\mathscr{D}} \frac{d^2x}{2\pi\hbar}\frac{x_1^2+x_2^2}{\hbar} - \frac{1}{2}\left(\int_{\mathscr{D}} \frac{d^2x}{2\pi\hbar}\right)^2 \\ &= \frac{1}{8\pi\hbar^2}\int d\phi\, \rho^4(\phi) - \frac{1}{2}\left(\int d\phi\, \frac{\rho^2(\phi)}{2\pi\hbar}\right)^2 \end{aligned}$$

Evaluating the integral over x_2 (up to its boundary) leads to an expression identical to the collective field Hamiltonian. To summarize we have seen that the energy of 1/2 BPS configurations in supergravity is given [3] by the collective Hamiltonian of the Fermi droplet, and that that the symplectic form of the droplets boundary coming from supergravity also reduces [14] to that commutation relations of collective field theory .

4. The Fermi droplet model/continued

We continue in this section with a more detailed description of the droplet model itself and the physical picture that it provides for ADS/CFT. We follow the model with Euclidean time. It was seen that EADS translates into a c=1 theory with the inverted harmonic oscillator potential:

$$S_{c=1} = \int d\tau \frac{1}{2}\mathrm{Tr}\Big[\Big(\frac{dM}{d\tau}\Big)^2 + M^2\Big] \tag{4.1}$$

and consider the operators

$$\Pi_\pm = (M \pm \dot{M})/\sqrt{2}. \tag{4.2}$$

The correlators are then given as

$$\langle \mathscr{O}^J_{(+;J_1',\dots,J_m')}(\tau')\, \mathscr{O}^J_{(-;J_1,\dots,J_n)}(\tau)\rangle = f(\{(J'),(J)\},N)D_1(\tau,\tau')^J \tag{4.3}$$

with

$$\mathscr{O}^J_{(\pm;J_1,\dots,J_n)}(\tau_1) \equiv \mathrm{Tr}\Big(\Pi_\pm^{J_1}\Big)\mathrm{Tr}\Big(\Pi_\pm^{J_2}\Big)\cdots\mathrm{Tr}\Big(\Pi_\pm^{J_n}\Big). \tag{4.4}$$

Note that we now have $D_1(\tau,\tau') \propto \exp(-|\tau-\tau'|)$.

Consider now the Hamiltonian formalism of collective field theory [11]. The equation (4.20) can be recast in the Hamiltonian form

$$\partial_\tau p_\pm = i[H, p_\pm] \tag{4.5}$$

by defining the effective Hamiltonian and the commutation relations as

$$H = \int dx\Big[\Big(\frac{p_+^3}{6} - \frac{(x^2+\mu)p_+}{2}\Big) - \Big(\frac{p_-^3}{6} - \frac{(x^2+\mu)p_-}{2}\Big)\Big], \tag{4.6}$$

$$[p_\pm(x,\tau), p_\pm(x',\tau)] = \mp i\delta'(x-x'). \tag{4.7}$$

Here, μ is an arbitrary constant, corresponding to an integration constant for the equation of motion. It represents the chemical potential and can be eliminated in terms of N.

Note also that we use the usual Lorentzian convention in writing down the equations of motion, by interpreting (4.5) as being due to the inverted harmonic potential $V(x) = -x$.

After a shift $\tilde{\phi}_\pm$:

$$p_\pm(x,\tau) = \pm(\sqrt{x^2+\mu} + \tilde{\phi}_\pm(x,\tau)), \tag{4.8}$$

$$H = \int dx \Big[\frac{1}{2}\sqrt{x^2+\mu}(\tilde{\phi}_+^2 + \tilde{\phi}_-^2) + \frac{1}{6}(\tilde{\phi}_+^3 + \tilde{\phi}_-^3)\Big], \tag{4.9}$$

and a change a change of variables $x = \mu \sinh\sigma$, $\tilde{\phi}_\pm = \left|\frac{d\sigma}{dx}\right|\phi_\pm$, one has

$$H = \int_0^\infty d\sigma \left[\frac{1}{2}(\phi_+^2 + \phi_-^2) + \frac{1}{6}\left|\frac{d\sigma}{dx}\right|^2 (\phi_+^3 + \phi_-^3)\right]. \tag{4.10}$$

