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Transcedentality Property of $N = 4$ Supersymmetric Gauge Theories

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We review the BFKL approach to the Regge processes in QCD. In the multi-colour QCD the equations for composite states of several Reggeized gluons in the leading logarithmic approximation turn out to be integrable. In the next-to-leading approximation some of remarkable properties of the BFKL dynamics remain to be valid in supersymmetric gauge theories. In particular the coefficients of the perturbative expansion of the eigenvalues of the kernels for the BFKL and DGLAP equations in N=4 SUSY have the maximal transcedentality level. With the use of the AdS/CFT correspondence we investigate relations between weak and strong coupling regimes in this model in the framework of the Beisert-Eden-Staudacher equation and the Pomeron - Graviton duality.

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1. Introduction

In the Born approximation the scattering amplitude in QCD at high energies $\sqrt{s}$ and fixed momentum transfers $q = \sqrt{-t}$ has a simple form showing in particular the helicity conservation

$$M_{AB}^{A'B'}(s,t)|_{Born} = g\, T_{A'A}^{c}\, \delta_{\lambda_{A'}\lambda_A}\, \frac{2s}{t}\, g\, T_{B'B}^{c}\, \delta_{\lambda_{B'}\lambda_B}\,. \tag{1.1}$$

In the leading logarithmic approximation (LLA) $g^2 \ln s \sim 1$ one obtains for this amplitude the Regge-type behavior [1]

$$M_{AB}^{A'B'}(s,t) = M_{AB}^{A'B'}(s,t)|_{Born}\, s^{\omega(t)}\,, \tag{1.2}$$

where the gluon Regge trajectory has an infrared divergency regularized by a gluon mass λ

$$\omega(-|q|^2) = -\frac{\alpha_c}{4\pi^2} N_c \int d^2k \frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_c}{2\pi} \ln \frac{|q^2|}{\lambda^2}\,. \tag{1.3}$$

The final state particles at high energies for the process $AB \to A'B'd_1...d_{n-1}$ in LLA are produced in the multi-Regge kinematics

$$s \gg s_r = (k_{r-1}+k_r)^2 \gg |q_r|^2\,,\ k_r = q_r - q_{r+1}\,. \tag{1.4}$$

Further, the gluon production amplitude in this region has the multi-Regge factorized form [1]

$$M_{2\to 1+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, g T_{c_2c_1}^{d_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... C(q_n,q_{n-1})\, \frac{s_n^{\omega_n}}{|q_n|^2}\,,\ \omega_r = \omega(-|q_r|^2)\,, \tag{1.5}$$

where the Reggeon-Reggeon-gluon vertex for the produced gluon with a definite helicity equals

$$C(q_2,q_1) = \frac{q_2\, q_1^*}{q_2 - q_1}\,. \tag{1.6}$$

We introduce the complex variables for the gluon transverse coordinates and momenta

$$\rho_k = x_k + iy_k\,,\ \rho_k^* = x_k - iy_k\,,\ p_k = i\frac{\partial}{\partial \rho_k}\,,\ p_k^* = i\frac{\partial}{\partial \rho_k^*}\,. \tag{1.7}$$

Then in LLA the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation for the Pomeron wave function can be written as follows [1]

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\,,\ \Delta = -\frac{\alpha_s N_c}{2\pi} \min E\,, \tag{1.8}$$

where Δ is the Pomeron intercept entering in the expression for the total cross-section $\sigma_t \sim s^{\Delta}$. In the operator representation the BFKL Hamiltonian is simplified [2]

$$H_{12} = \ln|p_1p_2|^2 + \frac{1}{p_1p_2^*}\left(\ln|\rho_{12}|^2\right) p_1p_2^* + \frac{1}{p_1^*p_2}\left(\ln|\rho_{12}|^2\right) p_1^*p_2 - 4\psi(1)\,, \tag{1.9}$$

where $\rho_{12} = \rho_1 - \rho_2$ and $\psi(x) = \Gamma'(x)/\Gamma(x)$. Here the kinetic energy is proportional to the sum of the gluon Regge trajectories $\omega(-|p_{1,2}|^2)$ and the potential energy $\sim \ln|\rho_{12}|^2$ is obtained by the Fourier transformation from the product of two vertices $C(q_2,q_1)$.

