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Entangled States, Yangian and Yang-Baxter Approach

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The GHZ type of entangled states are shown to be connected with Yang-Baxter approach and Yangian algebras.

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Contents

1. Introduction

The quantum entangled states leave room for spin-$\frac{1}{2}$ systems described in terms of other than the usual Lie algebras $SU(2)$. They are governed by either representations of braid group B_N, or Yangian algebras. The Yang-Baxter equation and Yangian algebra had been well established for long time[1−8]. On the other hand entangled states[9−13] play important role in quantum information. A special interest in this paper is to set up the close relationship between quantum entanglements and Yang-Baxter approach. We shall point out that such a connection is not only looking natural, but also deep singnificant.

2. Yangian and Entangled States

2.1 Yangian and Bell States

For a N spin$-\frac{1}{2}$ system, the Yangian can be realized in terms of

$$\mathbf{I} = \sum_{i=1}^{N} \mathbf{S}_i, \qquad (\mathbf{S}_i^2 = \frac{3}{4}); \tag{2.1}$$

$$\mathbf{J} = \sum_{i=1}^{N} \mu_i \mathbf{S}_i + \frac{\sqrt{-1}}{2} \sum_{i<j}^{N} \mathbf{S}_i \times \mathbf{S}_j. \tag{2.2}$$

The set $\mathbf{Y} = (\mathbf{I},\mathbf{J})$ forms Yangian associated with $SU(2)$. The operators $\mathbf{J}$ act on tensor space and μ_i are free complex parameters. The essential difference between Yangian and Lie algebra is in the parameters μ_i. It can be checked that $\mathbf{I}$ and $\mathbf{J}$ satisfy the relations given by Drinfeld[1−2] and the RTT relations[3−7] for the simplest rational solution of $R(u)$-matrix. The independent commutation relations are

$$[I_\alpha, I_\beta] = i\varepsilon_{\alpha\beta\gamma} I_\gamma, \qquad (\alpha,\beta,\gamma = 1,2,3) \tag{2.3}$$

$$[I_\alpha, J_\beta] = i\varepsilon_{\alpha\beta\gamma} J_\gamma, \tag{2.4}$$

and[8]

$$[J_3,[J_+,J_-]] = \frac{1}{4}(I_3 J_\pm - J_3 I_\pm) I_\pm, \tag{2.5}$$

where $J_\pm = J_1 \pm iJ_2$. Based on Eq. (2.3), Eq. (2.4) and Jacobian identities (2.5) exhaust all the commutation relations given by Drinfeld with $c_{\alpha\beta\gamma} = i\varepsilon_{\alpha\beta\gamma}$[8]. The realization (2.2) is general. A special model means a particular value of the parameter set $\{\mu_i\}$.

It is known that the spin triplet, spin singlet and Bell states for a two spin system read

$$\psi_{00} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \qquad \begin{array}{c} \psi_{11} = |\uparrow\uparrow\rangle \\ \psi_{10} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ \psi_{1-1} = |\downarrow\downarrow\rangle \end{array} \qquad \begin{array}{l} \Psi_1 = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) = |1\rangle \\ \Psi_2 = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle) = |2\rangle \\ \Psi_3 = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = |3\rangle \\ \Psi_4 = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = |4\rangle \end{array} \tag{2.6}$$

spin singlet spin triplet Bell states

We see that the states $\psi_{00} = \Psi_4$ and $\psi_{10} = \Psi_3$ are entangled states, whereas ψ_{11} and ψ_{1-1} can be decomposed into two direct products, i.e. they are not entangled states. However, taking the linear combination of ψ_{11} and ψ_{1-1} to yield Ψ_1 and Ψ_2, then they form entangled states. The Bell states possess the maximal degree of entanglement[9]. In parallel to the transitions for three states of the spin triplet we may ask what kind of operators can make transitions between two Ψ_i states $(i = 1,2,3,4)$. The answer is through $\mathbf{J}$.

It can directly check the following transitions: ($a = \mu_1, b = \mu_2$, and J_2 is taken as $\sqrt{-1}J_2$ for simplicity)

$$\begin{aligned} J_3|1\rangle &= (a+b)|2\rangle, & J_3|2\rangle &= (a+b)|1\rangle; \\ J_3|3\rangle &= (a-b+1)|4\rangle, & J_3|4\rangle &= (a-b-1)|3\rangle; \end{aligned} \tag{2.7}$$

$$\begin{aligned} J_1|1\rangle &= (a+b)|3\rangle, & J_1|2\rangle &= -(a-b+1)|4\rangle; \\ J_1|3\rangle &= (a+b)|1\rangle, & J_1|4\rangle &= -(a-b-1)|2\rangle; \end{aligned} \tag{2.8}$$

$$\begin{aligned} J_2|1\rangle &= (a-b+1)|4\rangle, & J_2|2\rangle &= -(a+b)|3\rangle; \\ J_2|3\rangle &= (a+b)|2\rangle, & J_2|4\rangle &= -(a-b-1)|1\rangle. \end{aligned} \tag{2.9}$$

where $|i\rangle = \Psi_i$ $(i = 1,2,3,4)$. Because $|1\rangle,\cdots,|4\rangle$ form orthogonal and normalized states, Eqs. (2.7)-(2.9) gives a 4×4 representation of $\mathbf{J}$. We shall show that the states $|i\rangle$ $(i = 1,2,3,4)$ are eigenstate of $\mathbf{J}^2$ with $ab = -\frac{1}{4}$.

The following graphs explain Eqs. (2.7)-(2.9).

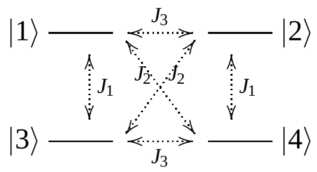

Fig 1. When $a-b=1$, $J_1|4\rangle = J_2|4\rangle = J_3|4\rangle = 0$;
when $a-b=-1$, $J_1|2\rangle = J_2|1\rangle = J_3|3\rangle = 0$.

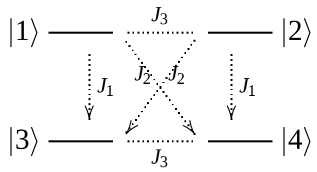

Fig 2. By taking the particular values of parameters a and b, some of the

transitions can be controlled in only one direction.

The Bell states are the pure entangled states for a system with two spins. They possess the maximal degree of entanglement and is the simplest member of Greenberger-Horne-Zeilinger (GHZ) entangled states for multi-spin system[10−12]. Before going to the general GHZ states let us introduce an operator $\hat{Q}$ which takes the four states $|i\rangle$ ($i=1,2,3,4$) as eigenstates. Therefore, it may describe the overall property for the four states.

Observing that

$$J_1^2|1\rangle = (a+b)^2|1\rangle, \qquad J_1^2|2\rangle = [(a-b)^2-1]|2\rangle;$$
$$J_1^2|3\rangle = (a+b)^2|3\rangle, \qquad J_1^2|4\rangle = [(a-b)^2-1]|4\rangle.$$

$$J_2^2|1\rangle = -[(a-b)^2-1]|1\rangle, \qquad J_2^2|2\rangle = -(a+b)^2|2\rangle;$$
$$J_2^2|3\rangle = -(a+b)^2|3\rangle, \qquad J_2^2|4\rangle = -[(a-b)^2-1]|4\rangle.$$

$$J_3^2|1\rangle = (a+b)^2|1\rangle, \qquad J_3^2|2\rangle = (a+b)^2|2\rangle;$$
$$J_3^2|3\rangle = [(a-b)^2-1]|3\rangle, \qquad J_3^2|4\rangle = [(a+b)^2-1]|4\rangle.$$

Noting that

$$\mathbf{J}^2 = \frac{1}{2}(J_1^2 - J_2^2 + J_3^2), \tag{2.10}$$

(here we use $\sqrt{-1}J_2$ instead of the usual J_2) and by taking

$$(a+b)^2 = [(a-b)^2-1], \tag{2.11}$$

i.e.

$$ab = -\frac{1}{4}, \tag{2.12}$$

we obtain

$$\mathbf{J}^2|i\rangle = \frac{3}{2}(a+b)^2|i\rangle \ \ (ab = -\frac{1}{4}), \ \ i = 1,2,3,4. \tag{2.13}$$

With the normalization we introduce $\hat{Q} \sim \mathbf{J}^2(ab = -\frac{1}{4})$:

$$\hat{Q}|i\rangle = \frac{2}{3}\frac{a^2}{(a^2-\frac{1}{4})^2}\mathbf{J}^2|i\rangle = |i\rangle, \ \ i = 1,2,3,4, \tag{2.14}$$

that acts on the four Bell states as the identity. It explores the common property of $|i\rangle$, i.e., $\hat{Q}$ indicates the fact that four Bell states have the same degree of entanglement.

On the other hand, if we take

$$(a+b)^2 = -[(a-b)^2 - 1], \tag{2.15}$$

i.e.

$$a^2 + b^2 = \frac{1}{2}, \tag{2.16}$$

the four states $|1\rangle, |2\rangle, |3\rangle, |4\rangle$ are divided into two sets (I) and (II) such that there are two states in each set. The sets (I) and (II) are the eigenstates of $J^2_\alpha, (\alpha = 1,2,3)$ with opposite signs of eigenvalues, namely, for $a^2+b^2 = \frac{1}{2}$ we have

$$\begin{aligned} &J_1^2|I\rangle = \lambda|I\rangle, J_1^2|II\rangle = (-\lambda)|II\rangle \\ &J_2^2|I'\rangle = \lambda|I'\rangle, J_2^2|II'\rangle = (-\lambda)|II'\rangle \\ &J_3^2|I''\rangle = \lambda|I''\rangle, J_3^2|I''\rangle = (-\lambda)|I''\rangle \end{aligned}$$

where

$$\begin{aligned} &\lambda = 2ab+1, \ \ |I\rangle = |1\rangle \ \ and \ \ |3\rangle, \ \ |II\rangle = |2\rangle \ \ and \ \ |4\rangle \\ &I' = |1\rangle \ \ and \ \ |4\rangle, \ \ II' = |2\rangle \ \ and \ \ |3\rangle, \\ &I'' = |1\rangle \ \ and \ \ |2\rangle, \ \ II'' = |3\rangle \ \ and \ \ |4\rangle \end{aligned} \tag{2.17}$$

So with the actions of J^2_α $(\alpha = 1,2,3)$ respectively, instead we have the following decomposition different from the usual decomposition $\underline{2}\times\underline{2} = \underline{3}\oplus\underline{1}$ for $SU(2)$**:**

$$\underline{2}\otimes\underline{2} = 2\oplus 2. \tag{2.18}$$

With the normalization we can introduce $\hat{T} \sim J^2_\alpha$, $(a^2+b^2 = \frac{1}{2}, \alpha = 1,2,3)$:

$$\hat{T}|i\rangle = \frac{1}{2ab+1}J^2_\alpha|i\rangle = |i\rangle, \forall\, |i\rangle \in \text{ set I } ; \hat{T}|i\rangle = \frac{1}{2ab+1}J^2_\alpha|i\rangle = -|i\rangle, \forall\, |i\rangle \in \text{ set II.} \tag{2.19}$$

It shows that under the actions of $\hat{T}$, the whole tensor space $\frac{1}{2}\otimes\frac{1}{2}$ is decomposed into two blocks corresponding to the eigenvalues 1 and -1 of $\hat{T}$ respectively. It differs from the parity $P = 1$ for spin triplet and -1 for spin singlet.

The further extension of Bell states goes along two lines. One is extended to the fundamental representation of $SU(n)$ algebras. The other is to multi-spin system.