The mode expansion in the interaction representation,

$$\phi_\pm(\sigma,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\xi e^{-i\xi(\tau\mp\sigma)}\alpha(\xi) \tag{4.11}$$

with $[\alpha(\xi),\alpha(\xi')] = -\omega\delta(\xi+\xi')$ which can be identified with the normal-mode operators of $\chi_\pm$ introduced above in the asymptotic region $|\sigma| \sim q \to \infty$, leads to

$$H = H_2 + H_3(\tau) \tag{4.12}$$

$$H_2 = \int_0^\infty d\xi\, \alpha(\xi)\alpha(-\xi) \tag{4.13}$$

$$H_3(\tau) = \frac{1}{6}\int_{-\infty}^{\infty} d^3\xi\, f(\xi_1+\xi_2+\xi_3)\, e^{-i(\xi_1+\xi_2+\xi_3)\tau}\alpha(\xi_1)\alpha(\xi_2)\alpha(\xi_3) \tag{4.14}$$

with

$$f(\xi) = \int_{-\infty}^{\infty} d\sigma \frac{1}{\mu\cosh^2\sigma} e^{i\xi\sigma} = \frac{2\pi\xi}{\mu\sinh\pi\xi}. \tag{4.15}$$

The form factor defining the cubic vertex of this collective field theory is signified by a non-trivial functional dependence on the momenta and the absence of momentum conservation. This is very peculiar from the viewpoint of Yang-Mills correlators where one had a conservation. This is one of the puzzles that will be explained away in the discussion that follows. We note that the nontrivial momentum dependent vertex is quite central for the correctness of the theory. Its precise form is crucial for the ability to generate higher point tree and loop amplitudes. This check was confirmed in explicit calculations[17] were agreement with 2d string theory was demonstrated.

4.1 Scattering

There is a simple method [18] for deriving the S-matrix in this theory which we now shortly summarize.

One considers the classical equations of motion which can be exactly solved. The simplest is to recall the description from supergravity. The shape of a general droplet configuration can be parametrized as:

$$x_1 = a(\psi)\cosh\psi, \quad x_2 = a(\psi)\sinh\psi, \tag{4.16}$$

$$a(\psi) = r_0 + \tilde{a}(\psi). \tag{4.17}$$

The function $\tilde{a}(\psi)$ describes the droplet shape, it can be used to parametrize fluctuations above the ground state. Explicit time dependence is exhibited by the coordinate transformation: $\tilde{\psi} = \psi - \tau$. This gives the form

$$x_1 = a(\tilde{\psi})\cosh(\tau + \tilde{\psi}), \quad x_2 = a(\tilde{\psi})\sinh(\tau + \tilde{\psi}). \tag{4.18}$$

which is explicitly time dependent. Note also that in this representation one has a simple harmonic oscillator equation of motion

$$\frac{d^2 x_i}{d\tau^2} = x_i, \tag{4.19}$$

which is obtained by a Wick rotation from the corresponding equation of AdS .

Correspondence with the fermion phase space is given by the identification $x_1 \to x, x_2 \to p$. One can then show the profile in the phase space $(x, p_\pm(x,\tau))$ obeys the (nonlinear) collective field equation

$$\frac{\partial}{\partial\tau} p_\pm = x - p_\pm \frac{\partial}{\partial x} p_\pm \tag{4.20}$$

where the suffix $+$ and $-$ denotes the two regions $p_+ > 0$ and $p_- < 0$, and the upper and lower boundary respectively: $p_\pm = \pm\sqrt{x^2 - a^2}$. The S-matrix is by definition given by considering the relation between two asymptotic regions $\tau \to \pm\infty$.

For sufficiently large x one has

$$x = e^q, \quad p_\pm = \pm e^q \mp e^{-q}\varepsilon_\pm(q,\tau). \tag{4.21}$$

and also $x \sim \frac{a(\psi)}{2} e^{-\tau-\psi}$ and $\sim \frac{a(\psi)}{2} e^{\tau+\psi}$ for $\tau \to \mp\infty$ respectively, the ε fields behave, with respect to the dependence on τ, as $\varepsilon_\pm(q,\tau) \sim \varepsilon_\pm(\tau \mp q)$.

One then calculates (from the known exact solution) the travel time for an incoming wave at (large x) to turn and reach the same location as an outgoing wave :

$$\tau_f - \tau_i = 2q + \log\frac{a(\psi)^2}{4}, \quad \varepsilon_-(q,\tau_i) = \varepsilon_+(q,\tau_f) = \frac{a(\psi)^2}{2} \tag{4.22}$$

This leads to a functional equation

$$\varepsilon_+(\tau - q) = \varepsilon_-(\tau - q - \log\frac{\varepsilon_+(\tau - q)}{2}). \tag{4.23}$$

relating the in-coming and the out-going packet.

This is nothing but the S-matrix .