The BFKL Hamiltonian is invariant under the Möbius transformation [3]

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d} \,. \tag{1.10}$$

There are two Casimir operators of the Möbius group

$$M^2 = \left(\sum_{r=1}^{2} \vec{M}^{(r)}\right)^2 = \rho_{12}^2 \, p_1 \, p_2 \,, \; M^{*2} = (M^2)^* \,. \tag{1.11}$$

Their eigenvalue equations

$$M^2 f_{m,\widetilde{m}} = m(m-1) f_{m,\widetilde{m}} \,, \; M^{*2} f_{m,\widetilde{m}} = \widetilde{m}(\widetilde{m}-1) f_{m,\widetilde{m}} \tag{1.12}$$

define the conformal weights

$$m = 1/2 + i\nu + n/2 \,, \; \widetilde{m} = 1/2 + i\nu - n/2 \tag{1.13}$$

with real ν and integer n for the principal series of unitary representations.

The Hamiltonian has the property of the holomorphic separability [4]

$$H_{12} = h_{12} + h_{12}^* \,, \tag{1.14}$$

where the holomorphic Hamiltonian h_{12} is given below

$$h_{12} = \ln(p_1 p_2) + \frac{1}{p_1} \, (\ln \rho_{12}) \, p_1 + \frac{1}{p_2} \, (\ln \rho_{12}) \, p_2 - 2\psi(1) \,. \tag{1.15}$$

2. Integrability of the multi-colour BFKL dynamics

Let us investigate the Bartels-Kwiecinskii-Praszalowicz equation [5] for the n-gluon state

$$E \, \Psi(\vec{\rho}_1, ..., \vec{\rho}_n) = \sum_{k<l} \frac{T_k^a T_l^a}{(-N_c)} H_{k,l} \, \Psi(\vec{\rho}_1, ..., \vec{\rho}_n) \,, \tag{2.1}$$

where T_k^a is the gauge group generator acting on the colour index of the gluon k.

The BKP equation is especially simple in the multi-colour QCD, where the Hamiltonian is simplified as follows $H = \frac{1}{2} \sum_k H_{k,k+1}$. It is invariant under the Möbius and duality transformations [6]

$$\rho_{r,r+1} \to p_r \to \rho_{r-1,r} \,. \tag{2.2}$$

The corresponding wave function has the property of the holomorphic factorization [4]

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, ..., \vec{\rho}_n) = \sum_{r,s} a_{r,s} \, \Psi_r(\rho_1, ..., \rho_n) \, \Psi_s(\rho_1^*, ..., \rho_n^*) \,, \tag{2.3}$$

where the coefficients $a_{r,s}$ are chosen from the condition of its single-valuedness.

The holomorphic Hamiltonian h commutes with the integrals of motion [2, 7]

$$q_r = \sum_{k_1 < k_2 < ... < k_r} \rho_{k_1 k_2} \rho_{k_2 k_3} ... \rho_{k_r k_1} \, p_{k_1} p_{k_2} ... p_{k_r} \,, \; [q_r, h] = 0 \,. \tag{2.4}$$

The integrability of the BFKL dynamics [7] is related to the fact, that H coincides with the local Hamiltonian of the Heisenberg spin model [8].

In particular for the Pomeron wave function ($n=2$) one can obtain the simple expression [3]

$$f_{m,\widetilde{m}}(\overrightarrow{\rho_1},\overrightarrow{\rho_2};\overrightarrow{\rho_0}) = \left(\frac{\rho_{12}}{\rho_{10}\,\rho_{20}}\right)^m \left(\frac{\rho_{12}^*}{\rho_{10}^*\,\rho_{20}^*}\right)^{\widetilde{m}} \tag{2.5}$$

with the corresponding energy having the holomorphic separability property

$$E_{m,\widetilde{m}} = \varepsilon_m + \varepsilon_{\widetilde{m}} \quad , \quad \varepsilon_m = \psi(m) + \psi(1-m) - 2\psi(1)\,,\;\; \psi(x) = \frac{d}{dx}\ln\Gamma(x)\,. \tag{2.6}$$

Further, the intercept of the BFKL Pomeron is positive [1]

$$\Delta = 4\,\frac{\alpha_s}{\pi} N_c \ln 2 \tag{2.7}$$

and, as a result, one obtains the violation of the Froissart bound

$$\sigma \sim s^{\Delta} > c \ln^2 s\,. \tag{2.8}$$

One should restore the s-channel unitarity for scattering amplitudes. The consistent way to solve this problem is to use the effective field theory for the Reggeized gluons [9], [10].