2.2 Yangian and $SU(3)$ entangled states

The Gell-mann matrices λ_μ satisfy

$$[F_\lambda, F_\mu] = i f_{\lambda\mu\nu} F_\nu \ (\lambda,\mu,\nu = 1,\cdots,8), \tag{2.20}$$

where $F_\mu = \frac{1}{2}\lambda_\mu$. To the later convenience, we denote λ_μ by

$$H_1 = \sqrt{2}\lambda_3,\ H_2 = \sqrt{2}(-\frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{2}\lambda_8), E_1 = I_+ = \frac{1}{\sqrt{2}}(\lambda_1 + i\lambda_2),\ E_{-1} = I_- = \frac{1}{\sqrt{2}}(\lambda_1 - i\lambda_2),$$

$$E_2 = U_+ = \frac{1}{\sqrt{2}}(\lambda_6 + i\lambda_7),\ E_{-2} = U_- = \frac{1}{\sqrt{2}}(\lambda_6 - i\lambda_7), E_3 = V_- = \frac{1}{\sqrt{2}}(\lambda_4 + i\lambda_5)$$

and $E_{-3} = V_+ = \frac{1}{\sqrt{2}}(\lambda_4 - i\lambda_5)$. Explicitly, we have

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ E_{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ E_{-2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ E_{-3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.21}$$

Denoting by $|i,j\rangle = |i\rangle_1|j\rangle_2$ $(i,j = 0,1,2)$ in the tensor space of $SU(3)\otimes SU(3)^*$ where the basis $|i\rangle_1 = u_0, u_1$ and u_2 in $SU(3)$ fundamental representation (quark states) and $|j\rangle_2 = u_0^*, u_1^*$ and u_2^* are dual base (antiquark states), the $SU(3)$ entangled states with the maximal degree of entanglement were given by [13] as follows.

$$\begin{aligned}
\psi_1^{(1)} &= \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) \\
\psi_2^{(1)} &= \frac{1}{\sqrt{3}}(|00\rangle + \omega|11\rangle + \omega^2|22\rangle) \\
\psi_3^{(1)} &= \frac{1}{\sqrt{3}}(|00\rangle + \omega^2|11\rangle + \omega|22\rangle)
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
\psi_1^{(2)} &= \frac{1}{\sqrt{3}}(|01\rangle + |12\rangle + |20\rangle) \\
\psi_2^{(2)} &= \frac{1}{\sqrt{3}}(|01\rangle + \omega|12\rangle + \omega^2|20\rangle) \\
\psi_3^{(2)} &= \frac{1}{\sqrt{3}}(|01\rangle + \omega^2|12\rangle + \omega|20\rangle)
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
\psi_1^{(3)} &= \frac{1}{\sqrt{3}}(|02\rangle + |10\rangle + |21\rangle) \\
\psi_2^{(3)} &= \frac{1}{\sqrt{3}}(|02\rangle + \omega|10\rangle + \omega^2|21\rangle) \\
\psi_3^{(3)} &= \frac{1}{\sqrt{3}}(|02\rangle + \omega^2|10\rangle + \omega|21\rangle)
\end{aligned} \tag{2.24}$$

where $\omega^3 = 1$. The construction of Eqs. (2.22)-(2.24) is easy to be understood. Since $\psi_1^{(1)}$ is a maximally entangled state, any decomposable unitary transformations $U_1 \otimes U_2$ generate equivalent ones, where U_1 and U_2 acts on the space of $SU(3)$ and $SU(3)^*$, respectively. Taking $U_1 =$ diag $(1,\omega,\omega^2)$ or diag $(1,\omega^2,\omega)$ and $U_2 = \mathbf{1}$ we find $\psi_2^{(1)}$ and $\psi_3^{(2)}$. To ensure the unitarity and maximal degree of entanglement, the U_1 is unique.

To find the Yangian operators which make transitions for $\psi_j^{(i)}(i,j=1,2,3)$, we introduce (see the Appendix) the Yangian operators acting on the tensor space $SU(3)\otimes SU(3)^*$ as[14–16]

$$\mathbf{Y} = \{F_\mu, J_\mu = aF_\mu^{(1)} + bF_\mu^{(2)} + \frac{1}{2}c_{\mu\nu\sigma}F_\nu^{(1)}F_\sigma^{(2)}\} \tag{2.25}$$

where $c_{\mu\nu\sigma} = if_{\mu\nu\sigma}$ are the structure constants of $SU(3)$.

Further we introduce the linear combination of $SU(3)$ generators to form the new basis ($\omega^3 = 1$)

$$H_1' = \begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix},\ H_2' = \begin{pmatrix}1&0&0\\0&\omega^2&0\\0&0&\omega\end{pmatrix},\ E_1' = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\ E_{-1}' = \begin{pmatrix}0&0&1\\\omega&0&0\\0&\omega^2&0\end{pmatrix},$$

$$E_2' = \begin{pmatrix}0&1&0\\0&0&\omega\\\omega^2&0&0\end{pmatrix},\ E_{-2}' = \begin{pmatrix}0&0&1\\\omega^2&0&0\\0&\omega&0\end{pmatrix},\ E_3' = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\ E_{-3}' = \begin{pmatrix}0&1&0\\0&0&\omega^2\\\omega&0&0\end{pmatrix}. \tag{2.26}$$

i.e.

$$\begin{gathered}H_1' = H_1 - \omega^2 H_2,\ \ H_2' = H_1 - \omega H_2,\\ E_1' = E_1 + E_2 + E_{-3},\ E_2' = E_1 + \omega E_2 + \omega^2 E_{-3},\ E_{-3}' = E_1 + \omega^2 E_2 + \omega E_{-3},\\ E_3' = E_3 + E_{-1} + E_{-2},\ E_{-1}' = E_3 + \omega E_{-1} + \omega^2 E_{-2},\ E_{-2}' = E_3 + \omega^2 E_{-1} + \omega E_{-2}.\end{gathered} \tag{2.27}$$

The set Eq. (2.27) forms $sl(3)$ algebra. In terms of Eq. (2.27) the Yangian operators $J(H_1'), J(H_2'), J(E_1'), \cdots$ can be defined. According to the action of Yangian operators on the tensor space $SU(3)\otimes SU(3)^*$ $((1,0)\otimes(0,1))$ that is expressed in terms of H_1, H_2 and $E_{\pm i}(i=1,2,3)$ given in the Appendix, after lengthy calculations we find the following results.

$$\begin{aligned}J(H_1')\psi_1^{(1)} &= \gamma\psi_2^{(1)} & J(H_2')\psi_1^{(1)} &= \gamma\psi_3^{(1)}\\ J(H_1')\psi_2^{(1)} &= (a-b)\psi_3^{(1)} & J(H_2')\psi_2^{(1)} &= \tau\psi_1^{(1)}\\ J(H_1')\psi_3^{(1)} &= \tau\psi_1^{(1)} & J(H_2')\psi_3^{(1)} &= (a-b)\psi_2^{(1)}\end{aligned} \tag{2.28}$$

$$\begin{aligned}J(H_1')\psi_1^{(2)} &= \beta\psi_2^{(2)} & J(H_2')\psi_1^{(2)} &= \alpha\psi_3^{(2)}\\ J(H_1')\psi_2^{(2)} &= \beta\psi_3^{(2)} & J(H_2')\psi_2^{(2)} &= \alpha\psi_1^{(2)}\\ J(H_1')\psi_3^{(2)} &= \beta\psi_1^{(1)} & J(H_2')\psi_3^{(2)} &= \alpha\psi_2^{(2)}\end{aligned} \tag{2.29}$$

$$\begin{aligned}J(H_1')\psi_1^{(3)} &= \alpha\psi_2^{(3)} & J(H_2')\psi_1^{(3)} &= \beta\psi_3^{(3)}\\ J(H_1')\psi_2^{(3)} &= \alpha\psi_3^{(3)} & J(H_2')\psi_2^{(3)} &= \beta\psi_1^{(3)}\\ J(H_1')\psi_3^{(3)} &= \alpha\psi_1^{(3)} & J(H_2')\psi_3^{(3)} &= \beta\psi_2^{(3)}\end{aligned} \tag{2.30}$$

where $\alpha = a - b\omega^2, \beta = a - b\omega, \gamma = a - b - \frac{3}{2}, \tau = a - b + \frac{3}{2}, \omega^3 = 1$.

$$\begin{array}{lll} J(E'_1)\psi_1^{(1)} = \gamma\psi_1^{(2)} & J(E'_2)\psi_1^{(1)} = \gamma\psi_2^{(2)} & J(E'_{-3})\psi_1^{(1)} = \gamma\psi_3^{(2)} \\ J(E'_1)\psi_2^{(1)} = \omega\alpha\psi_2^{(2)} & J(E'_2)\psi_2^{(1)} = \omega\alpha\psi_3^{(2)} & J(E'_{-3})\psi_2^{(1)} = \omega\alpha\psi_1^{(2)} \\ J(E'_1)\psi_3^{(1)} = \omega^2\beta\psi_3^{(2)} & J(E'_2)\psi_3^{(1)} = \omega^2\beta\psi_1^{(2)} & J(E'_{-3})\psi_3^{(1)} = \omega^2\beta\psi_2^{(2)} \end{array} \tag{2.31}$$

$$\begin{array}{lll} J(E'_1)\psi_1^{(2)} = (a-b)\psi_1^{(3)} & J(E'_2)\psi_1^{(2)} = \beta\psi_2^{(3)} & J(E'_{-3})\psi_1^{(2)} = \alpha\psi_3^{(3)} \\ J(E'_1)\psi_2^{(2)} = \omega\alpha\psi_2^{(3)} & J(E'_2)\psi_2^{(2)} = \omega(a-b)\psi_3^{(3)} & J(E'_{-3})\psi_2^{(2)} = \omega\beta\psi_1^{(3)} \\ J(E'_1)\psi_3^{(2)} = \omega^2\beta\psi_3^{(3)} & J(E'_2)\psi_3^{(2)} = \omega^2\alpha\psi_1^{(3)} & J(E'_{-3})\psi_3^{(2)} = \omega^2(a-b)\psi_2^{(3)} \end{array} \tag{2.32}$$

$$\begin{array}{lll} J(E'_1)\psi_1^{(3)} = \tau\psi_1^{(1)} & J(E'_2)\psi_1^{(3)} = \omega\alpha\psi_2^{(1)} & J(E'_{-3})\psi_1^{(3)} = \omega^2\beta\psi_3^{(1)} \\ J(E'_1)\psi_2^{(3)} = \omega\alpha\psi_2^{(1)} & J(E'_2)\psi_2^{(3)} = \omega\beta\psi_3^{(1)} & J(E'_{-3})\psi_2^{(3)} = \omega\tau\psi_1^{(1)} \\ J(E'_1)\psi_3^{(3)} = \omega^2\alpha\psi_3^{(1)} & J(E'_2)\psi_3^{(3)} = \omega^2\tau\psi_1^{(1)} & J(E'_{-3})\psi_3^{(3)} = \omega\alpha\psi_2^{(1)}. \end{array} \tag{2.33}$$

$$\begin{array}{lll} J(E'_3)\psi_1^{(1)} = \gamma\psi_1^{(3)} & J(E'_{-1})\psi_1^{(1)} = \gamma\psi_2^{(3)} & J(E'_{-2})\psi_1^{(1)} = \gamma\psi_3^{(3)} \\ J(E'_2)\psi_2^{(1)} = \omega^2\beta\psi_2^{(3)} & J(E'_{-1})\psi_2^{(1)} = \omega^2\beta\psi_3^{(3)} & J(E'_{-2})\psi_2^{(1)} = \omega^2\beta\psi_1^{(3)} \\ J(E'_3)\psi_3^{(1)} = \omega\alpha\psi_3^{(3)} & J(E'_{-1})\psi_3^{(1)} = \omega\alpha\psi_1^{(3)} & J(E'_{-2})\psi_3^{(1)} = \omega\alpha\psi_2^{(3)} \end{array} \tag{2.34}$$

$$\begin{array}{lll} J(E'_3)\psi_1^{(2)} = \gamma\psi_1^{(1)} & J(E'_{-1})\psi_1^{(2)} = \beta\psi_2^{(1)} & J(E'_{-2})\psi_1^{(2)} = \alpha\psi_3^{(1)} \\ J(E'_2)\psi_2^{(2)} = \omega^2\beta\psi_2^{(1)} & J(E'_{-1})\psi_2^{(2)} = \omega^2\alpha\psi_3^{(1)} & J(E'_{-2})\psi_2^{(2)} = \omega^2\tau\psi_1^{(1)} \\ J(E'_3)\psi_3^{(2)} = \omega\alpha\psi_3^{(1)} & J(E'_{-1})\psi_3^{(2)} = \omega\tau\psi_1^{(1)} & J(E'_{-2})\psi_3^{(2)} = \omega\beta\psi_2^{(1)} \end{array} \tag{2.35}$$

$$\begin{array}{lll} J(E'_3)\psi_1^{(3)} = (a-b)\psi_1^{(2)} & J(E'_{-1})\psi_1^{(3)} = \alpha\psi_2^{(2)} & J(E'_{-2})\psi_1^{(3)} = \beta\psi_3^{(2)} \\ J(E'_2)\psi_2^{(3)} = \omega^2\beta\psi_2^{(2)} & J(E'_{-1})\psi_2^{(3)} = \omega^2(a-b)\psi_3^{(2)} & J(E'_{-2})\psi_2^{(3)} = \omega^2\alpha\psi_1^{(2)} \\ J(E'_3)\psi_3^{(3)} = \omega\alpha\psi_3^{(2)} & J(E'_{-1})\psi_3^{(3)} = \omega\beta\psi_1^{(2)} & J(E'_{-2})\psi_3^{(3)} = \omega(a-b)\psi_2^{(2)} \end{array} \tag{2.36}$$

In Eqs. (2.28)-(2.36) all the $\psi_j^{(m)}(m, j = 1,2,3)$ are given by Eqs. (2.22)-(2.24) that are with the maximal degree of entanglement. The Eqs. (2.28)-(2.36) also show that the $\psi_j^{(m)}$ provide a nice set of basis which makes the representation of $Y(sl(3))$ given in the Appendix in simple way.