In terms of the normal-mode operators in the momentum representation, we have

$$\alpha_\pm(\eta) = \sum_{p=1}^{\infty}\left(\frac{2}{c_0^2}\right)^{p-1}\frac{\Gamma(1 \mp i\eta)}{\Gamma(2 \mp i\eta - p)}\frac{1}{p!}\int d^p\xi\,\delta(\eta - \sum\xi_i)\Big(\prod \alpha_\mp(\xi_i)\Big) \tag{4.24}$$

and the S-matrix in the classical (tree) approximation is given by

$$S(\sum\omega_i \to \sum\omega_i') = \langle 0|\prod_i \alpha_-(-\omega_i')\prod_j \alpha_+(\omega_j)|0\rangle. \tag{4.25}$$

For example, for $n \to 1$ scattering ($n \geq 2$), the S-matrix elements are, up to the delta-function of energy conservation ($\omega = \omega_1 + \cdots \omega_n$) which we will always suppress in what follows,

$$\langle 0|\alpha_-(-\omega)\alpha_+(\omega_1)\cdots\alpha_+(\omega_n)|0\rangle \Rightarrow \left(\frac{2}{c_0^2}\right)^{n-1}(-i\omega)(-i\omega-1)\cdots(-i\omega-n+2)\,\omega_1\cdots\omega_n. \quad (4.26)$$

In applying these results to the $1/2$ BPS case, we have to Wick-rotate the momentum (=$\pm$energy on the mass-shell) as $\xi = \omega \to iJ$ with J being the R-charge.

4.2 Correlators as S-matrix amplitudes

We now come to the discussion of the duality implied by the $c = 1$ model which as we have seen plays a central role in the higher dimensional ADS/CFT correspondence. We will describe the 'scattering interpretation' given in [19]. It exactly identifies correlators of 1/2 BPS operators as scattering amplitudes of the 2 dimensional noncritical string.

Consider the simplest correlator $m = n = 1$,

$$\langle \mathrm{Tr}\overline{\mathrm{Z}}^J \mathrm{Tr}\mathrm{Z}^J\rangle = JN^J. \quad (4.27)$$

should be interpreted as the trivial $(1 \to 1)$ S-matrix element

$$\langle 0|\alpha_-(-iJ)\alpha_+(iJ)|0\rangle = J. \quad (4.28)$$

As we have noted before one also has a δ-function factor $-i\delta(J-J) = -i\delta(0)$ imposing energy conservation, which is common for all S-matrix elements. It is this energy conserving delta function of d=2 scattering amplitudes that compares with the R-charge conservation appearing in the 1/2 BPS correlators.Then, with the normalization $\alpha(iJ) \leftrightarrow \frac{1}{N^{J/2}}\mathrm{Tr}\mathrm{Z}^J$ for incoming states and $\alpha(-iJ) \leftrightarrow \frac{1}{N^{J/2}}\mathrm{Tr}\overline{\mathrm{Z}}^J$ for outgoing states we see agreement between the S-matrix element and the correlator.

The further examples (in the leading planar approximation of the large N limit) for $n = 2,3,4$ ($J = \sum_{i=1}^n J_i$).

$$f(J,\{J_1,J_2\},N)_{planar} = JJ_1J_2N^{-1}, \quad (4.29)$$

$$\begin{aligned} f(J,\{J_1,J_2,J_3\},N)_{planar} &= J\Big(J_1J_2J_3(J_1-1)+J_1J_2J_3(J_2-1) \\ &+J_1J_2J_3(J_3-1)+2J_1J_2J_3\Big)N^{-2} = J_1J_2J_3J(J-1)N^{-2} \end{aligned} \quad (4.30)$$

again agree simply with the corresponding expressions for the noncritical string. The case of the 3-point function is well known. It was noticed in [20] and also earlier in [21] that three point correlators contain a factor similar to the vertex of collective field theory. The puzzle was that it was a momentum conserving vertex characteristic of fermions on the circle. Its origin is clarified by the S-matrix interpretation of[19], one has an on-shell condition and the energy conserving delta function associated with the the S-matrix.

There is an elegant way to to argue that an arbitrary 1/2 BPS correlators of Yang-Mills theory coincides in tree approximation with the S-matrix of. Using the coherent-state representation in which $\Pi_-(\Pi_+) \sim A^\dagger(A)$ are regarded as generalized coordinate (z) and momentum (α), respectively.

$$\begin{aligned} \mathrm{Tr}\left(\Pi_-^J\right) &\rightarrow \int \frac{dz}{2\pi}\int^{\alpha} d\alpha\, z^J = \int \frac{dz}{2\pi} z^J \alpha(z) = \alpha_{-J}, \\ \mathrm{Tr}\left(\Pi_+^J\right) &\rightarrow \int \frac{dz}{2\pi}\int^{\alpha} d\alpha\, \alpha^J = \int \frac{dz}{2\pi}\, \frac{\alpha(z)^{J+1}}{J+1}. \end{aligned} \tag{4.31}$$