3. DGLAP and BFKL equations in $N=4$ SUSY

The parton distributions are expressed in terms of the corresponding unintegrated quantities

$$f_a(x,Q^2) \;=\; \int_{k_\perp^2<Q^2} dk_\perp^2\, \varphi_a(x,k_\perp^2)\,. \tag{3.1}$$

With the use of the Mellin transformation

$$f_a(j,Q^2) = \int_0^1 dx\, x^{j-1}\, f_a(x,Q^2) \tag{3.2}$$

the DGLAP equation [11] for $f_a(x,Q^2)$ is written as a renormalization group equation for $f_a(j,Q^2)$ with the kernel expressed in terms of the anomalous dimension matrix γ_{ab}

$$\frac{d}{d\ln Q^2}\, f_a(j,Q^2) = \sum_b \gamma_{ab}(j)\, f_b(j,Q^2)\,. \tag{3.3}$$

The momenta $f_a(j,Q^2)$ are proportional to matrix elements of the light-cone components of the local twist-2 operators which are tensors or pseudo-tensors

$$O^a = \tilde{n}^{\mu_1}...\tilde{n}^{\mu_j}\, O^a_{\mu_1,...,\mu_j}\;,\;\; \tilde{\mathscr{O}}^a = \tilde{n}^{\mu_1}...\tilde{n}^{\mu_j}\, \tilde{\mathscr{O}}^a_{\mu_1,...,\mu_j}\,. \tag{3.4}$$

The anomalous dimensions are the same for the different tensor projections

$$\tilde{n}^{\mu_1}...\tilde{n}^{\mu_{1+\omega}}\, O^a_{\mu_1,...,\mu_{1+\omega},\sigma_1,...,\sigma_{|n|}}\, l_\perp^{\sigma_1}...l_\perp^{\sigma_{|n|}}\,. \tag{3.5}$$

Solutions of the BFKL equation are classified by the anomalous dimension $\gamma = \frac{1}{2} + i\nu$ and the conformal spin $|n|$. The conformal spin coincides with the number of transverse indices of the tensor O^a.

In the next-to-leading approximation the eigenvalue of the BFKL kernel is given below [12]

$$\omega = \omega_0(n, \gamma) + 4\,\hat{a}^2\,\Delta(n, \gamma)\,,\; \hat{a} = g^2 N_c/(16\pi^2)\,, \tag{3.6}$$

where in QCD $\Delta(n,\gamma)$ contains non-analytic functions $\delta_{n,0}$ and $\delta_{n,2}$, but in $N = 4$ SUSY these Kroniker symbols are cancelled [14].

Moreover, in the $N = 4$ model we obtain, that $\Delta(n,\gamma)$ has the Hermitian separability property

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}\,,\; M = \gamma + \frac{|n|}{2}\,, \tag{3.7}$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\; \beta'(z) = \frac{1}{4}\left[\Psi'\Big(\frac{z+1}{2}\Big) - \Psi'\Big(\frac{z}{2}\Big)\right] \tag{3.8}$$

and is expressed in terms of special functions belonging to the maximal transcedentality class [14]

$$\phi(M) = 3\zeta(3) + \Psi''(M) - 2\Phi(M) + 2\beta'(M)\Big(\Psi(1) - \Psi(M)\Big)\,, \tag{3.9}$$

where

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{\beta'(k+1)}{k+M} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M}\left(\Psi'(k+1) \;-\; \frac{\Psi(k+1) - \Psi(1)}{k+M}\right). \tag{3.10}$$

Let us return now to the DGLAP equation [11]. One loop anomalous dimension matrix for twist-2 operators in $N = 4$ SUSY was firstly calculated in Ref. [15]. The examples of such operators are given below

$$\mathscr{O}^g_{\mu_1,\ldots,\mu_j} = \hat{S} G^a_{\rho\mu_1} D_{\mu_2} D_{\mu_3} \ldots D_{\mu_{j-1}} G^a_{\rho\mu_j}\,, \tag{3.11}$$

$$\tilde{\mathscr{O}}^g_{\mu_1,\ldots,\mu_j} = \hat{S} G^a_{\rho\mu_1} D_{\mu_2} D_{\mu_3} \ldots D_{\mu_{j-1}} \tilde{G}^a_{\rho\mu_j}\,, \tag{3.12}$$

$$\mathscr{O}^q_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\Psi}^a \gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_j} \Psi^a\,, \tag{3.13}$$

$$\tilde{\mathscr{O}}^q_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\Psi}^a \gamma_5 \gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_j} \Psi^a\,, \tag{3.14}$$

$$\mathscr{O}^\varphi_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\Phi}^a D_{\mu_1} D_{\mu_2} \ldots D_{\mu_j} \Phi^a\,. \tag{3.15}$$

