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Functional relations in the eight-vertex model

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We observe that the exactly solved *eight-vertex solid-on-solid model* contains an hitherto unnoticed arbitrary field parameter, similar to the horizontal field in the six-vertex model. The parameter is required to describe a continuous spectrum of the *unrestricted* solid-on-solid model, which has an infinite-dimensional space of states even for a finite lattice. The introduction of the continuous field parameter allows us to completely review the theory of functional relations in the eight-vertex/SOS-model from a uniform analytic point of view.

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1. Introduction

The theory of integrable quantum systems originated from Bethe's seminal work [1] has numerous important applications in physical and mathematical sciences. The mathematical foundation of this theory is based to a great extent on powerful analytic and algebraic techniques discovered by Baxter in his pioneering papers [2–5] on the exact solution of the eight-vertex lattice model. This paper concerns one of these techniques — the method of functional relations. Over the last three decades, since Baxter's original works [2–5], this method has been substantially developed and applied to a large number of various solvable models. However, the status of this method in the eight-vertex model itself with an account of all subsequent developments has not been recently reviewed. This paper is intended to (partially) fill this gap. Here we will adopt an analytic approach exploiting the existence of an (hitherto unnoticed) continuous field parameter in the solvable eight-vertex solid-on-solid model of ref. [4].

For the purpose of the following discussion it will be useful to first summarize the key results of [2–5]. Here we will use essentially the same notations as those in [2]. Consider the homogeneous eight-vertex (8V) model on a square lattice of N columns, with periodic boundary conditions. The model contains three arbitrary parameters u, η and $\mathsf{q} = e^{i\pi\tau}$, $\operatorname{Im}\tau > 0$, which enter the parametrization of the Boltzmann weights (the parameter q enters as the nome for the elliptic theta-functions). The parameters η and q are considered as constants and the spectral parameter u as a complex variable. We assume that the parameter η is real and positive, $0 < \eta < \pi/2$, which corresponds to the disordered regime [6] of the model.

The row-to-row transfer matrix of the model, $\mathbf{T}(u)$, possesses remarkable analytic properties. Any of its eigenvalues, $\mathsf{T}(u)$, is both (i) an entire function of the variable u, and (ii) satisfies Baxter's famous functional equation,

$$\mathsf{T}(u)\,\mathsf{Q}(u) = \mathsf{f}(u-\eta)\,\mathsf{Q}(u+2\eta) + \mathsf{f}(u+\eta)\,\mathsf{Q}(u-2\eta), \tag{1.1}$$

where [1]

$$\mathsf{f}(u) = \big(\vartheta_4(u\,|\,\mathsf{q})\big)^N, \tag{1.2}$$

and $\mathsf{Q}(u)$ is an entire quasi-periodic function of u, such that

$$\mathsf{Q}(u+\pi) = \pm(-1)^{N/2}\,\mathsf{Q}(u), \qquad \mathsf{Q}(u+2\pi\tau) = \mathsf{q}^{-2N}\,e^{-2iuN}\,\mathsf{Q}(u). \tag{1.3}$$

These analytic properties completely determine all eigenvalues of the transfer matrix $\mathbf{T}(u)$. Indeed, Eq.(1.1) implies that the zeroes $u_1, u_2, \ldots, u_n$, of $\mathsf{Q}(u)$ satisfy the Bethe Ansatz equations,

$$\frac{\mathsf{f}(u_k+\eta)}{\mathsf{f}(u_k-\eta)} = -\frac{\mathsf{Q}(u_k+2\eta)}{\mathsf{Q}(u_k-2\eta)}, \qquad \mathsf{Q}(u_k) = 0, \qquad k = 1,\ldots,n\,. \tag{1.4}$$

These equations, together with the periodicity relations (1.3), define the entire function $\mathsf{Q}(u)$ (there will be many solutions corresponding to different eigenvectors). Once $\mathsf{Q}(u)$ is known the eigenvalue $\mathsf{T}(u)$ is evaluated from (1.1).

[1]Here we use the standard theta-functions [7], $\vartheta_i(u\,|\,\mathsf{q})$, $i = 1,\ldots,4$, $\mathsf{q} = e^{i\pi\tau}$, $\operatorname{Im}\tau > 0$, with the periods π and $\pi\tau$. Our spectral parameter u is shifted with respect to that in [2] by a half of the imaginary period, see Sect 2.5.1 for further details.

The entire functions $\mathsf{Q}(u)$ appearing in (1.1) are, in fact, eigenvalues of another matrix, $\mathbf{Q}(u)$, called the **Q**-matrix. Originally it was constructed [2] in terms of some special transfer matrices. A different, but related, construction of the **Q**-matrix was given in [3] and later on used in the book [6]. An alternative approach to the 8V-model was developed in [4, 5] where Baxter invented the *"eight-vertex" solid-on-solid* (SOS) model and solved it exactly by means of the co-ordinate Bethe Ansatz. This approach provided another derivation of the same result (1.1)-(1.4), since the 8V-model is embedded within the SOS-model.

Baxter's **Q**-matrix (or the **Q**-operator) possesses various exceptional properties and plays an important role in many aspects of the theory of integrable systems. A complete theory of the **Q**-operator in the 8V-model is not yet developed. However for the simpler models related with the quantum affine algebra $U_q(\widehat{sl_2})$ (where the fundamental **L**-operators [8] are intertwined by the **R**-matrix of the six-vertex model) the properties of the **Q**-operator are very well understood [9]. In this case the **Q**-operators (actually, there are two different **Q**-operators, $\mathbf{Q}_+$ and $\mathbf{Q}_-$) are defined as traces of certain monodromy matrices associated with infinite-dimensional representations of the so-called q-oscillator algebra. The main algebraic properties of the **Q**-operators can be concisely expressed by a single factorization relation

$$\mathbf{T}_j^+(u) = \mathbf{Q}_+(u+(2j+1)\eta)\mathbf{Q}_-(u-(2j+1)\eta) \tag{1.5}$$

where $\mathbf{T}_j^+(u)$ is the transfer matrix associated with the infinite-dimensional highest weight representation of $U_q(sl_2)$ with an arbitrary (complex) weight $2j$. Remarkably, this relation alone leads to a simple derivation of *all* functional relations involving various "fusion" transfer matrices and **Q**-operators [9,10]. For this reason Eq.(1.5) can be regarded as a *fundamental fusion relation*: once it is derived, no further algebraic work is required.

An important part of the theory of the **Q**-operators belongs to their analytic properties with respect to a certain parameter, which we call here the "field parameter". In the context of conformal field theory (considered in [9, 10]) this is the "vacuum parameter", which determines the Virasoro highest weight Δ; in the six-vertex model it corresponds to the horizontal field. In fact, the very existence of two different solutions [9,11] of the TQ-equation (1.1) can be simply illustrated by the fact that the spectrum of the transfer matrix does not depend on the sign of the field, whereas the spectrum of the **Q**-operator does.

It is well known that it is impossible to introduce an arbitrary field parameter into the "zero-field" or "symmetric" eight vertex model of [2] without destroying its integrability. However, such parameter is intrinsically present in the solvable SOS-model. It does not enter the Boltzmann weights, but arises from a proper definition of the space of states of the model. To realize this recall that the SOS-model [5] is an interaction-round-a-face model where the face variables ℓ_i (called the heights) take arbitrary integer values $-\infty < \ell_i < +\infty$. Its transfer matrix acts in an infinite-dimensional space of states even for a finite lattice. It has a continuous spectrum, parameterized by the eigenvalue of the operator which simultaneously increments all height variables, $\ell_i \to \ell_i + 1$, on the lattice. Indeed, taking into account the results of [12, 13], it is not difficult to conclude that the calculations of [5] require only a very simple modification to deduce that the eigenvalues of the SOS transfer matrix enjoy the same TQ-equation (1.1), but require different periodicity properties

$$\mathsf{Q}_\pm(u+\pi) = e^{\pm i\varphi}\,\mathsf{Q}_\pm(u), \qquad \mathsf{Q}_\pm(u+2\pi\tau) = \mathsf{q}^{-2N}\,e^{\pm\psi}\,e^{-2iuN}\,\mathsf{Q}_\pm(u), \tag{1.6}$$

where the exponent φ is arbitrary. It is determined by the eigenvalue $\omega = e^{2i\eta\varphi/\pi}$ of the height translation operator[2] (the second exponent ψ is dependent on φ). It is natural to assume, that the functions $\mathsf{Q}_\pm(u)$ solving these equations, are eigenvalues of the **Q**-operators for the SOS-model. Of course, it would be very desirable to obtain their explicit definition (and generalize the algebraic result (1.5) to the SOS-model), however, many properties of these operators can already be deduced from the information about their eigenvalues.

In this paper we will develop the analytic theory of the functional relations for the SOS-model starting from the eigenvalue equations (1.1) and (1.6). Bearing in mind that the TQ-equation (1.1) arises from very non-trivial algebraic "fusion" relations [2], it is not surprising that it implies all other functional relations. The required calculations are essentially the same as those in [9, 10], apart from trivial modifications arising in the context of lattice models.