It is relevant to note that there is also a (dual) coherent state representation in which

$$\begin{aligned} \mathrm{Tr}\Pi_-^J &= \beta_J = \int \frac{dz}{2\pi} z^{-J}\beta(z) \\ \mathrm{Tr}\left(\Pi_+^J\right) &= \int \frac{dz}{2\pi}\frac{\beta(z)^{J+1}}{J+1} \end{aligned} \tag{4.32}$$

The correlator then becomes

$$\langle 0| \int \frac{dz_1}{2\pi}\, \frac{\alpha(z_1)^{J_1'+1}}{J_1'+1} \cdots \int \frac{dz_m \alpha(z_m)^{J_m'+1}}{J_m'+1}\; \alpha_{-J_1}\alpha_{-J_2}\cdots\alpha_{-J_n}|0\rangle.$$

In this picture, the operators $\mathrm{Tr}\left(\Pi_-^J\right)$ are simply creation operators while the operators $\mathrm{Tr}\left(\Pi_+^{J'}\right)$ are nontrivial polynomials. The relationship between the two representations is given through the $(c=1)$ S-matrix operator:

$$\mathrm{Tr}\left(\Pi_+^J\right) = S^{-1}\alpha_J S.$$

Consequently, the above matrix elements are

$$\langle \quad \rangle = \langle 0|\alpha_{J_1'}\cdots\alpha_{J_m'} S\alpha_{-J_1}\alpha_{-J_2}\cdots\alpha_{-J_n}|0\rangle$$

This gives the basis for the correspondence between the 1/2-BPS correlators and $c=1$ S-matrix at the planar approximation.

It is relevant to have this correspondence at higher genus also. On the bulk side, evaluating of loop effects in the (E)AdS background is an unsolved problem. In gauge theory and simple matrix models, the existence of a fermionic representation allows for calculations at arbitrary finite N and integer J.

For the correlators one has exact general formulas [22], given by linear combinations of the ratios of Gamma functions, such as (say $n = 1,2$)

$$\langle \mathrm{Tr}(\overline{Z}^J)\mathrm{Tr}(Z^J)\rangle = \frac{1}{J+1}\Big(\frac{\Gamma(N+J+1)}{\Gamma(N)} - \frac{\Gamma(N+1)}{\Gamma(N-J)}\Big), \tag{4.33}$$

$$\begin{aligned} \langle \mathrm{Tr}(\overline{Z}^J)\mathrm{Tr}(Z^{J_1})\mathrm{Tr}(Z^{J_2})\rangle = \frac{1}{J_1+J_2+1}\Big(&\frac{\Gamma(N+J_1+J_2+1)}{\Gamma(N)} + \frac{\Gamma(N+1)}{\Gamma(N-J_1-J_2)} \\ &- \frac{\Gamma(N+J_1+1)}{\Gamma(N-J_2)} - \frac{\Gamma(N+J_2+1)}{\Gamma(N-J_1)}\Big), \quad etc \end{aligned} \tag{4.34}$$

We can perform an all orders comparison for example of the two point correlator with the $1\rightarrow 1$ c=1 theory amplitude, at large-J. The exact c=1 string amplitudes can be deduced through the fermionic picture. In the coherent state representation, one has the Hamiltonian for a single fermion

$$h = \left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right) - \mu. \tag{4.35}$$

The wave-function can be taken as functions of a_- or (the dual picture) of a_+. One has the simple transform from one basis to another

$$\psi_-(a_+) = \frac{1}{\sqrt{2\pi}} \int da_- e^{-a_+a_-} \psi_+(a_-). \tag{4.36}$$

The wave-functions both obey

$$\frac{1}{2}\left(\hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+\right)\psi_\pm = \left(i\partial_\tau + \mu\right)\psi_\pm. \tag{4.37}$$

In the coherent state representation with $\hat{a}_- = a\,\hat{a}_+ = \partial/\partial a$, one has the equation

$$\left(a\frac{\partial}{\partial a} + \frac{1}{2} - \mu\right)\psi_{+,k}(a) = k\psi_{+,k}(a). \tag{4.38}$$

From the viewpoint of the fermion phase-space (x, p), creation-annihilation coordinates are the null plane coordinates $a_\pm \to x_\pm \equiv (p \pm x)/\sqrt{2}$. The discussion of scattering theory in the fermionic picture can be found in [23, 24].