Furthermore, obviously, for the lower index j of the states $\psi_j^{(m)}(j,m = 1,2,3)$, the above operators are the following permutations, respectively

$$J(H'_1) : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \ J(H'_2) : \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, J(E'_1) : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \ J(E'_2) : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$

$$J(E'_{-3}) : \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \ J(E'_3) : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \ J(E'_{-1}) : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \ J(E'_{-2}) : \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

An interesting discrete operator similar to $\mathbf{J}^2(ab = -\frac{1}{4})$ shown in Eq. (2.12) for $SU(2)$ can be found. It is the extension of $\mathbf{J}^2$ for $Y(sl(3))$ corresponding to the Casimir operator of $sl(3)$ under the inner product Trxy :

$$\Omega = \frac{2}{3}(H_1 \otimes H_1 + H_2 \otimes H_2) + \frac{1}{3}(H_1 \otimes H_2 + H_2 \otimes H_1) + \sum_{i=1}^{3}(E_i \otimes E_{-i} + E_{-i} \otimes E_i) \tag{2.37}$$

that acts on the octet wave function $w_{1,1} = |02\rangle = \frac{1}{\sqrt{3}}(\psi_1^2 + \omega\psi_2^{(2)} + \omega^2\psi_3^{(2)})$ (in fact, $\mathbf{J}^2$ acts on the whole vector space $V(1,1)$ by a constant) and singlet one $v_{00} = \psi_1^{(1)}$ yields

$$\mathbf{J}^2 w_{1,1} = [\frac{8}{3}(a^2+b^2) + \frac{2}{3}ab - \frac{3}{4}]w_{1,1} = \rho w_{1,1}, \tag{2.38}$$

$$\mathbf{J}^2 w_{0,0} = [\frac{8}{3}(a^2+b^2) - \frac{16}{3}ab - 6]w_{0,0}. \tag{2.39}$$

By identifying both of eigenvalues we arrive at

$$ab = -\frac{7}{8}, \tag{2.40}$$

i.e., under the choice of Eq. (2.40) the total nine states share the same eigenvalues of $\mathbf{J}^2$. After renormalization we have $\hat{Q}_s = \frac{1}{\rho}\mathbf{J}^2$ as the identity multiplication. Making transformation $\{w_{1,1}, \cdots, v_{0,0}\} \to \psi_j^{(m)}(m, j = 1,2,3)$ and Eq. (2.21)→ (2.26), then under the new basis given by Eq. (2.26), we still have $\mathbf{J}^2\psi_j^{(m)} = \rho\psi_j^{(m)}$ if $ab = -\frac{7}{8}$. Since $\psi_j^{(m)}$ describe the maximal degree of entanglement and all the wave functions (including the Octet and singlet) share the same eigenvalues, we can imagine that the $\mathbf{J}^2$ may be related to the maximal degree of entanglement. Obviously, the $\mathbf{J}^2$ with $ab = -\frac{7}{8}$ is the natural extension of $\mathbf{J}^2$ with $ab = -\frac{3}{4}$ for $SU(2)$.

The above discussions can be viewed as a realization of the representation theory for $Y(sl(3))$[14–16] based on the entangled states.

2.3 Maximally entangled states for $SU(n)$

The maximally entangled states for $SU(3)$ can be extended to the tensor product $SU(n) \otimes SU(n)^*$, that is, the tensor product of the fundamental representation $\lambda_1 = (1,0,\cdots,0)$ of $SU(n)$ and its dual representation $\lambda_n = \lambda_1^* = (0,\cdots,0,1)$. We choose the basis of $sl(n)$ as follows.

$$H_1' = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \omega^{n-1} \end{pmatrix}, \ H_2' = (H_1')^2 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega^2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \omega^{2(n-1)} \end{pmatrix}, \ \cdots,$$

$$H_{n-1}' = (H_1')^{n-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega^{n-1} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \omega \end{pmatrix}, \ \omega^n = 1. \tag{2.41}$$

$$E_1^{(1)'} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ E_2^{(1)'} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \omega & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & \omega^{n-2} \\ \omega^{n-1} & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \cdots,$$

$$E_n^{(1)'} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \omega^{n-1} & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & \omega^2 \\ \omega & 0 & 0 & \cdots & 0 \end{pmatrix}; \qquad (2.42)$$

$$\cdots, \ \cdots, \ \cdots, \ \cdots$$

$$E_1^{(n-1)'} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, E_2^{(n-1)'} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ \omega & 0 & \cdots & 0 & 0 \\ 0 & \omega^2 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & \omega^{n-1} & 0 \end{pmatrix}, \cdots,$$

$$E_n^{(n-1)'} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ \omega^{n-1} & 0 & \cdots & 0 & 0 \\ 0 & \omega^{n-2} & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & \omega & 0 \end{pmatrix}. \qquad (2.43)$$

In this subsection, for any index $a > n$ we take the value $a' \leq n$ satisfying $a = a'(\text{mod}\, n)$ and we still denote it by a. The wave functions take the forms[13]

$$\begin{aligned} \Psi_k^{(m)} &= |1,m\rangle + \omega^{k-1}|2,m+1\rangle + \cdots + \omega^{(n-1)(k-1)}|n,m-1\rangle \\ &= \sum_{j=1}^{n} \omega^{(k-1)(j-1)}|j,m+j-1\rangle, (k,m = 1,2,\cdots,n), \end{aligned} \qquad (2.44)$$

where $|i,j\rangle = |i\rangle_1|j\rangle_2$ $(i,j = 1,2,\cdots,n)$ are basis states in the tensor space of $SU(n) \otimes SU(n)^*$, $|i\rangle_1 = 1,\cdots,n$ are base of the fundamental representation with the weight $\lambda_1 = (1,0,\cdots,n)$ and $|j\rangle_2 = 1^*,\cdots,n^*$ are the dual base.

We can choose a standard basis $\{I_\alpha\}$ of $sl(n)$ under the inner product $B(x,y) = \mathrm{Tr}xy$, $\forall x,y \in sl(n)$. Then the Yangian operators acting on the tensor space $SU(n) \otimes SU(n)^*$ are given by (see Eq. (2.25) and the Appendix)

$$\mathbf{Y} = \{I_\mu, J_\mu = aI_\mu^{(1)} + bI_\mu^{(2)} + \frac{1}{2}c_{\mu\nu\sigma}I_\nu^{(1)}I_\sigma^{(2)}\} \qquad (2.45)$$

where $c_{\mu\nu\sigma}$ are the structure constants of $sl(n)$ under the basis $\{I_\alpha\}$. On account of the base of $sl(n)$ given by Eqs. (2.41)-(2.43) and the wave functions given by Eq. (2.44), the actions of the Yangian operators are given as follows.

$$J(H_i') \begin{cases} \Psi_k^{(1)} = (a-b-\frac{n}{2})\Psi_{i+1}^{(1)}, & k=1 \\ \Psi_k^{(1)} = (a-b+\frac{n}{2})\Psi_1^{(1)}, & k=n-1-i \\ \Psi_k^{(1)} = (a-b)\Psi_{k+i}^{(1)}, & k \neq 1, n+1-i \\ \Psi_k^{(m)} = [a-b\omega^{i(m-1)}]\Psi_{k+i}^{(m)}, & m=2,\cdots,n \end{cases} \qquad (2.46)$$

$$J(E_i^{(j)'})\begin{cases}\Psi_1^{(1)}=(a-b-\frac{n}{2})\Psi_i^{(1+j)}\\ \Psi_k^{(1)}=(a\omega^{j(k-1)}-b)\Psi_{k+i-1}^{(j+1)},k=2,\cdots,n\\ \Psi_{n+2-i}^{(n+1-j)}=(a-b+\frac{n}{2})\omega^{j(k-1)}\Psi_1^{(1)}\\ \Psi_k^{(n+1-j)}=(a\omega^{j(k-1)}-b\omega^{n+1-i})\Psi_{k+i-1}^{(1)},k\neq n+2-i\\ \Psi_k^{(m)}=(a\omega^{j(k-1)}-b\omega^{(i-1)(m-1)})\Psi_{k+i-1}^{(m+j)},m\neq 1,n+1-j\end{cases} \tag{2.47}$$

where $i=1,\cdots,n$; $j=1,\cdots,n-1$. It can be checked that the above generalized relations can reduce to the given actions for $SU(2)$ and $SU(3)$.

Furthermore, rewriting the above basis Eqs. (2.41)-(2.43) as follows.

$$T_i^{(1)}=H'_{i-1},\ i=2,\cdots,n;\ T_i^{(j)}=E_i^{(j+1)'},i=1,\cdots,n,j=2,\cdots,n. \tag{2.48}$$

we have

$$T_i^{(j)}=\sum_{k=1}^{n}\omega^{(i-1)(k-1)}e_{k,j+k-1},\ \text{for}\ \ j=1,i\neq 1, \tag{2.49}$$

where $(e_{km})_{ij}=\delta_{ki}\delta_{mj}$. Then for the entangled wave functions given by Eq. (2.44), the Eqs. (2.46)-(2.49) can be rewritten as (for $i\neq 1, j\neq 1$)

$$J(T_i^{(j)})\Psi_k^{(m)}=[a\omega^{(j-1)(k-1)}-b\omega^{(i-1)(m-1)}+\frac{n}{2}\delta_{i+k-1,1}\delta_{m+j-1,1}\omega^{(j-1)(k-1)}-\frac{n}{2}\delta_{k,1}\delta_{m,1}]\Psi_{i+k-1}^{(m+j-1)}. \tag{2.50}$$

When $n=2$, they reduce to $J(T_2^{(1)})=J_3$, $J(T_1^{(2)})=J_1$, $J(T_2^{(2)})=J_2$, $\Psi_2^{(2)}=|1\rangle$, $\Psi_1^{(2)}=|2\rangle$, $\Psi_2^{(1)}=-|3\rangle$, $\Psi_2^{(1)}=-|4\rangle$ and yield all the results in the subsection § 2.1.

When $n=3$ they reduce to $T_2^{(1)}=H'_1$, $T_3^{(1)}=H'_2$, $T_1^{(2)}=E'_1$, $T_2^{(2)}=E'_2$, $T_3^{(2)}=E'_{-3}$, $T_1^{(3)}=E'_3$, $T_2^{(3)}=E'_{-1}$, $T_3^{(3)}=E'_{-2}$ and $\Psi_k^{(m)}=\psi_k^{(m)}$ are exactly the entangled states for $SU(3)$. It yields all the results in the subsection § 2.2.