The diagonalization of the anomalous dimension matrices γ and $\widetilde{\gamma}$ gives the result

$$\begin{vmatrix} -4S_1(j-2) & 0 & 0 \\ 0 & -4S_1(j) & 0 \\ 0 & 0 & -4S_1(j+2) \end{vmatrix}\,,\; \begin{vmatrix} -4S_1(j-1) & 0 \\ 0 & -4S_1(j+1) \end{vmatrix}\,, \tag{3.16}$$

containing one universal function γ_{uni} for the super-multiplet of twist-2 operators

$$\gamma^{(0)}_{uni}(j) = -4S_1(j-2)\,,\; S_1(j) = \sum_{i=1}^{j} \frac{1}{i}\,. \tag{3.17}$$

Note, that this function has the maximal possible transcedentality, which is related to an integrability of evolution equations for matrix elements of quasi-partonic operators in $N = 4$ SUSY [15].

4. Two- and three- loop universal anomalous dimension in $N=4$

Using the fact that the eigenvalue of the BFKL equation is expressed in terms of the most complicated special functions and the hypothesis that all singularities of the anomalous dimension can be obtained from this equation [13], we can argue [14], that the perturbative expansion of the universal anomalous dimension

$$\gamma_{uni}(j) = \hat{a}\gamma_{uni}^{(0)}(j) + \hat{a}^2\gamma_{uni}^{(1)}(j) + \hat{a}^3\gamma_{uni}^{(2)}(j) + \dots \tag{4.1}$$

should contain in each order of the perturbation theory only harmonic sums with the highest possible transcedentality. Such assumption allows us to find the universal anomalous dimension in two loops [14] from the corresponding QCD expressions

$$\frac{1}{8}\gamma_{uni}^{(1)}(j+2) = 2S_1(j)\left(S_2(j)+S_{-2}(j)\right) - 2S_{-2,1}(j) + S_3(j) + S_{-3}(j)\,, \tag{4.2}$$

where

$$S_r(j) = \sum_{i=1}^{j}\frac{1}{i^r}\,,\; S_{-r}(j) = \sum_{i=1}^{j}\frac{(-1)^i}{i^r}\,,\; S_{-2,1} = \sum_{m=1}^{j}\frac{(-1)^m}{m^2}S_1(m)\,. \tag{4.3}$$

This result was verified by direct calculations of the anomalous dimension matrix [16].

Further, recently the three-loop anomalous dimension matrix for QCD was calculated [17]. It allowed us to extract the universal anomalous dimension in three loops for $N=4$ SUSY using the above hypothesis of the maximal transcedentality [18]

$$\begin{aligned}
\frac{1}{32}\gamma_{uni}^{(2)}(j+2) = {} & 24S_{-2,1,1,1} - 12\left(S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}\right) \\
& +6\left(S_{-4,1} + S_{-3,2} + S_{-2,3}\right) - 3S_{-5} - 2S_3S_{-2} - S_5 \\
& -2S_1^2\left(3S_{-3} + S_3 - 2S_{-2,1}\right) - S_2\left(S_{-3} + S_3 - 2S_{-2,1}\right) \\
& -S_1\left(8\overline{S}_{-4} + \overline{S}_{-2}^2 + 4S_2\overline{S}_{-2} + 2S_2^2\right) \\
& -S_1\left(3S_4 - 12\overline{S}_{-3,1} - 10\overline{S}_{-2,2} + 16\overline{S}_{-2,1,1}\right)\,,
\end{aligned} \tag{4.4}$$

where the generalized harmonic sums are the functions of j given below

$$S_{a,b,c,\cdots}(j) = \sum_{m=1}^{j}\frac{1}{m^a}S_{b,c,\cdots}(m)\,,\; S_{-a,b,\cdots}(j) = \sum_{m=1}^{j}\frac{(-1)^m}{m^a}S_{b,\cdots}(m)\,, \tag{4.5}$$

$$\overline{S}_{-a,b,c\cdots}(j) = (-1)^j S_{-a,b,\ldots}(j) + S_{-a,b,\cdots}(\infty)\Big(1-(-1)^j\Big). \tag{4.6}$$

5. Comparison with other approaches

The three-loop anomalous dimension for $N=4$ SUSY at $j=1+\omega\to 1$

$$\gamma_{uni}^{N=4}(j) = \hat{a}\frac{4}{\omega} - 32\zeta_3\,\hat{a}^2 + 32\zeta_3\,\hat{a}^3\frac{1}{\omega} + \dots \tag{5.1}$$

is in an agreement with the predictions of the BFKL equation [14].