The eigenvalues $\mathsf{Q}_\pm(u)$ are two linear independent "Bloch wave" solutions [9,11] of the finite difference equation (1.1) for the same $\mathsf{T}(u)$. Their quantum Wronskian $\mathsf{W}(\varphi)$, defined as,

$$2i\,\mathsf{W}(\varphi)\,\mathsf{f}(u) = \mathsf{Q}_+(u+\eta)\,\mathsf{Q}_-(u-\eta) - \mathsf{Q}_+(u-\eta)\,\mathsf{Q}_-(u+\eta)\,, \tag{1.7}$$

is a complicated function of φ, η and q, depending on the eigenvalue $\mathsf{T}(u)$. The Bloch solutions $\mathsf{Q}_\pm(u)$ are well defined provided the exponent φ does not take some "singular values" (see Eq.(2.17) below), where $\mathsf{W}(\varphi)$ vanishes. Otherwise Eq.(1.1) has only one quasi-periodic solution, while the second linear independent solution does not possess any simple periodicity properties.

All singular cases (in fact, they split into different classes) can be effectively studied with a limiting procedure starting from a non-singular value of φ. In the simplest case, when η is generic and φ approaches the points $\varphi = k\pi$, $k\in\mathbb{Z}$, the solutions $\mathsf{Q}_+(u)$ and $\mathsf{Q}_-(u)$ smoothly approach the same value (which for even N coincides with the eigenvalue $\mathsf{Q}(u)$ of the 8V-model).

A more complicated situation occurs when the field φ tends to a singular value, say $\varphi = 0$, while the parameter η simultaneously approaches some rational fraction of π, where the transfer matrix of the 8V-model has degenerate eigenvalues. The limiting value of $\mathsf{T}(u)$ is always uniquely defined. However, if $\mathsf{T}(u)$ is a degenerate eigenvalue, the limiting values of $\mathsf{Q}_\pm(u)$ are not uniquely defined. They have "complete exact strings" of zeroes whose position can be made arbitrary by changing the direction of the two-parameter (η,φ)-limit. Obviously, this reflects a non-uniqueness of eigenvectors for degenerate states [14]. An immediate consequence of this phenomenon is that, for rational η, there is no unique algebraic definition of the **Q**-operator in the symmetric 8V-model. This explains an important observation of [15], that Baxter's two **Q**-operators, constructed in [2] and [3], are actually different operators, with different eigenvalues for degenerate eigenstates.

Further, the eigenvalues $\mathsf{Q}_\pm(u)$, considered as functions of φ, have rather complicated analytic properties. Besides having the (relatively simple) singular points discussed above, they are multi-valued functions with algebraic branching points in the complex φ-plane. These analytic properties are studied in an extended version of this article [16].

[2] In [5] Baxter restricted the parameter η to the "rational" values $L\eta = m_1\pi + m_2\pi\tau,\ L,m_1,m_2\in\mathbb{Z}$ and considered a finite-dimensional subspace of the whole space of states, regarding the values of heights *modulo L*. In this case the phase factors $\omega = e^{2i\eta\varphi/\pi}$ take quantized values $\omega^L = 1$ (see [12, 14] for further discussion of this point). Apart from providing the conceptual advantage of a finite-dimensional space of states, the above restriction on η and ω was not used anywhere else in [5] and, therefore, can be removed. The transfer matrix of the 8V-vertex model (reformulated as the SOS-model) acts only in the finite-dimensional subspace of the SOS space of states, corresponding to a discrete set of exponents $\varphi = k\pi$ and $\psi = 0$ (the value of N is assumed to be even).

2. Functional relations in the eight-vertex SOS model.

2.1 Overview

In this section we will outline the analytic theory of the functional relations in the SOS-model (which also covers the symmetric 8V-model). Actually most of the functional relations discussed below are quite universal and apply to a wider class of related model. They include the six-vertex model in a field [17], the restricted solid-on-solid (RSOS) model [18] and some integrable models of quantum field theory: the $c<1$ conformal field theory [9] and the massive sine-Gordon model in a finite volume [19].

Let $\mathsf{T}(u)$ and $\mathsf{Q}(u)$ denote the eigenvalues of the transfer matrix and the **Q**-operator respectively and η is an arbitrary real parameter in the range

$$0<\eta<\pi/2\ . \tag{2.1}$$

In all subsequent derivations we will use only one general assumption about the properties of the eigenvalues:

We assume that $\mathsf{T}(u)$ *and* $\mathsf{f}(u)$ *are* entire periodic *function of the variable u,*

$$\mathsf{T}(u+\pi)=\mathsf{T}(u), \qquad \mathsf{f}(u+\pi)=\mathsf{f}(u)\ , \tag{2.2}$$

and that the function $\mathsf{Q}(u)$ *solving the TQ-equation,*

$$\mathsf{T}(u)\,\mathsf{Q}(u)=\mathsf{f}(u-\eta)\,\mathsf{Q}(u+2\eta)+\mathsf{f}(u+\eta)\,\mathsf{Q}(u-2\eta)\ , \tag{2.3}$$

is a also an entire *(but not necessarily periodic) function of u.*

For every particular model the above requirements are supplemented by additional, model-specific analyticity properties of $\mathsf{Q}(u)$ (such as, for example, the imaginary period relation (1.3) for the 8V-model). These properties are discussed at the end of this section.

As explained in the Introduction, once the additional analyticity properties are fixed, the functional equation (2.3) completely determines all eigenvalues $\mathsf{T}(u)$ and $\mathsf{Q}(u)$. For certain applications, however, it is more convenient to use other functional equations in addition to (or instead of) (2.3). We will show that all such additional functional relations in the SOS-model (and in the related models mentioned above) follow elementary from two ingredients:

(i) the TQ-equation itself (Eqs. (2.2) and (2.3) above), and

(ii) the fact that for the same eigenvalue $\mathsf{T}(u)$ this equation has two different [9, 11] linearly independent solutions for $\mathsf{Q}(u)$ which are entire functions of u.

The only property of the function $\mathsf{f}(u)$ essentially used in this Section is its periodicity (2.2). For technical reasons we will also assume that $\mathsf{f}(u-\eta)$ and $\mathsf{f}(u+\eta)$ do not have common zeroes. This is a very mild assumption, excluding rather exotic row-inhomogeneous models, which are beyond the scope of this paper.

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2.2 General functional relations.

Since $\mathsf{T}(u)$ is an entire function, Eq.(2.3) implies that the zeroes $u_1, u_2, \ldots, u_n$ of any eigenvalue $\mathsf{Q}(u)$ satisfy the same set of the Bethe Ansatz equations

$$\frac{\mathsf{f}(u_k+\eta)}{\mathsf{f}(u_k-\eta)} = -\frac{\mathsf{Q}(u_k+2\eta)}{\mathsf{Q}(u_k-2\eta)}, \qquad \mathsf{Q}(u_k)=0, \qquad k=1,\ldots,n\,, \tag{2.4}$$

where the number of zeroes, n, is determined by the model-specific analyticity properties.

For any given eigenvalue $\mathsf{T}(u)$ introduce an infinite set of functions $\mathsf{T}_k(u)$, $k=3,4,\ldots\infty$, defined by the recurrence relation

$$\mathsf{T}_k(u+\eta)\,\mathsf{T}_k(u-\eta) = \mathsf{f}(u+k\eta)\,\mathsf{f}(u-k\eta)+\mathsf{T}_{k-1}(u)\,\mathsf{T}_{k+1}(u), \qquad k\geq 2\,, \tag{2.5}$$

where

$$\mathsf{T}_0(u)\equiv 0, \qquad \mathsf{T}_1(u)\equiv \mathsf{f}(u), \qquad \mathsf{T}_2(u)\equiv \mathsf{T}(u)\,. \tag{2.6}$$

This relation can be equivalently rewritten as

$$\mathsf{T}(u)\,\mathsf{T}_k(u+k\eta) = \mathsf{f}(u-\eta)\,\mathsf{T}_{k-1}(u+(k+1)\eta)+\mathsf{f}(u+\eta)\,\mathsf{T}_{k+1}(u+(k-1)\eta)\,, \tag{2.7a}$$

or as

$$\mathsf{T}(u)\,\mathsf{T}_k(u-k\eta) = \mathsf{f}(u+\eta)\,\mathsf{T}_{k-1}(u-(k+1)\eta)+\mathsf{f}(u-\eta)\,\mathsf{T}_{k+1}(u-(k-1)\eta)\,. \tag{2.7b}$$

Using the definition (2.5) one can easily express $\mathsf{T}_k(u)$ in terms of $\mathsf{T}(u)$ as a determinant

$$\mathsf{T}_k(u) = \big(\mathsf{f}^{(k)}(u)\big)^{-1}\ \det\Big\|\,\mathsf{M}_{ab}(u+k\eta)\,\Big\|_{1\leq a,b\leq k-1}, \qquad k\geq 2, \tag{2.8}$$

where the $(k-1)$ by $(k-1)$ matrix $\mathsf{M}(u)_{ab}$, $1\leq a,b\leq k-1$, is given by

$$\mathsf{M}_{ab}(u) = \delta_{a,b}\,\mathsf{T}(u-2a\eta)-\delta_{a,b+1}\,\mathsf{f}(u-(2a+1)\eta)-\delta_{a+1,b}\,\mathsf{f}(u-(2a-1)\eta)\,, \tag{2.9}$$

while the normalization factor reads

$$\mathsf{f}^{(k)}(u) = \prod_{\ell=0}^{k-3}\ \mathsf{f}(u-(k-3-2\ell)\eta)\,. \tag{2.10}$$

Finally, expressing $\mathsf{T}(u)$ from (2.3) through the corresponding eigenvalue $\mathsf{Q}(u)$ one arrives to the formula

$$\mathsf{T}_k(u) = \mathsf{Q}(u-k\eta)\,\mathsf{Q}(u+k\eta)\sum_{\ell=0}^{k-1}\frac{\mathsf{f}(u+(2\ell-k+1)\eta)}{\mathsf{Q}(u+(2\ell-k)\eta)\,\mathsf{Q}(u+(2\ell-k+2)\eta)}\,, \tag{2.11}$$

valid for $k=1,2,\ldots,\infty$. Note, that the Bethe Ansatz equations (2.4) guarantee that all the higher $\mathsf{T}_k(u)$ with $k\geq 3$ are entire functions of u as well as $\mathsf{T}(u)$. It is worth noting that these functions are actually eigenvalues of the "higher" transfer matrices, obtained through the algebraic fusion procedure [20]. In our analytic approach this information is, of course, lost. Nevertheless it will be useful to have in mind that the index k in the notation $\mathsf{T}_k(u)$ refers to the dimension of the "auxiliary" space in the definition of the corresponding transfer matrix. Another convenient scheme of notation for higher transfer matrices (used, e.g., in [10]) is based on (half-)integer spin labels j, such that $k = 2j+1$.