2.4 GHZ states with 3 spins

Now we turn to maximally entangled states formed by 3 spins. In this example there are 8 independent entangled states[10−11]:

$$\begin{aligned}|1\rangle\rangle&=\frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle+|\downarrow\downarrow\downarrow\rangle);\ |2\rangle\rangle=\frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle-|\downarrow\downarrow\downarrow\rangle);\\ |3\rangle\rangle&=\frac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\rangle+|\downarrow\downarrow\uparrow\rangle);\ |4\rangle\rangle=\frac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\rangle-|\downarrow\downarrow\uparrow\rangle);\\ |5\rangle\rangle&=\frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle+|\downarrow\uparrow\downarrow\rangle);\ |6\rangle\rangle=\frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle-|\downarrow\uparrow\downarrow\rangle);\\ |7\rangle\rangle&=\frac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\rangle+|\uparrow\downarrow\downarrow\rangle);\ |8\rangle\rangle=\frac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\rangle-|\uparrow\downarrow\downarrow\rangle).\end{aligned} \tag{2.51}$$

Similar to §2.1, $S_1\otimes S_2\otimes S_3$ can be decomposed into three sets:

(I) $|1\rangle\rangle$ and $|2\rangle\rangle$ that with $I_3=\pm\frac{3}{2}$;

(II) $|3\rangle\rangle$, $|5\rangle\rangle$ and $|7\rangle\rangle$;

(III) $|4\rangle\rangle$, $|6\rangle\rangle$ and $|8\rangle\rangle$.

The action of **J** is listed below (where $a=\mu_1, b=\mu_2$ and $c=\mu_3$ in Eq. (2.2)).

$$\begin{aligned}
&J_1|1\rangle\rangle=(c-1)|3\rangle\rangle+b|5\rangle\rangle+(a+1)|7\rangle\rangle; \;\; J_1|2\rangle\rangle=(c-1)|4\rangle\rangle+b|6\rangle\rangle+(a+1)|8\rangle\rangle;\\
&J_1|3\rangle\rangle=(c+1)|1\rangle\rangle+a|5\rangle\rangle+(b-1)|7\rangle\rangle; \;\; J_1|4\rangle\rangle=(c+1)|2\rangle\rangle-a|6\rangle\rangle-(b-1)|8\rangle\rangle;\\
&J_1|5\rangle\rangle=b|1\rangle\rangle+a|3\rangle\rangle+c|7\rangle\rangle; \;\; J_1|6\rangle\rangle=b|2\rangle\rangle-a|4\rangle\rangle-c|8\rangle\rangle;\\
&J_1|7\rangle\rangle=(a-1)|1\rangle\rangle+(b+1)|3\rangle\rangle+c|5\rangle\rangle; \;\; J_1|8\rangle\rangle=(a-1)|2\rangle\rangle-(b+1)|4\rangle\rangle-c|6\rangle\rangle;
\end{aligned} \tag{2.52}$$

$$\begin{aligned}
&J_2|1\rangle\rangle=(1-c)|4\rangle\rangle-b|6\rangle\rangle-(a+1)|8\rangle\rangle; \;\; J_2|2\rangle\rangle=(1-c)|3\rangle\rangle-b|5\rangle\rangle-(a+1)|7\rangle\rangle;\\
&J_2|3\rangle\rangle=(c+1)|2\rangle\rangle+a|6\rangle\rangle+(b-1)|8\rangle\rangle; \;\; J_2|4\rangle\rangle=(c+1)|1\rangle\rangle-a|5\rangle\rangle-(b-1)|7\rangle\rangle;\\
&J_2|5\rangle\rangle=b|2\rangle\rangle+a|4\rangle\rangle+c|8\rangle\rangle; \;\; J_2|6\rangle\rangle=b|1\rangle\rangle-a|3\rangle\rangle-c|7\rangle\rangle;\\
&J_2|7\rangle\rangle=(a-1)|2\rangle\rangle+(b+1)|4\rangle\rangle+c|6\rangle\rangle; \;\; J_2|8\rangle\rangle=(a-1)|1\rangle\rangle-(b+1)|3\rangle\rangle-c|5\rangle\rangle;
\end{aligned} \tag{2.53}$$

$$\begin{aligned}
&J_3|1\rangle\rangle=(a+b+c)|2\rangle\rangle; \;\; J_3|2\rangle\rangle=(a+b+c)|1\rangle\rangle;\\
&J_3|3\rangle\rangle=(a+b-c)|4\rangle\rangle-|6\rangle\rangle-|8\rangle\rangle; \;\; J_3|4\rangle\rangle=(a+b-c)|3\rangle\rangle-|5\rangle\rangle-|7\rangle\rangle;\\
&J_3|5\rangle\rangle=|4\rangle\rangle+(a-b+c)|6\rangle\rangle-|8\rangle\rangle; \;\; J_3|6\rangle\rangle=|3\rangle\rangle+(a-b+c)|5\rangle\rangle-|7\rangle\rangle;\\
&J_3|7\rangle\rangle=|4\rangle\rangle+|6\rangle\rangle+(-a+b+c)|8\rangle\rangle; \;\; J_3|8\rangle\rangle=|3\rangle\rangle+|5\rangle\rangle+(-a+b+c)|7\rangle\rangle.
\end{aligned} \tag{2.54}$$

By (2.54), we obtain the action of J_3^2 as follows.

$$\begin{aligned}
&J_3^2|1\rangle\rangle=(a+b+c)^2|1\rangle\rangle;\\
&J_3^2|2\rangle\rangle=(a+b+c)^2|2\rangle\rangle;\\
&J_3^2|3\rangle\rangle=[(a+b-c)^2-2]|3\rangle\rangle-(2a+1)|5\rangle\rangle+(1-2b)|7\rangle\rangle;\\
&J_3^2|4\rangle\rangle=[(a+b-c)^2-2]|4\rangle\rangle-(2a+1)|6\rangle\rangle+(1-2b)|8\rangle\rangle;\\
&J_3^2|5\rangle\rangle=(2a-1)|3\rangle\rangle+[(a-b+c)^2-2]|5\rangle\rangle-(2c+1)|7\rangle\rangle;\\
&J_3^2|6\rangle\rangle=(2a-1)|4\rangle\rangle+[(a-b+c)^2-2]|6\rangle\rangle-(2c+1)|8\rangle\rangle;\\
&J_3^2|7\rangle\rangle=(2b+1)|3\rangle\rangle+(2c-1)|5\rangle\rangle+[(-a+b+c)^2-2]|7\rangle\rangle;\\
&J_3^2|8\rangle\rangle=(2b+1)|4\rangle\rangle+(2c-1)|6\rangle\rangle+[(-a+b+c)^2-2]|8\rangle\rangle.
\end{aligned} \tag{2.55}$$

Therefore there are the following transitions.

I $\longleftrightarrow$ II, III

$$\begin{matrix} J_1|1\rangle\rangle \longrightarrow |3\rangle\rangle \\ J_2|2\rangle\rangle \longrightarrow |3\rangle\rangle \\ J_1|2\rangle\rangle \longrightarrow |4\rangle\rangle \\ J_2|1\rangle\rangle \longrightarrow |4\rangle\rangle \end{matrix} \quad \text{for} \quad \begin{matrix} a=-1 \\ b=0 \\ c\neq 1 \end{matrix}\;; \quad \begin{matrix} J_1|3\rangle\rangle \longrightarrow |1\rangle\rangle \\ J_2|4\rangle\rangle \longrightarrow |1\rangle\rangle \\ J_1|4\rangle\rangle \longrightarrow |2\rangle\rangle \\ J_2|3\rangle\rangle \longrightarrow |2\rangle\rangle \end{matrix} \quad \text{for} \quad \begin{matrix} a=0 \\ b=1 \\ c\neq -1 \end{matrix}\;; \tag{2.56}$$

$$\begin{matrix} J_1|1\rangle\rangle \longrightarrow |5\rangle\rangle \\ J_2|2\rangle\rangle \longrightarrow |5\rangle\rangle \\ J_1|2\rangle\rangle \longrightarrow |6\rangle\rangle \\ J_2|1\rangle\rangle \longrightarrow |6\rangle\rangle \end{matrix} \quad \text{for} \quad \begin{matrix} a=-1 \\ b\neq 0 \\ c=1 \end{matrix}\;; \quad \begin{matrix} J_1|5\rangle\rangle \longrightarrow |1\rangle\rangle \\ J_2|6\rangle\rangle \longrightarrow |1\rangle\rangle \\ J_1|6\rangle\rangle \longrightarrow |2\rangle\rangle \\ J_2|5\rangle\rangle \longrightarrow |2\rangle\rangle \end{matrix} \quad \text{for} \quad \begin{matrix} a=0 \\ b\neq 0; \\ c=0 \end{matrix} \tag{2.57}$$

$$\begin{array}{l} J_1|1\rangle\rangle \longrightarrow |7\rangle\rangle \\ J_2|2\rangle\rangle \longrightarrow |7\rangle\rangle \\ J_1|2\rangle\rangle \longrightarrow |8\rangle\rangle \\ J_2|1\rangle\rangle \longrightarrow |8\rangle\rangle \end{array} \text{ for } \begin{array}{c} a \neq -1 \\ b = 0 \\ c = 1 \end{array}; \quad \begin{array}{l} J_1|7\rangle\rangle \longrightarrow |1\rangle\rangle \\ J_2|8\rangle\rangle \longrightarrow |1\rangle\rangle \\ J_1|8\rangle\rangle \longrightarrow |2\rangle\rangle \\ J_2|7\rangle\rangle \longrightarrow |2\rangle\rangle \end{array} \text{ for } \begin{array}{c} a \neq 1 \\ b = -1 \\ c = 0 \end{array}; \qquad (2.58)$$

$$\text{II} \longleftrightarrow \text{II, III} \longleftrightarrow \text{III}$$

$$\begin{array}{l} J_3^2|3\rangle\rangle \longrightarrow |5\rangle\rangle \\ J_3^2|4\rangle\rangle \longrightarrow |6\rangle\rangle \end{array} \text{ for } \begin{array}{c} a \neq -\frac{1}{2} \\ b = \frac{1}{2} \\ c = a + \frac{1}{2} \pm \sqrt{2} \end{array}; \quad \begin{array}{l} J_3^2|3\rangle\rangle \longrightarrow |7\rangle\rangle \\ J_3^2|4\rangle\rangle \longrightarrow |8\rangle\rangle \end{array} \text{ for } \begin{array}{c} a = -\frac{1}{2} \\ b \neq \frac{1}{2} \\ c = b - \frac{1}{2} \pm \sqrt{2} \end{array}; \qquad (2.59)$$

$$\begin{array}{l} J_3^2|5\rangle\rangle \longrightarrow |3\rangle\rangle \\ J_3^2|6\rangle\rangle \longrightarrow |4\rangle\rangle \end{array} \text{ for } \begin{array}{c} a \neq \frac{1}{2} \\ b = a - \frac{1}{2} \pm \sqrt{2}; \\ c = -\frac{1}{2} \end{array} \quad \begin{array}{l} J_3^2|5\rangle\rangle \longrightarrow |7\rangle\rangle \\ J_3^2|6\rangle\rangle \longrightarrow |8\rangle\rangle \end{array} \text{ for } \begin{array}{c} a = \frac{1}{2} \\ b = c + \frac{1}{2} \pm \sqrt{2}; \\ c \neq -\frac{1}{2} \end{array} \qquad (2.60)$$

$$\begin{array}{l} J_3^2|7\rangle\rangle \longrightarrow |3\rangle\rangle \\ J_3^2|8\rangle\rangle \longrightarrow |4\rangle\rangle \end{array} \text{ for } \begin{array}{c} a = b + \frac{1}{2} \pm \sqrt{2} \\ b \neq -\frac{1}{2} \\ c = \frac{1}{2} \end{array}; \quad \begin{array}{l} J_3^2|7\rangle\rangle \longrightarrow |5\rangle\rangle \\ J_3^2|8\rangle\rangle \longrightarrow |6\rangle\rangle \end{array} \text{ for } \begin{array}{c} a = c - \frac{1}{2} \pm \sqrt{2} \\ b = -\frac{1}{2} \\ c \neq \frac{1}{2} \end{array}. \qquad (2.61)$$

Similar to the Bell states, we are able to find the conditions satisfied by the parameters a, b and c such that all the 8 entangled states are eigenstates of the operator $\mathbf{J}^2(a,b,c)$. The calculation is lengthy. We first set $b = a + c$ and then find the sufficient condition

$$c^2 = i\{i\frac{15\lambda - 3}{4} \pm [\frac{135}{8}(1+\lambda)]^{\frac{1}{2}}\}/[6(\lambda+1)]h^2, \qquad (2.62)$$

where $\lambda = \pm\sqrt{3}i$ and h is a free parameter. The fact that $\mathbf{J}^2$ gets the same eigenvalue for the 8 entangled states indicates that the operator $\mathbf{J}^2$ may be related to the maximal degree of entanglement.