The fermionic wave-functions $\psi_\pm(x_+)$ obey the analogous Schrödinger eqs. The equation is solved by

$$\psi_{+,k}(x_-) = A_k x_-^{-i(k-\mu)}\Theta(x_-) + B_k(-x_-)^{-i(k-\mu)}\Theta(-x_-) \tag{4.39}$$

with an analogous solution for ψ_-. The (Gaussian) transform then leads to the relation between the *in* and *out* fermion wave-function with the following reflection coefficient (see ref(O'Loughlin))

$$R(k-\mu) = \frac{\Gamma\left(\frac{1}{2} - i(k-\mu)\right)}{\sqrt{2\pi}}\left[e^{i\frac{\pi}{2}\left(\frac{1}{2}-(k-\mu)\right)} + \gamma e^{-i\frac{\pi}{2}(k-\mu)}\right] \tag{4.40}$$

where γ specifies the boundary conditions.

One next uses bosonization to make contact with the collective field

$$\psi_\pm^\dagger(z)\psi_\pm(z) = \alpha_\pm(z) \tag{4.41}$$

and consequently the $\hat{S}$-matrix is determined by

$$\langle 0|\prod_{k=1}^{m}\sum_n \left(R^*\psi^\dagger\right)_n (R\psi)_{l_k+n}\,\alpha_{-j_1}\alpha_{-j_2}\cdots\alpha_{-j_n}|0\rangle. \tag{4.42}$$

The fermion method gives the following exact expression [23] for the two-point amplitude

$$\frac{\partial}{\partial\mu}\langle V_q V_{-q}\rangle = \Gamma(-|q|)^2\,\mathrm{Im}\left\{e^{i\pi|q|/2}\left(\frac{\Gamma\left(|q|+\frac{1}{2}-i\mu\right)}{\Gamma\left(\frac{1}{2}-i\mu\right)} - \frac{\Gamma\left(\frac{1}{2}-i\mu\right)}{\Gamma\left(-|q|+\frac{1}{2}-i\mu\right)}\right)\right\} \tag{4.43}$$

In the large q limit, one has that

$$\frac{\partial}{\partial\mu}R(q,-q) = \mu^{-1}|q|\,R(q,-q)$$

which follows from the fact that the factor $\mu^{|q|}$ gives the dominant effect. Consequently at large q (only) we can use the formula

$$R(q,-q) \approx \frac{1}{|q|}\mu \frac{\partial}{\partial \mu} R(q,-q) \tag{4.44}$$

and

$$\Gamma(-|q|)^{-2}R(q,-q) \approx \frac{1}{2|q|i}\left\{ e^{i\frac{\pi}{2}|q|}\left(\frac{\Gamma(q+\frac{1}{2}-i\mu)}{\Gamma(\frac{1}{2}-i\mu)} - \frac{\Gamma(\frac{1}{2}-i\mu)}{\Gamma(-q+\frac{1}{2}-i\mu)}\right) \right.$$
$$\left. - e^{-i\frac{\pi}{2}|q|}\left(\frac{\Gamma(q+\frac{1}{2}+i\mu)}{\Gamma(\frac{1}{2}+i\mu)} - \frac{\Gamma(\frac{1}{2}+i\mu)}{\Gamma(-q+\frac{1}{2}+i\mu)}\right)\right\}.$$

This can be seen to be in agreement with the gauge-theory correlator $G(J)$, under the replacement $-\mu^2 \to N^2$ with an overall factor $\pm i$. The sign is consistent with the correspondence we found at the planar level.

5. Extensions

What we have reviewed is a fairly complete duality map involving the $1/2$ BPS sector of the theory. A central role in this map is played by the harmonic oscillator matrix model. Through Euclidean continuation one has a connection with the old $c = 1$ model and the $d = 2$ string. This presents then a simple physical basis for the gauge/gravity correspondence analogous to the one of the noncritical string.

Extensions which are in progress or are being contemplated are the following. On the Yang-Mills side one is interested in understanding the large N dynamics of a matrix system with more that a single matrix. In particular the study of $1/4$ or $1/8$ BPS states requires the participation of other Higgs matrices. Some recent progress in extending the map involving two(or more) matrices at the level of linearized fluctuations can be found in [25]. On the gravity side there has been definite progress in reduction of the full supergravity to a $1/4$ and $1/8$ BPS sector [26]. A future direction is finding a way to perform a comparison between the two sides. Recently very definite progress was accomplished in understanding full string theory on $AdS_5 \times S^5$ [27, 28]. It is likely that some of the results found in the SUGRA BPS sector could be of relevance to string theory. For the study of 'bubbling' configurations in open string field theory the reader should consult [29].

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