In a generic case Eq.(2.3) has two linear independent "Bloch wave" solutions $\mathsf{Q}_\pm(u)$, defined by their quasi-periodicity properties,

$$\mathsf{Q}_\pm(u+\pi) = e^{\pm i\varphi}\mathsf{Q}_\pm(u), \tag{2.12}$$

where the exponent φ depends on the eigenvalue $\mathsf{T}(u)$. These solutions satisfy the quantum Wronskian relation

$$2i\mathsf{W}(\varphi)\,\mathsf{f}(u) = \mathsf{Q}_+(u+\eta)\,\mathsf{Q}_-(u-\eta) - \mathsf{Q}_+(u-\eta)\,\mathsf{Q}_-(u+\eta)\ , \tag{2.13}$$

where $\mathsf{W}(\varphi)$ does not depend on u. Indeed, equating the two alternative expressions for $\mathsf{T}(u)$,

$$\mathsf{T}(u)\,\mathsf{Q}_+(u) = \mathsf{f}(u-\eta)\,\mathsf{Q}_+(u+2\eta) + \mathsf{f}(u+\eta)\,\mathsf{Q}_+(u-2\eta)\ , \tag{2.14}$$

and

$$\mathsf{T}(u)\,\mathsf{Q}_-(u) = \mathsf{f}(u-\eta)\,\mathsf{Q}_-(u+2\eta) + \mathsf{f}(u+\eta)\,\mathsf{Q}_-(u-2\eta)\ , \tag{2.15}$$

and writing $\mathsf{W}(\varphi)$ as $\mathsf{W}(\varphi|u)$ (to assume its possible u-dependence, which cannot be ruled out just from the definition (2.13)) one gets $\mathsf{W}(\varphi|u+\eta) = \mathsf{W}(\varphi|u-\eta)$. On the other hand, Eqs.(2.13), (2.2) and (2.12) imply a different periodicity relation $\mathsf{W}(\varphi|v+\pi) = \mathsf{W}(\varphi|v)$. For generic real η, these two periodicity relations can only be compatible if $\mathsf{W}(\varphi|u)$ is independent of u,

$$\mathsf{W}(\varphi|u) \equiv \mathsf{W}(\varphi)\ . \tag{2.16}$$

When η and φ are in general position, the eigenvalues $\mathsf{Q}_\pm(u)$ are locally analytic functions of η, therefore, by continuity, Eq.(2.16) at generic φ holds also when η/π is a rational number. However, when φ takes special values (for example, in the symmetric 8-vertex model) Eq.(2.16) for rational η/π cannot be established by the analytic arguments only.

Obviously, the condition (2.12) defines $\mathsf{Q}_\pm(u)$ up to arbitrary u-independent normalization factors. Using this freedom, it is convenient to assume the normalization[3] such that neither of $\mathsf{Q}_\pm(u)$ vanishes identically (as a function of u) or diverges at any value of φ. Then the quantum Wronskian $\mathsf{W}(\varphi)$ will take finite values, but still can vanish at certain isolated values of the exponent φ. These values are called *singular* in the sense that there is only one quasi-periodic solution (2.12), while the second linear independent solution of (2.3) does not possess the simple periodicity properties (2.12). As argued in [9], the singular exponents take values in the "dangerous" set

$$\varphi_{dang} = k\pi + \frac{\pi^2}{2\eta}\,\ell\ , \qquad k,\ell\in\mathbb{Z}\ . \tag{2.17}$$

However, each eigenvalue has its own set of singular exponents, being a subset of (2.17).

[3]In the context of the 8V/SOS-model this is the most natural normalization. The eigenvalues $\mathsf{Q}_\pm(u)$ are factorized in products of theta functions and the variation of φ only affects positions of zeroes. Obviously, the transfer matrix eigenvalues, $\mathsf{T}(u)$, do not have any singularities in φ.

Evidently, $\mathsf{Q}(u)$ in (2.11) can be substituted by any of the two Bloch solutions $\mathsf{Q}_{\pm}(u)$, so there are two alternative expression for each $\mathsf{T}_k(u)$. Further, multiplying (2.14) and (2.15) by $\mathsf{Q}_{-}(u-2\eta)$ and $\mathsf{Q}_{+}(u-2\eta)$ respectively, subtracting resulting equations and using (2.13) one obtains

$$2i\mathsf{W}(\varphi)\,\mathsf{T}(u)=\mathsf{Q}_{+}(u+2\eta)\,\mathsf{Q}_{-}(u-2\eta)-\mathsf{Q}_{+}(u-2\eta)\,\mathsf{Q}_{-}(u+2\eta)\;. \tag{2.18}$$

The last result, combined with the determinant formula (2.8), gives

$$2i\mathsf{W}(\varphi)\,\mathsf{T}_k(u)=\mathsf{Q}_{+}(u+k\eta)\,\mathsf{Q}_{-}(u-k\eta)-\mathsf{Q}_{+}(u-k\eta)\,\mathsf{Q}_{-}(u+k\eta), \tag{2.19}$$

where $k=0,1,2,\ldots,\infty$.

All the functional relations presented above are general corollaries of the TQ-equation (2.2), (2.3).

2.3 Rational values of η

Let us now assume that

$$2L\eta=m\pi,\qquad 1\leq m\leq L-1,\qquad L\geq 2, \tag{2.20}$$

where m and L are mutually prime integers. Evidently,

$$2k\eta\neq 0\pmod{\pi},\qquad 1\leq k\leq L-1\;. \tag{2.21}$$

Combining the expression (2.11) with (2.2) and (2.12), one immediately obtains the following functional relation,

$$\mathsf{T}_{L+k}(u)=2\cos(m\varphi)\,\mathsf{T}_k(u+\tfrac{1}{2}m\pi)+\mathsf{T}_{L-k}(u),\qquad k=1,2,\ldots\;. \tag{2.22}$$

which shows that, for the rational η of the form (2.20), all higher $\mathsf{T}_k(u)$ with $k\geq L$ are the linear combinations of a finite number of the lower $\mathsf{T}_k(u)$ with $k\leq L$. This relation is a simple corollary of the TQ-equation. It always holds for the rational values of η and does not require the existence of the second Bloch solution in (2.12) (indeed, Eq.(2.22) is independent of the sign of φ). Setting $k=1$ in (2.22) one obtains

$$\mathsf{T}_{L+1}(u)=2\cos(m\varphi)\,\mathsf{f}(u+\tfrac{1}{2}m\pi)+\mathsf{T}_{L-1}(u)\;. \tag{2.23}$$

This allows one to bring Eq.(2.5) with $k=L$ to the form

$$\mathsf{T}_L(u+\eta)\,\mathsf{T}_L(u-\eta)=\Big(\mathsf{f}(u+\tfrac{1}{2}m\pi)+e^{im\varphi}\,\mathsf{T}_{L-1}(u)\Big)\Big(\mathsf{f}(u+\tfrac{1}{2}m\pi)+e^{-im\varphi}\,\mathsf{T}_{L-1}(u)\Big) \tag{2.24}$$

where the periodicity (2.2) of the function $\mathsf{f}(u)$ was taken into account. Thus, for the rational η, the equations (2.5) with $k=2,3,\ldots,L-1$ together with Eq.(2.24) form a closed system of functional equations for a set of $L-1$ eigenvalues $\{\mathsf{T}(u),\mathsf{T}_3(u),\ldots,\mathsf{T}_L(u)\}$. Given that all $\mathsf{T}_k(u)$ with $k\geq 3$ are recursively defined through $\mathsf{T}(u)$, this system of equation leads to a single equation involving $\mathsf{T}(u)$ only. Indeed, substituting the determinant formulae (2.8) into (2.23) one obtains

$$\det\Big\|\,\overline{\mathsf{M}}_{ab}(u)\,\Big\|_{1\leq a,b\leq L}=0\;, \tag{2.25}$$

where the L by L matrix reads

$$\overline{\mathsf{M}}_{ab}(u)=\mathsf{M}_{ab}(u)-\omega\,\delta_{a,1}\,\delta_{b,L}\,\mathsf{f}(u-3\eta)-\omega^{-1}\,\delta_{a,L}\,\delta_{b,1}\,\mathsf{f}(u+\eta) \tag{2.26}$$

with $\mathsf{M}_{ab}(u)$ given by (2.9) and $\omega=e^{\pm im\varphi}$.