2.5 Extended GHZ states with 4 spins

There are $2^4 = 16$ independent entangled states as follows.

$$\begin{array}{ll}
|1) = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle); & |2) = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\downarrow\rangle); \\
|3) = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\downarrow\rangle + |\downarrow\downarrow\downarrow\uparrow\rangle); & |4) = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\downarrow\rangle - |\downarrow\downarrow\downarrow\uparrow\rangle); \\
|5) = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\uparrow\rangle + |\downarrow\downarrow\uparrow\downarrow\rangle); & |6) = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\uparrow\rangle - |\downarrow\downarrow\uparrow\downarrow\rangle); \\
|7) = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\uparrow\rangle + |\downarrow\uparrow\downarrow\downarrow\rangle); & |8) = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\uparrow\rangle - |\downarrow\uparrow\downarrow\downarrow\rangle); \\
|9) = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\downarrow\downarrow\rangle); & |10) = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\uparrow\rangle - |\uparrow\downarrow\downarrow\downarrow\rangle); \\
|11) = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle); & |12) = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\downarrow\uparrow\rangle); \\
|13) = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle); & |14) = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\downarrow\rangle); \\
|15) = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle); & |16) = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\uparrow\rangle).
\end{array} \qquad (2.63)$$

We list the actions of **J** on the above 16 states with $\mu_1 = a, \mu_2 = b, \mu_3 = c$ and $\mu_4 = d$ in Eq. (2.2) in the following.

$$
\begin{aligned}
J_1|1) &= (d-\frac{3}{2})|3) + (c-\frac{1}{2})|5) + (b+\frac{1}{2})|7) + (a+\frac{3}{2})|9);\\
J_1|2) &= (d-\frac{3}{2})|4) + (c-\frac{1}{2})|6) + (b+\frac{1}{2})|8) + (a+\frac{3}{2})|10);\\
J_1|3) &= (d+\frac{3}{2})|1) + (b-\frac{1}{2})11) + (a+\frac{1}{2})|13) + (c-\frac{3}{2})|15);\\
J_1|4) &= (d+\frac{3}{2})|2) + (b-\frac{1}{2})12) - (a+\frac{1}{2})|14) + (c-\frac{3}{2})|16);\\
J_1|5) &= (c+\frac{1}{2})|1) + (a+\frac{1}{2})11) + (b-\frac{1}{2})|13) + (d-\frac{1}{2})|15);\\
J_1|6) &= (c+\frac{1}{2})|2) - (a+\frac{1}{2})12) + (b-\frac{1}{2})|14) + (d-\frac{1}{2})|16);\\
J_1|7) &= (b-\frac{1}{2})|1) + (d-\frac{1}{2})11) + (c+\frac{1}{2})|13) + (a+\frac{1}{2})|15);\\
J_1|8) &= (b-\frac{1}{2})|2) + (d-\frac{1}{2})12) + (c+\frac{1}{2})|14) - (a+\frac{1}{2})|16);\\
J_1|9) &= (a-\frac{3}{2})|1) + (c+\frac{1}{2})11) + (d-\frac{1}{2})|13) + (b+\frac{3}{2})|15);\\
J_1|10) &= (a-\frac{3}{2})|2) - (c+\frac{1}{2})12) - (d-\frac{1}{2})|14) - (b+\frac{3}{2})|16);\\
J_1|11) &= (b+\frac{1}{2})|3) + (a-\frac{1}{2})5) + (d+\frac{1}{2})|7) + (c-\frac{1}{2})|9);\\
J_1|12) &= (b+\frac{1}{2})|4) - (a-\frac{1}{2})6) + (d+\frac{1}{2})|8) - (c-\frac{1}{2})|10);\\
J_1|13) &= (a-\frac{1}{2})|3) + (b+\frac{1}{2})5) + (c-\frac{1}{2})|7) + (d+\frac{1}{2})|9);\\
J_1|14) &= -(a-\frac{1}{2})|4) + (b+\frac{1}{2})6) + (c-\frac{1}{2})|8) - (d+\frac{1}{2})|10);\\
J_1|15) &= (c+\frac{3}{2})|3) + (d+\frac{1}{2})5) + (a-\frac{1}{2})|7) + (b-\frac{3}{2})|9);\\
J_1|16) &= (c+\frac{3}{2})|4) + (d+\frac{1}{2})6) - (a-\frac{1}{2})|8) - (b-\frac{3}{2})|10);
\end{aligned}
\tag{2.64}
$$

and the actions for $J_\alpha|i\rangle \;\; (\alpha = 2,3, \;\; i = 1,2,\cdots,8)$ are shown in the Appendix B1-B3.

Observing eq(2.64) and Appendix B we can get the transitions between GHZ states as given in the subsection §2.2.

In a short conclusion, for the two types of maximally entangled states where one type is formed in terms of fundamental representations of $SU(n)$ and the other is regarding multi-spin systems, we have showed the following statements.

(1) Yangian operators $J_\mu(a,b,\cdots)$ act on the whole Hilbert tensor space formed by all the maximally entangled states. They make the transitions between the singlet and multiplet. The role played by Yangian in entangled states is similar to what Lie algebras do within reduced subspaces with fixed $\mathbf{I}^2$. In this sense we may say that the maximally entangled states may provide an explicit

Yangian representation, i.e. they are described in terms of Hopf algebra. The point here is that Yangian is formed by 2-body operators.

(2) The operators of the Casimirs of Yangians with the special choice of parameters are the discrete eigen-operators of the considered entangled states. It may serve to describe the maximal degree of entanglement.

3. GHZ states and Yang-Baxter equation

For a given set of GHZ states (and the extension to the fundamental representations of $SU(n)$) in section 2 we have pointed out that the different GHZ states are connected through Yangian operators. Another point of view is to generate GHZ states based on the natural multi-spin states. For Bell states it was first pointed out by Kauffman and Lomonaco[17] that the matrix

$$B_4=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&-1&1&0\\-1&0&0&1\end{pmatrix}=\frac{1}{\sqrt{2}}(\mathbf{1}+\begin{pmatrix}&&&1\\&&1&\\&-1&&\\-1&&&\end{pmatrix}), \tag{3.1}$$

transforms the states $(\uparrow\uparrow,\uparrow\downarrow,\downarrow\uparrow,\downarrow\downarrow)^T$ to the Bell states $(\Psi_1,\Psi_3,-\Psi_4,-\Psi_2)^T$ given by Eq. (2.6). For the Bell states the related $\check{R}(x)$-matrix and Hamiltonian were found in Refs. [18-19]. We shall show that Eq. (3.1) can be extended to more general GHZ states.

3.1 Symplectic matrix and general GHZ states

Observing the Bell states given by Eq. (2.6), GHZ states for 4 spin system shown by Eq. (2.63) and the general GHZ states (maximally entangled states) given by [20]

$$\frac{1}{\sqrt{2}}(|m_1,m_2,\cdots,m_n\rangle \pm |-m_1,-m_2,\cdots,-m_n\rangle) \tag{3.2}$$

that are generated through the transformation matrix B_N:

$$B_N=\mathbf{1}+M,\quad (N=2^{2n}\ \text{ i.e., }\ N=4,16,\cdots,2^{2n}=m^2), \tag{3.3}$$

where

$$M=\frac{1}{\sqrt{2}}\begin{array}{c|cccccccc}
 & (s,s) & (s,s-1) & \cdots & (\frac{1}{2},-s) & (-\frac{1}{2},s) & \cdots & (-s,-s+1) & (-s,-s)\\
\hline
(s,s) & & & & & & & & 1\\
(s,s-1) & & & & & & & 1 & \\
\vdots & & & & & & \iddots & & \\
(\frac{1}{2},-s) & & & & & 1 & & & \\
(-\frac{1}{2},s) & & & & -1 & & & & \\
\vdots & & & \iddots & & & & & \\
(-s,-s+1) & & -1 & & & & & & \\
(-s,-s) & -1 & & & & & & &
\end{array} \tag{3.4}$$

i.e. M is a $N\times N$ matrix, where $N=(2s+1)^2$ is the number of spins, $s=\frac{1}{2},\frac{3}{2},\frac{5}{2},\cdots$. We label the matrix M in terms of the indices appearing in Eq. (3.4) for both the row and the column of M, then the elements are given by

$$M_{ij,kl}=\varepsilon(i)\delta_i^{-k}\delta_j^{-l},\ \ \varepsilon(i)=1,\ \text{for } i>0,\ -1,\ \text{for } i<0;\ i,j,k,l=s,s-1,\cdots,-s. \tag{3.5}$$

It is easy to find that

$$M^2 = -\mathbf{1}, \tag{3.6}$$

and calculation gives

$$M_i M_{i\pm1} = -M_{i\pm1} M_i, \ \ (M_i = M_{i,i+1}), \tag{3.7}$$

$$M_i M_j = M_j M_i \ \ \text{for} \ \ |i = j| \geq 2. \tag{3.8}$$

With the help of Eq. (3.3) and Eqs. (3.5)-(3.8), it can be proved that the matrix B_N $(N = 4, 16, \cdots)$ satisfies braid relation:

$$B_i B_{i+1} B_i = B_{i+1} B_i B_{i+1}. \tag{3.9}$$

The matrix B_N has two distinct eigenvalues

$$(B_N - \lambda_1)(B_N - \lambda_2) = 0, \tag{3.10}$$

where

$$\lambda_1 = \frac{1}{\sqrt{2}}(1+i), \ \lambda_1 = \frac{1}{\sqrt{2}}(1-i). \tag{3.11}$$

The Eq. (3.9) also admits a q-deformation solution

$$B(q) = \mathbf{1} + M(q), \tag{3.12}$$

where

$$M(q) = \sum_{i,j} \varepsilon(i) q_{ij} |ij\rangle\langle -i-j| \tag{3.13}$$

and q_{ij} satisfy

$$q_{ij} q_{-i-j} = 1, \ \ q_{ij} q_{-jk} q_{-ij} = q_{jk}. \tag{3.14}$$

For instance, for GHZ states with 4 spins ($s = \frac{3}{2}$, so $N = 16$), we conclude that the transformation matrices generating GHZ states with $(2s+1)$ components satisfy braid relation.

3.2 Yang-Baxterization of B_N-matrix

Since the B_N-matrix does have two distinct eigenvalues, it can be Yang-Baxterized to [21−22]

$$\check{R}(x) = B + xB^{-1} \tag{3.15}$$

that satisfies the Yang-Baxter equation (YBE):

$$\check{R}_i(x)\check{R}_{i+1}(xy)\check{R}_i(y) = \check{R}_{i+1}(y)\check{R}_i(xy)\check{R}_{i+1}(x). \tag{3.16}$$

For more explicit physical meaning we introduce a new variable u for

$$x = e^{2i\Theta_1} \tag{3.17}$$

through

$$u = \frac{1-x}{1+x} = -i\tan\Theta_1, \ \ v = \frac{1-y}{1+y} = -i\tan\Theta_2, \tag{3.18}$$

then the sum of two velocities of u and v corresponding to $xy = e^{2i(\Theta_1+\Theta_2)}$ should be

$$xy \longrightarrow \frac{u+v}{1+uv}, \tag{3.19}$$

and the YBE reads

$$\check{R}_i(\Theta_1)\check{R}_{i+1}(\Theta_1+\Theta_2)\check{R}_i(\Theta_2) = \check{R}_{i+1}(\Theta_2)\check{R}_i(\Theta_1+\Theta_2)\check{R}_{i+1}(\Theta_1), \tag{3.20}$$

i.e.

$$\check{R}_i(u)\check{R}_{i+1}(\frac{u+v}{1+uv})\check{R}_i(v) = \check{R}_{i+1}(v)\check{R}_i(\frac{u+v}{1+uv})\check{R}_{i+1}(u). \tag{3.21}$$

The Eq. (3.19) is nothing but the sum of two velocities in special relativity with $c = 1$. It is easy to understand the Eq. (3.19) if one observes the more general form of B_4 shown in Eq. (3.1)[23]. It takes the form of Lorentz transformation.