Near the negative even points $j+2r=\omega\to 0$ one can verify, that the anomalous dimension satisfies the equation

$$\gamma_{uni}=4\,\frac{\hat{a}}{\omega}+\frac{\gamma_{uni}^2}{\omega}\,, \tag{5.2}$$

corresponding to the resummation of the double logarithmic terms $\sim\alpha/\omega^2$.

Further, one can calculate the universal anomalous dimension at large j

$$\gamma_{uni}=a(z)\,\ln j\,,\;z=\frac{\alpha N_c}{\pi}=4\hat{a} \tag{5.3}$$

up to three loops

$$a(z)=-z+\frac{\pi^2}{12}z^2-\frac{11}{720}\pi^4z^3+\dots. \tag{5.4}$$

On the other hand, using the AdS/CFT correspondence [19] between the superstring model on the anti-de-Sitter space and the $N=4$ supersymmetric Yang-Mills theory, A. Polyakov with collaborators predicted $a(z)$ in the strong coupling limit [20]

$$\lim_{z\to\infty}a(z)=-z^{1/2}+\frac{3\ln 2}{4\pi}+\dots. \tag{5.5}$$

In Ref. [16] the simple resummation of the perturbation theory for $a(z)$ was suggested in the form

$$\widetilde{a}=-z+\frac{\pi^2}{12}\widetilde{a}^2. \tag{5.6}$$

This prediction for a up to three loops is in a rather good agreement with the exact result

$$\widetilde{a}=-z+\frac{\pi^2}{12}z^2-\frac{1}{72}\pi^4z^3+\dots \tag{5.7}$$

and with its asymptotic behaviour at $z\to\infty$.

6. Equations for the anomalous dimension at $j\to\infty$

Let us introduce the new parameter ε related to z in eq. (5.3)

$$\varepsilon=\frac{1}{\sqrt{z}}. \tag{6.1}$$

It is obvious, that in the strong coupling regime $\varepsilon\to 0$. Eden and Staudacher (ES) expressed the coefficient $a(z)$ appearing in the asymptotic expression (5.3) for γ_{uni} in terms of the new function

$$a(z)=-\frac{2}{\varepsilon}\,f(0)\,, \tag{6.2}$$

satisfying the integral equation [21]

$$\varepsilon\,f(x)=\frac{t}{e^t-1}\left(\frac{J_1(x)}{x}-\int_0^\infty dy\,\frac{J_1(x)\,J_0(y)-J_1(y)\,J_0(x)}{x-y}\,f(y)\right)\,,\;t=\varepsilon x\,, \tag{6.3}$$

where $J_n(x)$ are the Bessel functions. The modified equation for $f(x)$ was derived recently by Beisert, Eden and Staudacher (BES) [22] with taking into account a phase arising from the crossing symmetry of the underlying S-matrix.

Using the Laplace transformation

$$f(x) = \int_{-i\infty}^{i\infty} \frac{dj}{2\pi i} e^{xj} \phi(j) \tag{6.4}$$

one can write the following anzatz for the solution of the ES equation

$$\phi(j) = \sum_{n=1}^{\infty} (\delta_{n,1} - a_{n,\varepsilon}) \sum_{s=1}^{\infty} \frac{\left(\sqrt{(j+s\varepsilon)^2+1} + j + s\varepsilon\right)^{-n}}{\sqrt{(j+s\varepsilon)^2+1}} . \tag{6.5}$$