2.3.1 Non-zero quantum Wronskian

Continuing the consideration of the rational case (2.20), let us additionally assume that both quasi-periodic solutions (2.12) exist and that their quantum Wronskian (2.13) is non-zero. It is worth noting that the functions $\mathsf{Q}_\pm(u)$ in this case cannot contain *complete exact strings*. A complete exact string (or, simply, a *complete string*) is a ring of L zeroes $u_1,\ldots,u_L$, where each consecutive zero differs from the previous one by 2η, closing over the period π,

$$u_{k+1} = u_k + 2\eta, \qquad k = 1,\ldots,L, \qquad u_{L+1} = u_1 \pmod{\pi} . \tag{2.27}$$

It is easy to see that any such string manifests itself as a factor in the RHS of (2.13), but not in its LHS (unless, of course, $\mathsf{W}(\varphi) = 0$).

It follows from (2.12) that

$$\mathsf{Q}_\pm(u+m\pi) = e^{\pm im\varphi}\mathsf{Q}_\pm(u) . \tag{2.28}$$

Using (2.19) and (2.20) one easily obtains the two equivalent relations,

$$e^{+im\varphi}\,\mathsf{T}_k(u) + \mathsf{T}_{L-k}(u+\tfrac{1}{2}m\pi) = \mathsf{C}(\varphi)\;\mathsf{Q}_+(u+k\eta)\,\mathsf{Q}_-(u-k\eta) , \tag{2.29a}$$

and

$$e^{-im\varphi}\,\mathsf{T}_k(u) + \mathsf{T}_{L-k}(u+\tfrac{1}{2}m\pi) = \mathsf{C}(\varphi)\;\mathsf{Q}_+(u-k\eta)\,\mathsf{Q}_-(u+k\eta) , \tag{2.29b}$$

where $k = 0,1,\ldots,L$ and

$$\mathsf{C}(\varphi) = \frac{\sin(m\varphi)}{\mathsf{W}(\varphi)} . \tag{2.30}$$

In particular, for $k = 0$ one gets,

$$\mathsf{T}_L(u+\tfrac{1}{2}m\pi) = \mathsf{C}(\varphi)\;\mathsf{Q}_+(u)\,\mathsf{Q}_-(u) . \tag{2.31}$$

Quote also one simple but useful[4] consequence of (2.29),

$$\log\frac{\mathsf{Q}_+(u+k\eta)}{\mathsf{Q}_-(u+k\eta)} - \log\frac{\mathsf{Q}_+(u-k\eta)}{\mathsf{Q}_-(u-k\eta)} = \log\left(\frac{e^{+im\varphi}\,\mathsf{T}_k(u) + \mathsf{T}_{L-k}(u+\frac{1}{2}m\pi)}{e^{-im\varphi}\,\mathsf{T}_k(u) + \mathsf{T}_{L-k}(u+\frac{1}{2}m\pi)}\right) \tag{2.32}$$

This first-order finite difference equation relates the ratio $\mathsf{Q}_+/\mathsf{Q}_-$ with the eigenvalues of the (higher) transfer matrices.

Introduce the meromorphic functions

$$\Psi_\pm(u) = e^{\pm im\varphi}\sum_{\ell=0}^{L-1}\frac{\mathsf{f}(u+(2\ell+1)\eta)}{\mathsf{Q}_\pm(u+2\ell\eta)\,\mathsf{Q}_\pm(u+(2\ell+2)\eta)}, \tag{2.33}$$

such that

$$\mathsf{T}_L(u+\tfrac{1}{2}m\pi) = \Big(\mathsf{Q}_+(u)\Big)^2\Psi_+(u) = \Big(\mathsf{Q}_-(u)\Big)^2\Psi_-(u) . \tag{2.34}$$

[4]Namely this relation with $k = 1$ was used in [21] to show that for rational values of η the expression for the non-linear mobility for the quantum Brownian particle in a periodic potential obtained in [9] exactly coincide with that of [22] found from the thermodynamic Bethe Ansatz.

With this definition all the relations (2.29) reduce to a single relation which again can be written in two equivalent forms

$$\Psi_+(u) = \mathsf{C}(\varphi)\,\frac{\mathsf{Q}_-(u)}{\mathsf{Q}_+(u)}, \qquad \Psi_-(u) = \mathsf{C}(\varphi)\,\frac{\mathsf{Q}_+(u)}{\mathsf{Q}_-(u)}\ . \tag{2.35}$$

Obviously,

$$\Psi_+(u)\Psi_-(u) = \Big(\mathsf{C}(\varphi)\Big)^2, \qquad \frac{\Psi_+(u)}{\Psi_-(u)} = \Big(\frac{\mathsf{Q}_-(u)}{\mathsf{Q}_+(u)}\Big)^2\ . \tag{2.36}$$

2.3.2 The RSOS regime and its vicinity

Further reduction of the functional relation in the rational case (2.20) occurs for certain special values of the field from the set

$$m\,\varphi = (r+1)\,\pi, \qquad r = 0,1,2,\dots\ . \tag{2.37}$$

Consider the effect of varying φ in the relation (2.31). The eigenvalue $\mathsf{T}_L(u)$ in the LHS will remain finite, so as the eigenvalues $\mathsf{Q}_\pm(u)$ in the RHS. The latter also do not vanish identically (as functions of u) at any value of φ (see the discussion of our normalization assumptions before (2.17) above). Therefore the coefficient $\mathsf{C}(\varphi)$, defined in (2.30), is always finite. This means that in the rational case (2.20), the quantum Wronskian, $\mathsf{W}(\varphi)$, can only vanish at zeroes of the numerator in (2.30). However, the converse is not true: $\mathsf{W}(\varphi)$ does not necessarily vanish when $\mathsf{C}(\varphi)=0$. Here we are interested in this latter case where

$$\mathsf{C}(\varphi) = 0, \qquad \mathsf{W}(\varphi) \neq 0 \tag{2.38}$$

with φ from the set (2.37). By definition we call it the *RSOS regime*. The relations (2.29) and (2.31) reduce to

$$\mathsf{T}_k(u) = (-1)^r\,\mathsf{T}_{L-k}(u+\tfrac{1}{2}m\pi), \qquad k = 1,\dots,L-1, \tag{2.39a}$$

and

$$\mathsf{T}_L(u) = 0. \tag{2.39b}$$

All these relations can be written as a single relation (in two equivalent forms involving only $\mathsf{Q}_+(u)$ or $\mathsf{Q}_-(u)$ respectively),

$$\Psi_+(u) = \Psi_-(u) = 0, \tag{2.40}$$

with $\Psi_\pm(u)$ defined by (2.33).

The special "truncation" relations (2.39), exactly coincide with those appearing in the RSOS-model [18]. These were obtained [23, 24] by the algebraic fusion procedure [25] and hold for all eigenvalues of the RSOS model. The above analysis shows that *all eigenvalues of the RSOS model are non-singular*. The quantum Wronskian of the Bloch solutions (2.12) is always non-zero (otherwise the coefficient $\mathsf{C}(\varphi)$ in (2.29) would not have vanished). For this reason the solutions of the Bethe Ansatz equations for the RSOS model cannot contain complete strings. Since, as argued in [14] the complete strings are necessary attributes of degenerate states, one arrives to a rather non-trivial statement: *the spectrum of the transfer matrix in the RSOS model is non-degenerate*.

Consider now the vicinity of the RSOS regime, when η and φ are approaching their limiting values given by (2.20) and (2.37) respectively. Interestingly, one can express some η- and φ-derivatives

$$\partial_\eta \mathsf{T}_k(u) = \frac{\partial}{\partial\eta}\mathsf{T}_k(u\,|\,\eta,\varphi), \qquad \partial_\varphi \mathsf{T}_k(u) = \frac{\partial}{\partial\varphi}\mathsf{T}_k(u\,|\,\eta,\varphi)\ , \tag{2.41}$$

calculated at the "RSOS point",

$$(\eta,\,\varphi) = (m\pi/2L,\,\pi(r+1)/m)\ , \tag{2.42}$$

in terms of the corresponding values of $\mathsf{Q}_\pm(u)$ and their first order u-derivatives

$$\mathsf{Q}'_\pm(u) = \frac{\partial}{\partial u}\mathsf{Q}_\pm(u|\eta,\varphi)\ . \tag{2.43}$$

Using (2.19) one obtains,

$$\begin{aligned}
&\partial_\eta\Big[\mathsf{T}_k(u)-(-1)^r\,\mathsf{T}_{L-k}(u+m\pi/2)\Big]+L\,\partial_v\mathsf{T}_k(u)=\\
&\qquad\qquad = \frac{L}{i\mathsf{W}(\varphi)}\Big[\mathsf{Q}'_+(u+k\eta)\,\mathsf{Q}_-(u-k\eta)-\mathsf{Q}_+(u-k\eta)\,\mathsf{Q}'_-(u+k\eta)\Big]\ ,\\
&\partial_\varphi\Big[\mathsf{T}_k(u)-(-1)^r\,\mathsf{T}_{L-k}(u+m\pi/2)\Big]=\\
&\qquad\qquad = \frac{m}{2\mathsf{W}(\varphi)}\Big[\mathsf{Q}_+(u+k\eta)\,\mathsf{Q}_-(u-k\eta)+\mathsf{Q}_+(u-k\eta)\,\mathsf{Q}_-(u+k\eta)\Big]\ ,
\end{aligned} \tag{2.44}$$

where the expressions in the RHS are calculated directly at the point (2.42). According to the definitions (2.6), $\mathsf{T}_0(u)$ and $\mathsf{T}_1(u)$ do not depend on η and φ at all, therefore, one can express η- and φ-derivatives of $\mathsf{T}_{L-1}(u)$ and $\mathsf{T}_L(u)$ at the RSOS point (2.42) in terms of the of values $\mathsf{Q}_\pm(u)$ and $\mathsf{Q}'_\pm(u)$.