In terms of the defined velocity u we obtain the solution of YBE

$$\check{R}(\Theta) = \mathbf{1} + uM = \mathbf{1} - i\tan\Theta M. \tag{3.22}$$

In difference from the familiar rational solution of YBE

$$\check{R}(u) = \mathbf{1} + uP, \tag{3.23}$$

and making comparison with Eq. (3.22) the permutation P is replaced by a symplectic structure M which takes the form shown by Eqs. (3.13) and (3.14).

To look for further physical meaning of Eq. (3.22) we should find the Hamiltonian for the system.

3.3 Hamiltonian and evolution

Since any $(2s+1)^2 \times (2s+1)^2$ matrix B_N has two eigenvalues

$$[B_N - \frac{1}{\sqrt{2}}(1+i)][B_N - \frac{1}{\sqrt{2}}(1-i)] = 0, \tag{3.24}$$

and

$$\check{R}_N(x) = B_N + xB_N^{-1} = \frac{1}{\sqrt{2}}[(1+x)\mathbf{1} + (1-x)M]. \tag{3.25}$$

Suppose Φ is a natural basis without entanglement, then GHZ states Ψ is given by

$$\Psi = B_N\Phi. \tag{3.26}$$

For instance when $N = 4$, we have[17,18]

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_3 \\ -\phi_4 \\ -\phi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \uparrow\uparrow \\ \uparrow\downarrow \\ \downarrow\uparrow \\ \downarrow\downarrow \end{pmatrix} = B_4 \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{pmatrix}. \tag{3.27}$$

The wave function $\Psi(x)$ should be normalized, so instead of the $\check{R}(x)$ we should use the normalized $B(x)$-matrix

$$B_N(x) = \rho(x)^{-1/2}\check{R}_N(x), \;\; \rho = 1 + x^2 \tag{3.28}$$

to generate $\Psi(x)$ such that

$$\Psi(x) = B_N(x)\Phi. \tag{3.29}$$

From Eq. (3.28) it follows

$$i\frac{\partial\Psi(x)}{\partial x} = i(\frac{\partial B_N(x)}{\partial x}B_N^{-1}(x))\Psi(x) = H(x)\Psi(x), \tag{3.30}$$

where

$$H(x) = i\frac{\partial B_N(x)}{\partial x}B_N^{-1}(x). \tag{3.31}$$

Taking Eq. (3.28) into account we get

$$H(x) = i\frac{\partial}{\partial x}(\rho(x)^{-\frac{1}{2}}\check{R}(x))(\rho(x)^{-\frac{1}{2}}\check{R}(x))^{-1} = -i\rho(x)^{-1}M. \tag{3.32}$$

To obtain time-independent (“real") Hamiltonian $H(x=1)$, a new variable θ is introduced through

$$\cos\theta = \rho(x)^{-\frac{1}{2}},\ \ \sin\theta = x\rho(x)^{-\frac{1}{2}}, \tag{3.33}$$

i.e.

$$\frac{dx}{d\theta} = 1+x^2 = \rho(x). \tag{3.34}$$

From Eq. (3.30) it follows

$$i\frac{\partial\Psi(x)}{\partial x}\frac{dx}{d\theta} = H(x)\frac{dx}{d\theta}\Psi(x), \tag{3.35}$$

We then obtain the Schrödinger equation with the Hamiltonian

$$H_N = -iM \tag{3.36}$$

$$i\frac{\partial\Psi(\theta)}{\partial\theta} = H_N\Psi(\theta), \tag{3.37}$$

where the Schrödinger equation works in the space of parameter θ that plays the role of “time" and

$$H = i\frac{\partial B(\theta)}{\partial\theta}B^{-1}(\theta) \quad (\theta=0). \tag{3.38}$$

For $N=4$,

$$H_4 = \frac{1}{\sqrt{2}}\sigma_y\otimes\sigma_x. \tag{3.39}$$

For general N we have

$$H_N = \frac{1}{\sqrt{2}}\sigma_y\otimes\overbrace{\sigma_x\otimes\sigma_x\otimes\cdots\otimes\sigma_x}^{n-1}. \tag{3.40}$$

In this picture the GHZ states can be generated through the evolution with the parameter θ. The wave function $\Psi(\theta)$ is then $e^{-iH_N\theta}\Psi(\theta=0)$.

For a given $\check{R}(x)$-matrix we are able to discuss algebras through the RFT approach. It can be made based on the direct calculation. We are not going to discuss the subject here.

For $N=4$ the Hamiltonian with the nearest neighbor interaction takes the form

$$H_{n,n+1} \sim \sigma_n^y\sigma_{n+1}^x. \tag{3.41}$$

In the usual manner we define the Hamiltonian on a chain

$$H = \sum_n H_{n,n+1} = J \sum_n \sigma_n^y \sigma_{n+1}^x \tag{3.42}$$

that is σ^x and σ^y at the alternative lattice. It is different from the Heisenberg chain derived based on $\check{R}_{n,n+1}(u) = \mathbf{1} + uP_{n,n+1}$ which was first introduced by C.N. Yang to describe the S-matrix for δ-interaction model[24]. In fact, Eq. (3.42) provides a unitary transformation.

3.4 Braid group matrix of GHZ with 3 spins

In the subsections (3.1)-(3.3) we have set up the relationship between GHZ states with even number of spins and braid group representations. The calculations is straightforward because of $N = m^2$ for B_N being $m \times m$ matrix, i.e., $4 \times 4, 16 \times 16, 64 \times 64, \cdots$. However, for the GHZ states with 3 spins, we meet the B_8-matrix, but 8 cannot be written as m^2. We should use the coupled Yang-Baxter equations to determine B_8 for the given B_4. The general braid relations where $S^{j\frac{1}{2}}, S^{\frac{1}{2}j}$ and $S^{\frac{1}{2}\frac{1}{2}}$ are involved had been given by Drinfeld-Jimbo formula[25−26]:

$$S_{12}^{j\frac{1}{2}} S_{23}^{j\frac{1}{2}} S_{12}^{\frac{1}{2}\frac{1}{2}} = S_{23}^{\frac{1}{2}\frac{1}{2}} S_{12}^{j\frac{1}{2}} S_{23}^{j\frac{1}{2}}; \tag{3.43}$$

$$S_{12}^{\frac{1}{2}j} S_{23}^{\frac{1}{2}\frac{1}{2}} S_{12}^{j\frac{1}{2}} = S_{23}^{j\frac{1}{2}} S_{12}^{\frac{1}{2}\frac{1}{2}} S_{23}^{\frac{1}{2}j}; \tag{3.44}$$

$$S_{12}^{\frac{1}{2}\frac{1}{2}} S_{23}^{\frac{1}{2}j} S_{12}^{\frac{1}{2}j} = S_{23}^{\frac{1}{2}j} S_{12}^{\frac{1}{2}j} S_{23}^{\frac{1}{2}\frac{1}{2}}; \tag{3.45}$$

Eqs. (3.43)-(3.45) determine $S^{j\frac{1}{2}}$ and $S^{\frac{1}{2}j}$ for the given $S^{\frac{1}{2}\frac{1}{2}}$. For the 6-vertex solution of $S^{\frac{1}{2}\frac{1}{2}}$ (4 by 4 matrix) the $S^{j\frac{1}{2}}$ and $S^{\frac{1}{2}j}$ can be explicitly given based on the general theory of $S^{j_1 j_2}$ of Drinfeld , Kirillov and Reshetikhin. However, when $S^{\frac{1}{2}\frac{1}{2}}$ takes the symplectic form shown by Eq. (3.1), we should find the new solutions of $S^{j\frac{1}{2}}$ and $S^{\frac{1}{2}j}$. The calculation is lengthy and shown in the Appendix C. The result is

$$B_8 = S^{\frac{3}{2}\frac{1}{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & & & & & 1 \\ & 1 & & & & & 1 & \\ & & 1 & & & 1 & & \\ & & & 1 & 1 & & & \\ & & & -1 & 1 & & & \\ & & -1 & & & 1 & & \\ & -1 & & & & & 1 & \\ 1 & & & & & & & -1 \end{pmatrix}, \tag{3.46}$$

and

$$S^{\frac{1}{2}\frac{3}{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & & & & & 1 \\ & 1 & & & & & -1 & \\ & & 1 & & & -1 & & \\ & & & 1 & 1 & & & \\ & & & -1 & 1 & & & \\ & & 1 & & & 1 & & \\ & 1 & & & & & -1 & \\ 1 & & & & & & & -1 \end{pmatrix}, \tag{3.47}$$

where the other elements of the matrices are zero. Noting that $S^{\frac{3}{2}\frac{1}{2}}S^{\frac{1}{2}\frac{3}{2}} \neq 1$. The matrix $B_8 = S^{\frac{3}{2}\frac{1}{2}}$ transforms the decomposable 3-spin states to GHZ states with 3-spins as follows.

$$\begin{pmatrix} |1\rangle \\ |3\rangle \\ |5\rangle \\ |7\rangle \\ -|8\rangle \\ -|6\rangle \\ -|4\rangle \\ |2\rangle \end{pmatrix} = S^{\frac{3}{2}\frac{1}{2}} \begin{pmatrix} |\uparrow\uparrow\uparrow\rangle \\ |\uparrow\uparrow\downarrow\rangle \\ |\uparrow\downarrow\uparrow\rangle \\ |\downarrow\uparrow\uparrow\rangle \\ |\uparrow\downarrow\downarrow\rangle \\ |\downarrow\uparrow\downarrow\rangle \\ |\downarrow\downarrow\uparrow\rangle \\ |\downarrow\downarrow\downarrow\rangle \end{pmatrix}, \tag{3.48}$$

where the LHS of Eq (3.48) is given by Eq. (2.51). Noting that changing a sign of individual wave function does not change the maximal degree of entanglement, we conclude that the LHS and RHS of Eq. (3.48) are connected each other by the solution of braid relations Eqs. (3.43)-(3.45) provided $S^{\frac{1}{2}\frac{1}{2}}$ is fixed to be B_4.

4. Conclusion

There exists the close relationship between the GHZ type of maximally entangled states and Yang-Baxter approach. On the one hand, the Yangian algebra describes transitions for entangled states and possibly $\mathbf{J}^2$ with special choice of dependent parameters is related to the maximal degree of entanglement. It is quite natural to introduce Yangian operators for GHZ states. On the other hand, the matrices transforming the natural spin-tensor states to GHZ type of entangled states obey braid relation. The idea is similar to the spinor structure of $SO(2n)$ representations and braiding[27−28], but here we meet sympletic structure.They can be Yang-Baxterized to yield $\check{R}(x)$-matrix and lead to Hamiltonian which governs the evolution process of wave function with respect to the variable θ. From the point of view of statistics[27] the Schr*ö*dinger equation we introduced may dictate the fractional statistics for different θ.

We hope that possibly the Yang-Baxter description of entangled states may help to set up a hidden connection between entangled states and topological quantum field theory[29−30]. Of course, the general definition of entangling degree has not been well established, so there may be long way to go.

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Appendix A: The Representations of the Yangian $Y(sl(3))$

In general, the definition of Yangian was first given by Drinfeld in Refs. [1-2]. Let $\{I_\lambda\}$ be an orthonormal basis of a simple Lie algebra $\mathscr{G}$ over **C** with respect some invariant inner product $(\ ,\ )$. The Yangian $Y(\mathscr{G})$ associated to $\mathscr{G}$ is the Hopf algebra over **C** generated (as an associative algebra) by elements I_λ and J_λ with relations (summation over repeated indices is always understood hereafter):

1) $[I_\lambda, I_\mu] = c_{\lambda\mu\nu} I_\nu,\ \ [I_\lambda,\ J_\nu] = c_{\lambda\mu\nu} J_\nu$ (A1)

2) $[J_\lambda,\ [J_\mu,\ I_\nu]] - [I_\lambda,\ [J_\mu,\ J_\nu]] = a_{\lambda\mu\nu\alpha\beta\gamma}\{I_\alpha,\ I_\beta,\ I_\gamma\};$ (A2)

3) $[[J_\lambda, J_\mu], [I_\sigma, J_\tau]] + [[J_\sigma,\ J_\tau],\ [I_\lambda,\ J_\mu]] = (a_{\lambda\mu\nu\alpha\beta\gamma} c_{\sigma\tau\nu} + a_{\sigma\tau\nu\alpha\beta\gamma} c_{\lambda\mu\nu})\{I_\alpha,\ I_\beta,\ J_\gamma\};$ (A3)

where the $c_{\lambda\mu\nu}$ are structure constants of $\mathscr{G}$, and

$$a_{\lambda\mu\nu\alpha\beta\gamma} = \frac{1}{24} c_{\lambda\alpha\sigma} c_{\mu\beta\tau} c_{\nu\gamma\rho} c_{\sigma\tau\rho},\ \ \{x_1,\ x_2,\ x_3\} = \sum_{i\neq j\neq k} x_i x_j x_k \text{ (symmetric summation)} \quad \text{(A4)}$$

4) co-product:

$\Delta(I_\lambda) = I_\lambda \otimes 1 + 1 \otimes I_\lambda,\ \ \Delta(J_\lambda) = J_\lambda \otimes 1 + 1 \otimes J_\lambda + \frac{1}{2} c_{\lambda\mu\nu} I_\nu \otimes I_\mu$ (A5)

In the case $\mathscr{G} = sl(2)$, (A1) implies (A2), and for $\mathscr{G} \neq sl(2)$, (A3) follows from (A1) and (A2).

When $\mathscr{G} = sl(n)$, we have the following conclusion. Let $\mathrm{Tr}(xy)$ be taken as the inner product of $sl(n)$, and $\{I_\lambda\}$ be the orthonormal basis. Then there is a homomorphism of algebras (but not a Hopf-algebra homomorphism)

$$\varepsilon : Y(sl(n)) \longrightarrow U(sl(n)) \tag{A6}$$

such that

$$\varepsilon(x) = x, \;\; \varepsilon(J(x)) = \frac{1}{4}\sum_{\lambda,\mu} \mathrm{Tr}(x(I_\lambda I_\mu + I_\mu I_\lambda))I_\lambda I_\mu \tag{A7}$$

It is shown in Ref. [15] that for the fundamental representations of $sl(n)$ with the weights

$$\lambda_1 = (1,0,\cdots,0), \lambda_2 = (0,1,\cdots,0), \cdots, \lambda_n = (0,0,\cdots,1), \tag{A8}$$

the action of $J(x)$ given by Eq. (A7) is just αx for any $x \in sl(n)$, where α is a constant (in the case $sl(2)$, $\alpha = 0$). So for any fundamental representation $V(\lambda_i)$ of $sl(n)$ and $a \in \mathbf{C}$, we can construct a representation of $Y(sl(n))$ with the actions given by

$$x \to x, J(x) \to ax, \tag{A9}$$

and we denote it by $V(\lambda_i)(a)$.

Next we turn to the case of $sl(3)$. According to the inner product Trxy, we can obtain an orthonormal basis of $sl(3)$:

$$\hat{H}_1 = \frac{1}{\sqrt{2}}H_1,\; \hat{H}_2 = \frac{1}{\sqrt{6}}(H_1 + 2H_2),\;\; \hat{E}_i = \frac{1}{\sqrt{2}}(E_i + E_{-i}),\; \hat{E}_{-i} = \frac{1}{\sqrt{-2}}(E_i - E_{-i}),\;\; i = 1,2,3, \tag{A10}$$

where H_1, H_2, E_i, E_{-i} are given in Eq. (2.15).

The homomorphism of algebras $\varepsilon : Y(sl(3)) \longrightarrow U(sl(3))$ as in (A7) is given by

$$\begin{aligned}
&x \longrightarrow x, \;\; \forall x \in sl(3)\\
J(H_1) &\longrightarrow \frac{1}{6}(H_1^2 + H_1H_2 + H_2H_1) - \frac{1}{4}(E_2E_{-2} + E_{-2}E_2) + \frac{1}{4}(E_3E_{-3} + E_{-3}E_3)\\
J(H_2) &\longrightarrow \frac{-1}{6}(H_2^2 + H_1H_2 + H_2H_1) + \frac{1}{4}(E_1E_{-1} + E_{-1}E_1) - \frac{1}{4}(E_3E_{-3} + E_{-3}E_3)\\
J(E_1) &\longrightarrow \frac{1}{12}((H_1 + 2H_2)E_1 + E_1(H_1 + 2H_2)) + \frac{1}{4}(E_3E_{-2} + E_{-2}E_3)\\
J(E_{-1}) &\longrightarrow \frac{1}{12}((H_1 + 2H_2)E_{-1} + E_{-1}(H_1 + 2H_2)) + \frac{1}{4}(E_{-3}E_2 + E_2E_{-3})\\
J(E_2) &\longrightarrow \frac{-1}{12}((2H_1 + H_2)E_2 + E_2(2H_1 + H_2)) + \frac{1}{4}(E_3E_{-1} + E_{-1}E_3)\\
J(E_{-2}) &\longrightarrow \frac{-1}{12}((2H_1 + H_2)E_{-2} + E_{-2}(2H_1 + H_2)) + \frac{1}{4}(E_{-3}E_1 + E_1E_{-3})
\end{aligned}$$

$$J(E_3) \longrightarrow \frac{1}{12}((H_1-H_2)E_3+E_3(H_1-H_2))+\frac{1}{4}(E_1E_2+E_2E_1)$$
$$J(E_{-3}) \longrightarrow \frac{1}{12}((H_1-H_2)E_{-3}+E_{-3}(H_1-H_2))+\frac{1}{4}(E_{-1}E_{-2}+E_{-2}E_{-1})$$

After the direct computation, we know that for the fundamental representation of $sl(3)$ with weight $(1,0)$, $\varepsilon(J(x))=\frac{5}{12}x$ and for the fundamental representation of $sl(3)$ with weight $(0,1)$ (the dual representation of $(1,0)$), $\varepsilon(J(x))=-\frac{5}{12}x$.

Therefore, according to the co-product given by Eq. (A5), we can consider the representation of Yangian $Y(sl(3))$

$$W=V(1,0)(a)\otimes V(0,1)(b)=V(1,0)(a)\otimes V(1,0)^*(b). \quad \text{(A11)}$$

In another form, we can rewrite it as in Eq. (2.19). As representations of $sl(3)$, we have

$$W=V(1,0)(a)\otimes V(1,0)(b)\cong V(1,1)\oplus V(0,0), \quad \text{(A12)}$$

where $V(1,1)$ can be regarded as the 8-dimensional adjoint representation of $sl(3)$ with the basis

$$\eta_{0,0}(H_1)=|00\rangle-|11\rangle,\ \eta'_{0,0}(H_2)=|11\rangle-|22\rangle,\ w_{2,-1}(E_1)=|01\rangle,\ w_{-1,2}(E_2)=|12\rangle,$$
$$w_{1,1}(E_3)=|02\rangle, w_{-2,1}(E_{-1})=|10\rangle,\ w_{1,-2}(E_{-2})=|21\rangle,\ w_{-1,-1}(E_{-3})=|20\rangle; \quad \text{(A13)}$$

and $V(0,0)\cong \mathbf{C}$ is the trivial representation of $sl(3)$ with the basis

$$v_{0,0}=|00\rangle+|11\rangle+|22\rangle. \quad \text{(A14)}$$

Under the above basis, the actions of $J(x)$ on $W=V(1,0)(a)\otimes V(0,1)(b)$ are given by

$$J(H_1)\to\begin{pmatrix}
\frac{1}{3}(a-b) & \frac{2}{3}(a-b) & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3}(a-b+\frac{3}{2})\\
\frac{1}{3}(a-b) & -\frac{1}{3}(a-b) & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3}(a-b+\frac{3}{2})\\
0 & 0 & a+b & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -(a+b) & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -b & 0\\
a-b-\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}; \quad \text{(A15)}$$

$$J(H_2)\to\begin{pmatrix}
\frac{1}{3}(a-b) & -\frac{1}{3}(a-b) & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3}(a-b+\frac{3}{2})\\
-\frac{2}{3}(a-b) & -\frac{1}{3}(a-b) & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3}(a-b+\frac{3}{2})\\
0 & 0 & -b & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & a+b & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -(a+b) & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0\\
0 & a-b-\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}; \quad \text{(A16)}$$

$$J(E_1) \to \begin{pmatrix} 0 & 0 & -(a+b) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3}(2a+b) & \frac{1}{3}(a-b) & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3}(a-b+\frac{3}{2}) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -b & 0 & 0 \\ 0 & 0 & a-b-\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{A17}$$

$$J(E_2) \to \begin{pmatrix} 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(a+b) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3}(a-b) & \frac{1}{3}(a+2b) & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3}(a-b+\frac{3}{2}) \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a-b-\frac{3}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{A18}$$

$$J(E_3) \to \begin{pmatrix} 0 & 0 & 0 & 0 & -b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3}(2a+b) & \frac{1}{3}(a+2b) & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3}(a-b+\frac{3}{2}) \\ 0 & 0 & 0 & 0 & a-b-\frac{3}{2} & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{A19}$$

$$J(E_{-1}) \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & a+b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b & 0 & 0 & 0 \\ -\frac{1}{3}(a+2b) & \frac{1}{3}(a-b) & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3}(a-b+\frac{3}{2}) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a-b-\frac{3}{2} & 0 & 0 & 0 \end{pmatrix}; \tag{A20}$$

$$J(E_{-2}) \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a+b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3}(a-b) & -\frac{1}{3}(2a+b) & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3}(a-b+\frac{3}{2}) \\ 0 & 0 & -b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a-b-\frac{3}{2} & 0 & 0 \end{pmatrix}; \tag{A21}$$

$$J(E_{-3}) \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b & 0 & 0 & 0 \\ -\frac{1}{3}(a+2b) & -\frac{1}{3}(2a+b) & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3}(a-b+\frac{3}{2}) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a-b-\frac{3}{2} & 0 \end{pmatrix} \quad \text{(A22)}$$

Through the direct calculation, it is easy to find that the actions of $Y(sl(3))$ on W can be regarded as

$$x \to \begin{pmatrix} \rho_{ad}(x) & 0 \\ 0 & 0 \end{pmatrix}, \quad J(x) \to \begin{pmatrix} \frac{2}{3}(a-b)J_{ad}(x)+\frac{1}{2}(a+b)\rho_{ad}(x) & (a-b+\frac{3}{2})M_{12}(x) \\ (a-b-\frac{3}{2})M_{21}(x) & 0 \end{pmatrix} \quad \text{(A23)}$$

where $\rho_{ad}(x)$ is the adjoint representation of $sl(3)$, $J_{ad}(x)$ is the action of $\varepsilon(J(x))$ given by Eq. (A7) on the adjoint representation, $M_{12}(x)$ and $M_{21}(x)$ are the 1×8 and 8×1 matrices, respectively.