2.4 Zero field case

Consider now the zero field limit $\varphi=0$. Let us return to the case of an irrational η/π where the spectrum of the transfer matrix is non-degenerate. The eigenvalues $\mathsf{Q}_\pm(u)$, corresponding to the same eigenstate smoothly approach the same value at $\varphi=0$. Moreover, adjusting a φ-dependent normalization of $\mathsf{Q}_\pm(u)$ one can bring their small φ expansion to the form

$$\mathsf{Q}_\pm(u) = \mathsf{Q}_0(u) \mp \varphi\,\overline{\mathsf{Q}}_0(u)/2 + O(\varphi^2), \qquad \varphi\to 0\ , \tag{2.45}$$

where

$$\mathsf{Q}_0(u) = \mathsf{Q}_\pm(u)|_{\varphi=0}, \qquad \overline{\mathsf{Q}}_0(u) = -2\,\frac{d\,\mathsf{Q}_+(u)}{d\varphi}\Big|_{\varphi=0} = 2\,\frac{d\,\mathsf{Q}_-(u)}{d\varphi}\Big|_{\varphi=0}\ . \tag{2.46}$$

From (2.12) it follows that

$$\mathsf{Q}_0(u+\pi) = \mathsf{Q}_0(u), \qquad \overline{\mathsf{Q}}_0(u+\pi) = \overline{\mathsf{Q}}_0(u) + 2i\mathsf{Q}_0(u)\ . \tag{2.47}$$

It it easy to see that the quasi-periodic part of $\overline{\mathsf{Q}}_0(u)$ is totally determined by $\mathsf{Q}_0(u)$,

$$\overline{\mathsf{Q}}_0(u) = \frac{2iu}{\pi}\,\mathsf{Q}_0(u) + \overline{\mathsf{Q}}_0^{(per)}(u)\ . \tag{2.48}$$

However, the periodic part

$$\overline{\mathsf{Q}}_0^{(per)}(u+\pi)=\overline{\mathsf{Q}}_0^{(per)}(u), \tag{2.49}$$

can only be determined up to an additive term proportional to $\mathsf{Q}_0(u)$. Indeed, consider the effect of an inessential normalization transformation

$$\mathsf{Q}_\pm(u)\to e^{\pm\alpha\varphi}\mathsf{Q}_\pm(u)\ , \tag{2.50}$$

where α is a constant. The value of $\mathsf{Q}_0(u)$ remains unchanged while the periodic part of $\overline{\mathsf{Q}}_0(u)$ transforms as

$$\overline{\mathsf{Q}}_0^{(per)}(u)\to\overline{\mathsf{Q}}_0^{(per)}(u)-2\alpha\,\mathsf{Q}_0(u)\ . \tag{2.51}$$

The quantum Wronskian relation (2.13) reduces to

$$\mathsf{Q}_0(u+\eta)\,\overline{\mathsf{Q}}_0(u-\eta)-\mathsf{Q}_0(u-\eta)\,\overline{\mathsf{Q}}_0(u+\eta)=2i\dot{\mathsf{W}}(0)\,\mathsf{f}(u)\ , \tag{2.52}$$

where

$$\mathsf{W}'(0)=\frac{d\,\mathsf{W}(\varphi)}{d\varphi}\Big|_{\varphi=0}\ . \tag{2.53}$$

The expression (2.19) now becomes

$$2i\mathsf{W}'(0)\mathsf{T}_k(u)=\mathsf{Q}_0(u+k\eta)\,\overline{\mathsf{Q}}_0(u-k\eta)-\mathsf{Q}_0(u-k\eta)\,\overline{\mathsf{Q}}_0(u+k\eta)\ . \tag{2.54}$$

It is easy to see that at $\varphi=0$ the TQ-equation (2.3) is satisfied if $\mathsf{Q}(u)$ there is replaced by either of $\mathsf{Q}_0(u)$ or $\overline{\mathsf{Q}}_0(u)$. The same remark applies to the more general equation (2.11).

The Bethe Ansatz equations (2.4) for the zeroes of $\mathsf{Q}_0(u)$ are the standard equations [2] arising in the analysis of the symmetric 8V-model. Exactly the same equations also hold for the zeroes of $\overline{\mathsf{Q}}_0(u)$, but their usefulness is very limited. Even though $\overline{\mathsf{Q}}_0(u)$ is an entire function of u, it lacks the simple periodicity (cf. (2.47)) and, therefore, does not possess any convenient product representation. Moreover, the transformation (2.51) affects the position of zeros of $\overline{\mathsf{Q}}_0(u)$, making them ambiguous. All this renders the Bethe Ansatz equations for $\overline{\mathsf{Q}}_0(u)$ useless. Fortunately, these equations are not really required for determination of $\overline{\mathsf{Q}}_0(u)$. Once the zeros of $\mathsf{Q}_0(u)$ are known the function $\overline{\mathsf{Q}}_0(u)$ is explicitly calculated from (2.52).

Additional functional relations arise in the rational case (2.20). These relations are straightforward corollaries of (2.29), (2.31) and (2.35). For instance, Eq.(2.29) gives

$$\mathsf{T}_k(u)+\mathsf{T}_{L-k}(u+m\pi/2)=\mathsf{C}(0)\ \mathsf{Q}_0(u+k\eta)\,\mathsf{Q}_0(u-k\eta),\qquad \varphi=0\ , \tag{2.55}$$

where

$$\mathsf{C}(0)=m/\mathsf{W}'(0)\ . \tag{2.56}$$

All these relations (with different k) can be equivalently re-written as a single relation

$$\sum_{\ell=0}^{L-1}\frac{\mathsf{f}(u+(2\ell+1)\eta)}{\mathsf{Q}_0(u+2\ell\eta)\,\mathsf{Q}_0(u+(2\ell+2)\eta)}=\mathsf{C}(0),\qquad \varphi=0\ , \tag{2.57}$$

which is the $\varphi=0$ version of (2.35). Setting $k=0$ in (2.55) one gets

$$\mathsf{T}_L(u+m\pi/2)=\mathsf{C}(0)\,(\mathsf{Q}_0(u))^2,\qquad \varphi=0\ . \tag{2.58}$$

Thus, at $\varphi = 0$ the eigenvalue $\mathsf{T}_L(u+m\pi/2)$ becomes a perfect square. It only has *double* zeroes, whose positions coincide with the zeroes of $\mathsf{Q}_0(u)$.

As is well known, in the rational case (2.20) the transfer matrix of the 8V-model has a degenerate spectrum (for sufficiently large values of $N \geq 2L$). We would like to stress here that the above relations (2.55)–(2.58) hold only for *non-degenerate* states. Actually, the assumption made in the beginning of this subsection, that $\mathsf{Q}_+(u)$ coincides with $\mathsf{Q}_-(u)$ when $\varphi = 0$, is true only for non-degenerate states. Removing this assumption and taking $\varphi \to 0$ limit in (2.31), while keeping η fixed by (2.20), one obtains

$$\mathsf{T}_L(u+m\pi/2) = \mathsf{C}(0)\,\overline{\mathsf{Q}}_+(u)\overline{\mathsf{Q}}_-(u), \qquad \overline{\mathsf{Q}}_\pm = \lim_{\varphi\to 0} \mathsf{Q}_\pm(u)\ . \tag{2.59}$$

For a degenerate state the eigenvalues $\overline{\mathsf{Q}}_+(u)$ and $\overline{\mathsf{Q}}_-(u)$ can only differ by positions of *complete exact strings*. This ambiguity does not affect any transfer matrix eigenvalues $\mathsf{T}_k(u)$, since the complete strings trivially cancel out from (2.11). In principle, the complete strings can take arbitrary positions, however, for $\overline{\mathsf{Q}}_\pm(u)$ they take rather distinguished positions. Indeed, due to (2.59), the zeroes of $\overline{\mathsf{Q}}_\pm(u)$ manifest themselves as zeroes of $\mathsf{T}_L(u+m\pi/2)$ which are uniquely defined even for the degenerate states. From the above discussion it is clear that $\mathsf{T}_L(u)$ has either double zeroes or complete strings of zeroes. Further analysis of the degenerate case is contained in [16].

2.5 Particular models

So far our considerations were rather general and covered several related models at the same time. For each particular model, one needs to specify additional properties, namely, (i) the explicit form of the function $\mathsf{f}(u)$ and (ii) detailed analytic properties of the eigenvalues $\mathsf{Q}_\pm(u)$. In this Section we will do this for three different models: the 8V/SOS-model, the 6V-model and the $c<1$ conformal field theory.

2.5.1 The symmetric eight-vertex model

There are only eight "allowed" vertex configurations, shown in Fig.1, which have non-vanishing Boltzmann weights. These weights are not arbitrary; they parameterized by only four arbitrary constants a,b,c,d,

$$\omega_1 = \omega_2 = a, \qquad \omega_3 = \omega_4 = b, \qquad \omega_5 = \omega_6 = c, \qquad \omega_7 = \omega_8 = d\ . \tag{2.60}$$

The remaining eight configurations are forbidden; their Boltzmann weight is zero. Following [2] we parameterize the Boltzmann weights a, b, c, d as

$$\begin{aligned}
a &= \rho\ \vartheta_4(2\eta\,|\,\mathsf{q}^2)\ \vartheta_4(v-\eta\,|\,\mathsf{q}^2)\ \vartheta_1(v+\eta\,|\,\mathsf{q}^2),\\
b &= \rho\ \vartheta_4(2\eta\,|\,\mathsf{q}^2)\ \vartheta_1(v-\eta\,|\,\mathsf{q}^2)\ \vartheta_4(v+\eta\,|\,\mathsf{q}^2),\\
c &= \rho\ \vartheta_1(2\eta\,|\,\mathsf{q}^2)\ \vartheta_4(v-\eta\,|\,\mathsf{q}^2)\ \vartheta_4(v+\eta\,|\,\mathsf{q}^2),\\
d &= \rho\ \vartheta_1(2\eta\,|\,\mathsf{q}^2)\ \vartheta_1(v-\eta\,|\,\mathsf{q}^2)\ \vartheta_1(v+\eta\,|\,\mathsf{q}^2),
\end{aligned} \tag{2.61}$$

Note, that our notations are slightly different from those in Baxter's original papers [2–5]. The variables q, η, v and ρ used therein (hereafter denoted as q_B, η_B, v_B and ρ_B) are related to our

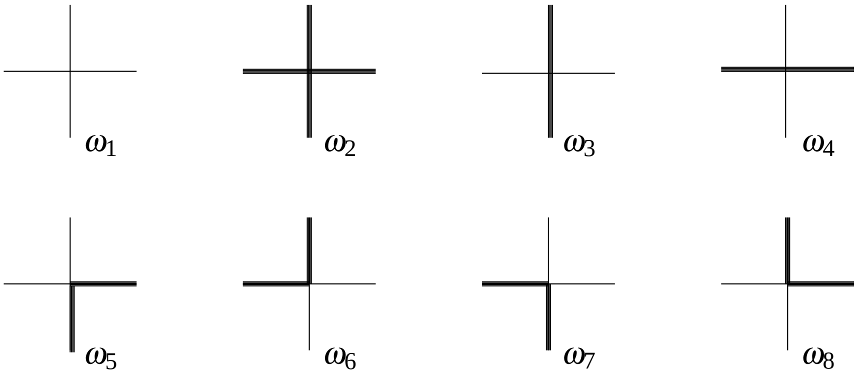

Figure 1: Eight allowed vertex configuration and their Boltzmann weights. Thin lines represent the "spin-up" states and the bold lines represent the "spin-down" states of the edge spins

variables q, η, v and ρ as

$$\mathsf{q}^2 = \mathsf{q}_B = e^{-\pi \mathsf{K}'_B/\mathsf{K}_B}, \qquad \eta = \frac{\pi\eta_B}{2\mathsf{K}_B}, \qquad v = \frac{\pi v_B}{2\mathsf{K}_B}, \qquad \rho = \rho_B\ , \tag{2.62}$$

where K_B and K'_B are the complete elliptic integrals associated to the nome q_B. Here we the fix the normalization of the Boltzmann weights as

$$\rho = 2\ \vartheta_2(0\,|\,\mathsf{q})^{-1}\ \vartheta_4(0\,|\,\mathsf{q}^2)^{-1}\ , \tag{2.63}$$

where

$$\vartheta_i(u\,|\,\mathsf{q}),\ i = 1,\ldots,4, \quad \mathsf{q} = e^{\mathrm{i}\pi\tau}, \qquad \mathrm{Im}\,\tau > 0, \tag{2.64}$$

are the standard theta functions [7] with the periods π and $\pi\tau$.

We denote the transfer matrix $\mathbf{T}$ and the $\mathbf{Q}$-matrix from [2,3] as $\mathbf{T}^B(v)$ and $\mathbf{Q}^B(v)$, remembering that our variable v is related to v_B by (2.62). Below we often use a shifted spectral parameter

$$u = v - \pi\tau/2\ , \tag{2.65}$$

simply connected to the variable v in (2.62). We also consider the re-defined matrices

$$\mathbf{T}(u) = (-i\,\mathsf{q}^{-1/4})^N\, e^{ivN}\,\mathbf{T}^B(v), \qquad \mathbf{Q}(u) = e^{ivN/2}\mathbf{Q}^B(v) \tag{2.66}$$

where N is the number of columns of the lattice. The eigenvalues $\mathsf{T}(u)$ and $\mathsf{Q}(u)$ of these new matrices enjoy the following periodicity properties

$$\mathsf{T}(u+\pi) = \mathsf{T}(u), \qquad \mathsf{T}(u+\pi\tau) = (-\mathsf{q})^{-N}\, e^{-2iuN}\,\mathsf{T}(u)\ , \tag{2.67}$$

and

$$\textit{8V-model:} \qquad \mathsf{Q}(u+\pi) = s\, e^{i\pi N/2}\,\mathsf{Q}(u), \qquad \mathsf{Q}(u+2\pi\tau) = \mathsf{q}^{-2N}\, e^{-2iNu}\,\mathsf{Q}(u)\ . \tag{2.68}$$

Here the "quantum number" $s=\pm 1$, is the eigenvalue of the operator $\mathscr{S}$, defined as

$$\mathscr{S}=\sigma_3^{(1)}\otimes\sigma_3^{(2)}\otimes\cdots\otimes\sigma_3^{(N)},\qquad \mathscr{R}=\sigma_1^{(1)}\otimes\sigma_1^{(2)}\otimes\cdots\otimes\sigma_1^{(N)}\,. \tag{2.69}$$

This operator always commutes with $\mathbf{T}(u)$ and $\mathbf{Q}(u)$.

Baxter's TQ-equation (Eq.(4.2) of [2] and Eq.(87) of [3]) now takes the form (2.3) with

$$\mathsf{f}(u)=\big(\vartheta_4(u\,|\,\mathsf{q})\big)^N\,. \tag{2.70}$$

The main reason for the above redefinitions is to bring the TQ-equation to the universal form (2.3), where $\mathsf{T}(u)$ and $\mathsf{f}(u)$ are periodic functions of u (see Eq.(2.2)) for an arbitrary, odd or even, number of sites, N. This also helps to facilitate the considerations of the scaling limit in our next paper [26].

Comparing the first equation in (2.68) with the periodicity of the Bloch solutions (2.12) one concludes that the exponents φ read

$$\varphi^{(8V)}=\begin{cases}0\quad(\mathrm{mod}\,\pi), & N=\mathrm{even}\\[2ex] \dfrac{\pi}{2}\quad(\mathrm{mod}\,\pi), & N=\mathrm{odd}\end{cases} \tag{2.71}$$

Thus, for an even N the exponents of the symmetric 8V-model, with the cyclic boundary conditions, always belong to the "dangerous" set (2.17). For an odd N the exponents (2.71) fall into this set only for certain rational values of η/π. A notable example is the case $\eta=\pi/3$, considered in [26–28].

The imaginary period relations in (2.67) and (2.68) certainly deserve a detailed consideration. First, note that in (2.68) we only stated the periodicity with respect to the *double* imaginary period $2\pi\tau$, which always holds in all cases when the 8V-model has been exactly solved[5]. Actually, this is a rather overcautious statement which can be easily specialized further. For the following discussion assume a generic (i.e., irrational) value of η/π. Then for even N the Bloch solutions (2.12) always coincide (just as in the zero-field case of Sect.2.4). For odd N there are always two linearly independent Bloch solutions for each eigenvalue $\mathsf{T}(u)$, one with $s=+1$ and one with $s=-1$ (remind that in this case each eigenvalue of the transfer matrix is double-degenerate [29]). The existence of the "imaginary" period imposes rather non-trivial restrictions on the properties of the eigenvalues. Indeed, the second relation in (2.67) immediately implies that the function

$$\widetilde{\mathsf{Q}}(u)=r\,\mathsf{q}^{N/2}\,e^{iuN}\mathsf{Q}(u+\pi\tau) \tag{2.72}$$

where r is a constant, satisfies the TQ-equation (2.3) as well as $\mathsf{Q}(u)$. Further, if $\mathsf{Q}(u)$ is a Bloch solution

$$\mathsf{Q}(u+\pi)=e^{i\varphi}\,\mathsf{Q}(u) \tag{2.73}$$

with some φ then $\widetilde{\mathsf{Q}}(u)$ is also such a solution with the exponent

$$\widetilde{\varphi}=\varphi+N\pi\quad(\mathrm{mod}\,2\pi). \tag{2.74}$$

[5]Ref. [2] applies to rational η and arbitrary values of N, while ref. [3] applies to arbitrary η and even values N. It is reasonable to assume that (2.68) holds in general, however, the case of an arbitrary η and an odd N has never been considered.

Obviously, there are two options, either $\widetilde{\mathsf{Q}}(u)$ is proportional to $\mathsf{Q}(u)$ or it is proportional to the other linearly independent Bloch solution with the negated exponent "$-\varphi$". The first option is realized for even N,

$$\textit{8V-model, N even:} \qquad \mathsf{Q}(u+\pi\tau) = r\,\mathsf{q}^{-N/2}\,e^{-iuN}\mathsf{Q}(u), \tag{2.75}$$

The constant $r = \pm 1$ is then the eigenvalue of the spin-reversal operator $\mathscr{R}$ defined in (2.69). The second option requires the exponent φ to be a half-an-odd integer fraction of π, it is realized for odd N,

$$\textit{8V-model, N odd:} \qquad \mathsf{Q}_{\pm}(u+\pi\tau) = \mathsf{q}^{-N/2}\,e^{-iuN}\mathsf{Q}_{\mp}(u). \tag{2.76}$$

The above relations (2.75) and (2.76) were derived for irrational values of η/π, however they also hold in the rational case (2.20), if no additional degeneracy of the eigenvalues of the transfer matrix occurs (apart from the one related with the spin-reversal symmetry for odd N). The functional relation (2.31) can be then written in the form

$$\mathsf{T}_L(u+\tfrac{1}{2}m\pi) = \mathsf{A}\,e^{iNu}\,\mathsf{Q}_+(u)\,\mathsf{Q}_+(u+\pi\tau) \tag{2.77}$$

where A is a constant. This relation is identical to the one conjectured in [15][6].