Appendix B: Actions of J for GHZ states with 4 spins

$$\begin{aligned}
J_2|1) &= -(d-\frac{3}{2})|4) - (c-\frac{1}{2})|6) - (b+\frac{1}{2})|8) - (a+\frac{3}{2})|10);\\
J_2|2) &= -(d-\frac{3}{2})|3) - (c-\frac{1}{2})|5) - (b+\frac{1}{2})|7) - (a+\frac{3}{2})|9);\\
J_2|3) &= (d+\frac{3}{2})|2) - (b-\frac{1}{2})12) + (a+\frac{1}{2})|14) - (c-\frac{3}{2})|16);\\
J_2|4) &= (d+\frac{3}{2})|1) - (b-\frac{1}{2})11) - (a+\frac{1}{2})|13) - (c-\frac{3}{2})|15);\\
J_2|5) &= (c+\frac{1}{2})|2) + (a+\frac{1}{2})12) - (b-\frac{1}{2})|14) - (d-\frac{1}{2})|16);\\
J_2|6) &= (c+\frac{1}{2})|1) - (a+\frac{1}{2})11) - (b-\frac{1}{2})|13) - (d-\frac{1}{2})|15);\\
J_2|7) &= (b-\frac{1}{2})|2) - (d-\frac{1}{2})12) - (c+\frac{1}{2})|14) + (a+\frac{1}{2})|16);\\
J_2|8) &= (b-\frac{1}{2})|1) - (d-\frac{1}{2})11) - (c+\frac{1}{2})|13) - (a+\frac{1}{2})|15);\\
J_2|9) &= (a-\frac{3}{2})|2) + (c+\frac{1}{2})12) + (d-\frac{1}{2})|14) + (b+\frac{3}{2})|16);\\
J_2|10) &= (a-\frac{3}{2})|1) - (c+\frac{1}{2})11) - (d-\frac{1}{2})|13) - (b+\frac{3}{2})|15);\\
J_2|11) &= (b+\frac{1}{2})|4) + (a-\frac{1}{2})6) + (d+\frac{1}{2})|8) + (c-\frac{1}{2})|10);\\
J_2|12) &= (b+\frac{1}{2})|3) - (a-\frac{1}{2})5) + (d+\frac{1}{2})|7) - (c-\frac{1}{2})|9);
\end{aligned} \quad (B1)$$

$$
\begin{aligned}
J_2|13) &= (a-\frac{1}{2})|4) + (b+\frac{1}{2})6) + (c-\frac{1}{2})|8) + (d+\frac{1}{2})|10);\\
J_2|14) &= -(a-\frac{1}{2})|3) + (b+\frac{1}{2})5) + (c-\frac{1}{2})|7) - (d+\frac{1}{2})|9);\\
J_2|15) &= (c+\frac{3}{2})|4) + (d+\frac{1}{2})6) + (a-\frac{1}{2})|8) + (b-\frac{3}{2})|10);\\
J_2|16) &= (c+\frac{3}{2})|3) + (d+\frac{1}{2})5) - (a-\frac{1}{2})|7) - (b-\frac{3}{2})|9);
\end{aligned}
$$

$$
\begin{aligned}
J_3|1) &= (a+b+c+d)|2);\\
J_3|2) &= (a+b+c+d)|1);\\
J_3|3) &= (a+b+c-d)|4) - |6) - |8) - 10);\\
J_3|4) &= (a+b+c-d)|3) - |5) - |7) - 9);\\
J_3|5) &= |4) + (a+b-c+d)|6) - |8) - |10);\\
J_3|6) &= |3) + (a+b-c+d)|5) - |7) - |9);\\
J_3|7) &= |4) + |6) + (a-b+c+d)|8) - |10);\\
J_3|8) &= |3) + |5) + (a-b+c+d)|7) - |9); \qquad (B2)\\
J_3|9) &= |4) + |6) + |8) + (-a+b+c+d)|10);\\
J_3|10) &= |3) + |5) + |7) + (-a+b+c+d)|9);\\
J_3|11) &= (a-b+c-d)|12) + 2|16);\\
J_3|12) &= (a-b+c-d)|11) - 2|13);\\
J_3|13) &= 2|12) + (a-b-c+d)|14) + 2|16);\\
J_3|14) &= (a-b-c+d)|13);\\
J_3|15) &= (a+b-c-d)|16);\\
J_3|16) &= -2|11) - 2|13) + (a+b-c-d)|15).
\end{aligned}
$$

$$
\begin{aligned}
J_3^2|1) &= (a+b+c+d)^2|1);\\
J_3^2|2) &= (a+b+c+d)^2|2);\\
J_3^2|3) &= [(a+b+c-d)^2-3]|3) - 2(a+b+1)|5) - 2(a+c)|7) - 2(b+c-1)|9);\\
J_3^2|4) &= [(a+b+c-d)^2-3]|4) - 2(a+b+1)|6) - 2(a+c)|8) - 2(b+c-1)|10);
\end{aligned}
$$

$$
\begin{aligned}
&J_3^2|5\rangle = 2(a+b-1)|3\rangle + [(a+b-c+d)^2-3]|5\rangle - 2(a+d+1)|7\rangle - 2(b+d)|9\rangle;\\
&J_3^2|6\rangle = 2(a+b-1)|4\rangle + [(a+b-c+d)^2-3]|6\rangle - 2(a+d+1)|8\rangle - 2(b+d)|10\rangle;\\
&J_3^2|7\rangle = 2(a+c)|3\rangle + 2(a+d-1)|5\rangle + [(a-b+c+d)^2-3]|7\rangle - 2(c+d+1)|9\rangle;\\
&J_3^2|8\rangle = 2(a+c)|4\rangle + 2(a+d-1)|6\rangle + [(a-b+c+d)^2-3]|8\rangle - 2(c+d+1)|10\rangle; \qquad (B3)\\
&J_3^2|9\rangle = 2(b+c)|3\rangle + 2(b+d)|5\rangle + 2(c+d-1)|7\rangle + [(-a+b+c+d)^2-3]|9\rangle;\\
&J_3^2|10\rangle = 2(b+c)|4\rangle + 2(b+d)|6\rangle + 2(c+d-1)|8\rangle + [(-a+b+c+d)^2-3]|10\rangle;\\
&J_3^2|11\rangle = [(a-b+c-d)^2-4]|11\rangle - 2(a-b+c-d+2)|13\rangle + 2(a+b-c-d)|15\rangle;\\
&J_3^2|12\rangle = [(a-b+c-d)^2-4]|12\rangle - 2(a-b-c+d)|14\rangle + 2(a-b+c-d-2)|16\rangle;\\
&J_3^2|13\rangle = 2(a-b+c-d-2)|11\rangle + [(a-b-c+d)^2-8]|13\rangle + 2(a+b-c-d)|15\rangle;\\
&J_3^2|14\rangle = 2(a-b-c+d)|12\rangle + (a-b-c+d)^2|14\rangle + 2(a-b-c+d)|16\rangle;\\
&J_3^2|15\rangle = -2(a+b-c-d)|11\rangle - 2(a+b-c-d)|13\rangle + (a+b-c-d)^2|15\rangle;\\
&J_3^2|16\rangle = -2(a-b+c-d+2)|12\rangle + 2(a-b-c+d)|14\rangle + [(a+b-c-d)^2-8]|16\rangle.
\end{aligned}
$$

Appendix C: Proof of B_8 to satisfy braid relations Eqs. (3.43)-(3.45)

For

$$S^{jj} = \frac{1}{\sqrt{2}}(\mathbf{1}+M), \;\; M^2 = \alpha\mathbf{1}, \tag{C1}$$

where j=half integer, then braid relation for S^{jj} reads

$$S^{jj}_{12}S^{jj}_{23}S^{jj}_{12} = S^{jj}_{23}S^{jj}_{12}S^{jj}_{23}. \tag{C2}$$

It leads to

$$M_{12}M_{23}M_{12} + M_{12} = M_{23}M_{12}M_{23} + M_{23}. \tag{C3}$$

Setting the matrix M to the form

$$M^{ab}_{cd} = p_a\delta_{ad}\delta_{bc}\delta_{a-b} + v_{ab}\delta_{a-c}\delta_{b-d}|_{a\neq -b}. \tag{C4}$$

The Eq. (B2) gives the relations

$$M_{12}M_{23}M_{12} = M_{23}, \quad M_{23}M_{12}M_{23} = M_{12}, \tag{C5}$$

or

$$M_iM_{i\pm1}M_i = M_{i\pm1}, \tag{C6}$$

with the allowed values of p_a and v_{ab}, provided they satisfy one of the following three relations.

$$\text{(a) All } p_a = v_{ab} = 1; \tag{C7}$$

$$\text{(b) } p_a = \begin{matrix} 1 & a>0 \\ -1 & a<0 \end{matrix} \quad v_{ab} = \begin{matrix} 1 & a>0 \\ -1 & b<0 \end{matrix}; \tag{C8}$$

$$\text{(c) } p_a = \begin{matrix} 1 & a>0 \\ -1 & a<0 \end{matrix} \quad v_{ab} = \begin{matrix} 1 & a>0, b\neq a \text{ or } b=a>0 \\ -1 & a>0, b\neq a \text{ or } b=a<0 \end{matrix}. \tag{C9}$$

The solutions (b) and (c) give the same $S^{\frac{1}{2}\frac{1}{2}}$ in which only v_{aa} is survived. Together with $p_a p_{-a} = v_{aa}v_{-a-a} = v_{ab}v_{-a-b} = -1$, we get

$$M^2 = -\mathbf{1}, \text{ and } (S^{\frac{1}{2}\frac{1}{2}})^2 - \sqrt{2}S^{\frac{1}{2}\frac{1}{2}} + \mathbf{1} = 0. \tag{C10}$$

Similarly, we set

$$S^{j\frac{1}{2}}{}^{a\alpha}_{\beta b} = p_{a\alpha}\delta_{ab}\delta_{\alpha\beta}|_{a\neq\pm\alpha} + q_{a\alpha}\delta_{a-b}\delta_{\alpha-\beta}|_{a\neq\pm\alpha} + (S^{\frac{1}{2}\frac{1}{2}})^{a\alpha}_{\beta b}|_{a,b\in(\frac{1}{2},-\frac{1}{2})}, \tag{C11}$$

and

$$S^{\frac{1}{2}j}{}^{\alpha a}_{\beta b} = f_{\alpha a}\delta_{\alpha\beta}\delta_{ab}|_{a\neq\pm\alpha} + g_{\alpha a}\delta_{\alpha-\beta}\delta_{a-b}|_{a\neq\pm\alpha} + (S^{\frac{1}{2}\frac{1}{2}})^{\alpha a}_{\beta b}|_{a,b\in(\frac{1}{2},-\frac{1}{2})}. \tag{C12}$$

Substituting Eqs. (C1), (C11) and (C12) into Eqs. (3.43)-(3.45) for $j=\frac{3}{2}$ after calculation one finds

$$\begin{aligned}
&p_{\frac{3}{2}\frac{1}{2}} = p_{\frac{3}{2}-\frac{1}{2}} = p_{-\frac{3}{2}\frac{1}{2}} = \frac{1}{\sqrt{2}},\ p_{-\frac{3}{2}-\frac{1}{2}} = -\frac{1}{\sqrt{2}};\\
&q_{\frac{3}{2}\frac{1}{2}} = q_{\frac{3}{2}-\frac{1}{2}} = q_{-\frac{3}{2}-\frac{1}{2}} = \frac{1}{\sqrt{2}},\ q_{-\frac{3}{2}\frac{1}{2}} = -\frac{1}{\sqrt{2}};\\
&f_{\frac{1}{2}\frac{3}{2}} = f_{-\frac{1}{2}\frac{3}{2}} = f_{\frac{1}{2}-\frac{3}{2}} = \frac{1}{\sqrt{2}},\ f_{-\frac{1}{2}-\frac{3}{2}} = -\frac{1}{\sqrt{2}};\\
&g_{\frac{1}{2}\frac{3}{2}} = g_{-\frac{1}{2}\frac{3}{2}} = g_{-\frac{1}{2}-\frac{3}{2}} = \frac{1}{\sqrt{2}},\ g_{-\frac{1}{2}\frac{3}{2}} = -\frac{1}{\sqrt{2}}.
\end{aligned} \tag{C13}$$

Taking the labeling

$$(a,\alpha) = (\frac{3}{2},\frac{1}{2}),(\frac{3}{2},-\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},-\frac{1}{2}),\cdots,(-\frac{3}{2},-\frac{1}{2}),$$

and

$$(\beta,b) = (\frac{1}{2},\frac{3}{2}),(-\frac{1}{2},\frac{3}{2}),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},-\frac{1}{2}),\cdots,(-\frac{1}{2},-\frac{3}{2}),$$

then we obtain

$$S^{\frac{3}{2}\frac{1}{2}} = \begin{pmatrix}
p_{\frac{3}{2}\frac{1}{2}} & & & & q_{\frac{3}{2}\frac{1}{2}} \\
& p_{\frac{3}{2}-\frac{1}{2}} & & q_{\frac{3}{2}-\frac{1}{2}} & \\
& & S^{\frac{1}{2}\frac{1}{2}} & & \\
& q_{-\frac{3}{2}\frac{1}{2}} & & p_{-\frac{3}{2}\frac{1}{2}} & \\
q_{-\frac{3}{2}-\frac{1}{2}} & & & & p_{-\frac{3}{2}-\frac{1}{2}}
\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}
1 & & & & & & & 1 \\
& 1 & & & & & 1 & \\
& & 1 & & & 1 & & \\
& & & 1 & 1 & & & \\
& & & -1 & 1 & & & \\
& & -1 & & & 1 & & \\
& -1 & & & & & 1 & \\
1 & & & & & & & -1
\end{pmatrix}. \tag{C14}$$