2.5.2 The solid-on-solid model

The main idea of this paper is to study deformations of the eigenvalues $\mathsf{T}(u)$ and $\mathsf{Q}(u)$ under continuous variations of the exponents φ from their discrete values (2.71). As explained in the Introduction the resulting eigenvalues correspond to the unrestricted SOS-model. We will therefore assume the more general periodicity relations (1.6) for the Bloch wave solutions $\mathsf{Q}_{\pm}(u)$, which hold for both odd and even N,

$$\textit{SOS-model:} \qquad \mathsf{Q}_{\pm}(u+\pi) = e^{\pm i\varphi}\,\mathsf{Q}_{\pm}(u), \qquad \mathsf{Q}_{\pm}(u+2\pi\tau) = \mathsf{q}^{-2N}\,e^{\pm\psi}\,e^{-2iuN}\,\mathsf{Q}_{\pm}(u), \tag{2.78}$$

where the exponent φ is arbitrary. The second exponent ψ is not an independent parameter, it is determined by φ (see the discussion in Section 4 of [16]).

The second relation in (2.78) can be further refined for even N

$$\textit{SOS-model, N even:} \qquad \mathsf{Q}_{\pm}(u+\pi\tau) = \mathsf{q}^{-N/2}\,e^{\pm\psi/2}\,e^{-iuN}\,\mathsf{Q}_{\pm}(u), \tag{2.79}$$

whereas the periodicity of $\mathsf{T}(u)$ remains the same (2.67) as in the 8V-model. However, there is no a general SOS-model analog of (2.76), as it is specific to half-odd exponents only. As a result Eq.(2.67) is replaced with

$$\textit{SOS-model, N odd:} \qquad \mathsf{T}(u+\pi) = \mathsf{T}(u), \qquad \mathsf{T}(u+2\pi\tau) = \mathsf{q}^{-4N}\,e^{-4iuN}\,\mathsf{T}(u)\,. \tag{2.80}$$

Strictly speaking the use of the term "SOS-model" here is justified for even N only [4]. Nonetheless, we will use this term to indicate arbitrary values of the field parameter φ in general.

[6] The conjecture of [15] also covers a special case of degenerate states for rational values of η, where the relation (2.76) holds for the eigenvalues of the **Q**-matrix of [2] for even N.

2.5.3 Six-vertex model in a horizontal field

The allowed vertex configurations of the six-vertex model form a subset of those shown in Fig.1. Namely, the Boltzmann weights ω_7 and ω_8 are equal to zero. The remaining six weights will be parameterized as

$$\begin{aligned} \omega_1 &= e^{+H-i\eta}\,a,\ \omega_2 = e^{-H-i\eta}\,a,\ \omega_3 = e^{+H+i\eta}\,b, \\ \omega_4 &= e^{-H+i\eta}\,b,\ \omega_5 = e^{iu-2i\eta}\,c,\ \ \omega_6 = e^{iu-2i\eta}\,c, \end{aligned} \tag{2.81}$$

where H stands for the horizontal field

$$a = \mathsf{h}(u+\eta), \qquad b = \mathsf{h}(u-\eta), \qquad c = \mathsf{h}(2\eta), \qquad \mathsf{h}(u) = 1 - e^{2iu}\ . \tag{2.82}$$

The above parametrization is simply related to that given in Eq.(12) of [14] (where the vertical field V is set to zero). The TQ-equation (eq.(11) of [14]) takes the form (2.2), (2.3), where

$$\mathsf{f}(u) = (\mathsf{h}(u))^N\ . \tag{2.83}$$

The Bloch solutions (2.12), corresponding to the eigenvectors of the transfer matrix with n "up-spins", can be written as

$$\mathsf{Q}_\pm(u) = e^{\pm i\varphi u/\pi}\,\mathsf{A}_\pm(e^{2iu})\ , \tag{2.84}$$

where $\mathsf{A}_+(x)$ and $\mathsf{A}_-(x)$ are polynomials in x of the degrees n and $(N-n)$, respectively, and

$$\varphi = \frac{i\pi HN}{2\eta} + \frac{\pi}{2}\,(N-2n)\ . \tag{2.85}$$

Introduce new variables[7]

$$x = e^{2iu}, \qquad q = e^{2i\eta}, \qquad z = e^{2i\eta\varphi/\pi}\ . \tag{2.86}$$

Regarding x as a new spectral parameter instead of u and writing $\mathsf{T}(u)$ and $\mathsf{f}(u)$ as $\mathsf{T}(x)$ and $\mathsf{f}(x)$, respectively, one can rewrite (2.3) in the form

$$\mathsf{T}(x)\,\mathsf{A}_\pm(x) = z^{\pm 1}\,\mathsf{f}(q^{-1}x)\,\mathsf{A}_\pm(q^2x) + z^{\mp 1}\,\mathsf{f}(qx)\,\mathsf{A}_\pm(q^{-2}x)\ , \tag{2.87}$$

where the polynomials $\mathsf{A}_\pm(x)$ are defined in (2.84). This form is particularly convenient for the 6V-model.

2.5.4 Conformal field theory

The continuous quantum field theory version of Baxter's commuting transfer matrices of the lattice theory was developed in [9, 10, 30]. These papers were devoted to the $c < 1$ conformal field theory (CFT). The parameters β and p used there define the central charge c and the Virasoro highest weight Δ,

$$c = 1 - 6(\beta - \beta^{-1})^2, \qquad \Delta = \left(\frac{p}{\beta}\right)^2 + \frac{c-1}{24}\ . \tag{2.88}$$

[7]The parameter q should not be confused with the nome q in the 8V-model

They are related to our η and φ as

$$2\eta = \beta^2\pi, \qquad \varphi = 2\pi p/\beta^2 \ . \tag{2.89}$$

The multiplicative spectral parameter λ used in those papers is related to our variable u as

$$\lambda^2 = -e^{-2iu} \ . \tag{2.90}$$

The eigenvalues of the CFT **Q**-operators $\mathsf{Q}_\pm(u)$ are entire functions of the variable u, satisfying the periodicity relation (2.12). Their leading asymptotics at large positive imaginary u read

$$\log \mathsf{Q}_\pm(u) = \frac{A}{\cos(\frac{\pi\eta}{\pi-2\eta})}\, e^{i\pi u/(\pi-2\eta)} + O(1), \qquad u\to+i\infty, \quad |\mathrm{Re}u|<\pi/2 \ , \tag{2.91}$$

where A is a known constant [9]. Here we assumed that η does not belong to the set

$$\eta = \frac{\pi}{2}\Big(1-\frac{1}{2k}\Big), \qquad k=1,2,\ldots,\infty \ . \tag{2.92}$$

At these special values of η the theory contains logarithmic divergences and the asymptotics (2.91) should be replaced with

$$\log \mathsf{Q}_\pm(u) = 2i(-1)^k A\, u\, e^{2iuk/\pi} + \mathbb{C}\, e^{2iuk} + O(1), \qquad u\to+i\infty, \quad |\mathrm{Re}u|<\pi/2 \ , \tag{2.93}$$

where $\mathbb{C}$ is a regularization-dependent constant. The factorization formulae read[8]

$$\mathsf{Q}_\pm(u) = e^{\pm iu\varphi/\pi}\,\mathsf{A}_\pm(u), \qquad \mathsf{A}_\pm(u) = \prod_{k=1}^{\infty}\big(1-e^{-2i(u-u_k^\pm)}\big) \ , \tag{2.94}$$

where the zeroes $u_1^\pm, u_2^\pm, \ldots$ accumulate at infinity along the straight line

$$u = \pi/2 + iy, \qquad y\to+\infty \ . \tag{2.95}$$

Finally, the function $\mathsf{f}(u)$ in the case of CFT should be set to one[9],

$$\mathsf{f}(u)\equiv 1 \ . \tag{2.96}$$

With these specializations the functional relations given above become identical to those previously obtained in [9, 10, 30].

2.6 Related developments and bibliography

The literature on the functional relation in solvable models is huge; therefore it would not be practical to mention all papers in the area. Our brief review is restricted only to a subset of publications directly related to the eight-vertex/six-vertex models and associated models of quantum field theory.

[8]Here we assumed that $0<\eta<\pi/4$. When $\pi/4<\eta<\pi/2$ the product in (2.94) should contain the standard Weierstrass regularization factors [9].

[9]Again, we have assumed that η does not fall into the set (2.92), otherwise $\mathsf{f}(u) = \exp(4A\eta e^{2iuk}/\pi)$.

2.6.1 Transfer matrix relations

In the above presentation the entire functions $\mathsf{T}_k(u)$ with $k \geq 3$ were defined by the recurrence relation (2.5), which allows one to express them solely in terms of $\mathsf{T}(u)$, as in (2.8). No other additional properties of $\mathsf{T}_k(u)$ were used. However, as is well known, these functions are eigenvalues of the higher transfer matrices, usually associated with the so-called fusion procedure. This algebraic procedure provides a derivation of the functional relations for the higher transfer matrices based on decomposition properties of products of representations of the affine quantum groups. Originally, all these "transfer matrix relations" were obtained essentially in this way. We would like to stress that the logic of these developments was exactly opposite to that employed in our review. The goal was to find new techniques, independent of the TQ-relation, rather than to deduce everything from the latter. The first important contribution was made by Stroganov [31]. He gave an algebraic derivation of the first nontrivial relation in (2.5) (with $k = 2$),

$$\mathsf{T}(u+\eta)\,\mathsf{T}(u-\eta) - \mathsf{f}(u+2\eta)\,\mathsf{f}(u-2\eta) = O\Big((u-u_0)^N\Big) \tag{2.97}$$

in the vicinity of the point $u = u_0$ where the transfer matrices $\mathsf{T}(u_0+\eta)$ and $\mathsf{T}(u_0-\eta)$ become shift operators. Remarkably, this single relation alone contains almost all information about the eigenvalues $\mathsf{T}(u)$. To illustrate this point consider, for instance, the 6V-model. For a chain of the length N each eigenvalue $\mathsf{T}(u)$ is a trigonometric polynomial of the degree N, determined by $N+1$ unknown coefficients. The mere fact that the LHS of (2.97) has an N-th order zero immediately gives N algebraic equations for these unknowns. Similar arguments, obviously, apply to the 8V-model. One additional equation is usually easy to find from some elementary considerations (e.g., from the large u asymptotics in the 6V-model). Further, in the thermodynamic limit, $N \to \infty$ with u kept fixed, Eq.(2.97) becomes a closed functional relation for the eigenvalues (its RHS vanishes). This is the famous "inversion relation" [31–33]. With additional analyticity assumptions it can be effectively used to calculate the eigenvalues of the transfer matrix at $N = \infty$. Recently, Eq.(2.97) was used to derive a new non-linear integral equation [34], especially suited for the analysis of high-temperature properties of lattice models.

Soon after [31] Stroganov derived [35] a particular case of (2.23) for the 6-vertex model with $\eta = \pi/6$ (i.e., for $L = 3$ and $m = 1$). He also found that for the case of an odd number of sites[10] Eq.(2.25) takes the form

$$\mathsf{T}(u-2\eta)\,\mathsf{T}(u)\,\mathsf{T}(u+2\eta) = \mathsf{f}(u)\mathsf{f}(u+2\eta)\,\mathsf{T}(u-2\eta) + \mathsf{f}(u-2\eta)\mathsf{f}(u+2\eta)\,\mathsf{T}(u) + \mathsf{f}(u-2\eta)\mathsf{f}(u)\,\mathsf{T}(u+2\eta)\,. \tag{2.98}$$

He then used this equation to obtain Bethe Ansatz type equations for the zeroes of $\mathsf{T}(u)$ and to reproduce Lieb's celebrated result [36] for the residual entropy of the two-dimensional ice. Unfortunately, these results were left unpublished.

The ideas of [31, 35] were further developed in the *analytic Bethe Ansatz* [37] where the TQ-equation (or an analogous equation) is used essentially as a formal substitution to solve the transfer matrix functional equations. The notion of "higher" or "fused" R-matrices was developed in [20] from the point of view of representation theory. These R-matrices were calculated in [38] for the 6V-model, in [39–42] for the 8V-model and in [25] for the SOS-model. The functional relations

[10] In our notations this corresponds to $\varphi = \pi/2 \pmod{\pi}$.

(2.7) were given in [38] for the 6V-model and in [24] for 8V/SOS-model. The determinant identity (2.25) was discussed in [24, 43]. An algebraic derivation of the truncation relations (2.39) for the RSOS model [18] was given in [24]. A particular case of this truncation for the hard hexagon model [44] was previously discovered in [23]. An algebraic derivation of (2.23) in the zero-field six-vertex model is given in [45]. The idea of calculation of φ- and η-derivatives (2.44) at the RSOS point given in Sect.2.3.2 is borrowed from [46] and [47].

Remarkably, the same functional equations (2.5) (along with all their specializations in the rational case) arise in a related, but different context of the thermodynamic Bethe Ansatz [48]; see [49] for its application to the 8V-model. Usually this approach in lattice models is associated with non-linear integral equations. Here we refer to the functional form of these equations discovered in [50]. Further discussion of the correspondence of the functional relation method with the thermodynamic Bethe Ansatz and its generalizations for excited states can be found in [19, 30, 51–54].

2.6.2 Q-matrix and TQ-relations

As noted before, a full algebraic theory of the **Q**-matrix in the 8V-model is not yet developed. The idea of the construction of the **Q**-matrix in terms of some special transfer matrices belongs to Baxter. It is a key element of his original solution of the 8V-model. Readers interested in details should familiarize themselves with the Appendix C of [2] (along with other four appendices and, of course, the main text of that paper, which contain a wealth of important information on the subject). The results of [2] only apply for certain rational values of η. The construction of [2] and the set of allowed values of η were recently revised in [15]. A different construction for the **Q**-matrix, which works for an arbitrary η, was given [4].

There are many related solvable models connected with the R-matrix of the 8V-model but having different **L**-operators and different *quantum spaces*. The general structure of the functional relations in all such models remains the same. In particular, they all possess a TQ-relation (though it may contain different scalar factors and require different analytic properties of the eigenvalues). In [4] Baxter also presented an extremely simple explicit formula for the matrix elements of the **Q**-matrix for the zero-field 6V-model in the sector with $N/2$ "up-spins" (the half filling). However, no such expression is known for the 8V-model, or the other sectors of the 6V-model. The quantum space of the 6V-model is build from the two-dimensional highest weight representation of $U_q(sl_2))$ at every site of the lattice. Curiously enough, if this representation is replaced with the general cyclic representation (arising at roots of unity, $q^L = 1$) then all matrix elements of the **Q**-matrix can be explicitly calculated [55] as a simple product involving only a two-spin interaction[11]. Remarkably, the resulting **Q**-matrix exactly coincides with the transfer matrix of the chiral Potts model [56–58]; this allows one to view the latter as a "descendant of the six-vertex model" [55]. The generalization of this construction to the eight-vertex and the Kashiwara-Miwa model [59] is considered in [60]. Further developments of the theory of the **Q**-matrix and related topics (along with many important applications to various solvable models) can be found in [61–70].

[11] The factorization of the matrix elements of the **Q**-matrix is typical for quantum space representations without highest and lowest weights.

Baxter's original idea of the construction of **Q**-operators which utilizes traces of certain monodromy matrices was extended in [9,10] for trigonometric models related with the quantum affine algebra $U_q(\widehat{sl}(2))$. It turned out that in the trigonometric case the situation is considerably simpler than for the 8-vertex model and the **Q**-operators coincide with some special transfer matrices. The corresponding **L**-operators are obtained as specializations of the universal **R**-matrix [71] to infinite-dimensional representations of the q-oscillator algebra in the "auxiliary space". Although the calculations of [9,10] were specific to the continuous quantum field theory, the same procedure can readily be applied to lattice models (see, e.g., [72–75] for the corresponding results for the 6V-model). In the case of the 6V-model with non-zero horizontal field this construction leads to two **Q**-matrices[12], whose eigenvalues $\mathsf{Q}_\pm$ are precisely the "Bloch wave" solutions of the TQ-equation.

Note that functional relations which involves bi-linear combinations of $\mathsf{Q}_\pm$, namely (2.19), (2.29) and (2.31) are universal in the sense that they do not involve the model-specific function $\mathsf{f}(u)$. These relations were derived in [9,10] in the context of the conformal field theory. Similar relations previously appeared in the chiral Potts model [55,76,77], though the correspondence is not exact because there is no an additive spectral parameter in that model. Eq.(2.19) in the eight-vertex and the XXX-models was found in [11] and [78]. A special case (2.77) of the relation (2.31) involving Baxter's original **Q**-matrix [2] for the 8V-model was conjectured in [15]. Another special (zero-field) case (2.55) of the same relation (2.31) in conformal field theory was conjectured in [53].

3. Conclusion

In this paper we developed some new ideas in the classical subject of Baxter's celebrated eight-vertex and solid-on-solid models. Our primary observation concerns a (previously unnoticed) arbitrary field parameter in the solvable solid-on-solid model. This parameter is analogous to the horizontal field in the six-vertex model. This fact might not be so surprising to experts, since all the hard work has been done before and one just needs to lay side-by-side the papers [5,12,13] to realize that an arbitrary field parameter is, in fact, required to describe the continuous spectrum of the unrestricted solid-on-solid model.

The introduction of an arbitrary field allowed us to develop a completely analytic theory of the functional relations in the 8V/SOS-model. As demonstrated in [16], the solutions of the Bethe Ansatz equations are multivalued functions of the field variable, having algebraic branching points. It is plausible that many (if not all) eigenvalues of the transfer matrix can be obtained from each other via analytic continuation in this variable.

It appears that the analytic structure of eigenvalues in the eight-vertex/SOS model certainly deserves further studies. Somewhat simpler (but still very interesting) structure arises in the six-vertex model and, especially, in the $c < 1$ conformal field theory [79]. In the latter case the Riemann surface of the eigenvalues closes within each level subspace of the Virasoro module.

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[12] As noted in [10], for the "half-filled" sector of the zero-field 6V-model these **Q**-matrices reduce to Baxter's expression [4] mentioned above, as they, of course, should.

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