The coefficients $a_{n,\varepsilon}$ satisfy the set of algebraic equations

$$a_{n,\varepsilon} = \sum_{n'=1}^{\infty} K_{n,n'}(\varepsilon) \left(\delta_{n',1} - a_{n',\varepsilon}\right) , \tag{6.6}$$

where the integral kernel is calculated explicitly [23]

$$K_{n,n'}(\varepsilon) = 2n \sum_{R=0}^{\infty} (-1)^R \frac{2^{-2R-n-n'}}{\varepsilon^{2R+n+n'}} \zeta(2R+n+n') \frac{(2R+n+n'-1)!\,(2R+n+n')!}{R!\,(R+n)!\,(R+n')!\,(R+n+n')!} . \tag{6.7}$$

One can verify from this expression, that in all orders of the perturbation theory for $a(z)$ the maximal transcedentality is valid and the coefficients in front of the products of ζ-functions are integer numbers. Note, that for the BES equation [22] the kernel $K_{n,n'}(\varepsilon)$ should be multiplied by the factor $i = \sqrt{-1}$ for odd values of the sum $n+n'$.

Using the new variable $z = j + \sqrt{j^2+1}$ one can write the dispersion representation [23]

$$\xi(z) = \int_L \frac{dz'}{2\pi i} \frac{\xi(z') - \xi(-1/z')}{z - z'} \tag{6.8}$$

for the function

$$\xi(z) = \frac{z^2+1}{2z} \left(\phi(j-\varepsilon) - \phi(j)\right) . \tag{6.9}$$

In the above dispersion representation for $\xi(z)$ the integration is performed along the unit circle in the anti-clock-wise direction and the point z is situated outside the circle. The corresponding discontinuity satisfies the linearized "unitarity" constraint [23]

$$\frac{\xi(z) - \xi(-1/z)}{2\sqrt{j^2+1}} = 1 - \sum_{s=1}^{\infty} \frac{\xi\left(j + s\varepsilon + \sqrt{(j+s\varepsilon)^2+1}\right)}{\sqrt{(j+s\varepsilon)^2+1}} . \tag{6.10}$$

For the case of the strong coupling regime $\varepsilon \to 0$ one can solve the integral equations. In particular, for the BES case [22] the anomalous dimension has the form [23]

$$\lim_{\varepsilon \to 0} a_{sing}^{BES} = -\frac{1}{\varepsilon} \frac{I_1(2/\varepsilon)}{I_0(2/\varepsilon)} \to -z \tag{6.11}$$

in an agreement with the string side prediction [20].

7. BFKL Pomeron and graviton in N=4 SUSY

It is possible to calculate the Pomeron intercept in the $N=4$ supersymmetric gauge theory at large coupling constants [18] (see also [24]). To begin with, one can simplify the eigenvalue for the BFKL kernel in the diffusion approximation as follows (see [12])

$$j = 2 - \Delta - D\nu^2 \,,\; \gamma = \frac{j}{2} + i\nu \,, \tag{7.1}$$

assuming, that the parameter Δ is small at large $z \sim \alpha$. Due to the energy-momentum conservation we have $\gamma|_{j=2} = 0$ and therefore γ can be expressed only in terms of the parameter Δ

$$\gamma = (j-2)\left(\frac{1}{2} - \frac{1/\Delta}{1+\sqrt{1+(j-2)/\Delta}}\right). \tag{7.2}$$

On the other hand, with the use of the AdS/CFT correspondence [19] the above eigenvalue equation can be written as the graviton Regge trajectory

$$j = 2 + \frac{\alpha'}{2}t \,,\; t = E^2/R^2 \,,\; \alpha' = \frac{R^2}{2}\Delta \,. \tag{7.3}$$

The behaviour of γ at $g \to \infty$ and $j \to \infty$ is known from the paper of Polyakov with collaborators [20]

$$\gamma_{|z\to\infty} = -\sqrt{j-2}\,\Delta^{-1/2}_{|j\to\infty} = \sqrt{\pi j}\, z^{1/4} \,. \tag{7.4}$$

Therefore one can obtain the Pomeron intercept at large couplings [18] (see also Ref. [24])

$$j = 2 - \Delta \,,\; \Delta = \frac{1}{\pi} z^{-1/2} \,. \tag{7.5}$$

To verify this result independently one can calculate the slope of the anomalous dimension at $j=2$

$$\gamma'(2) = \frac{1}{2} - \frac{1}{2\Delta} = -\frac{\pi^2}{6}z + \frac{\pi^4}{72}z^2 - \frac{\pi^6}{540} + \ldots \,. \tag{7.6}$$

Similar to the case $j \to \infty$ we use the following resummation procedure [16]

$$\frac{\pi^2}{6}z = -\widetilde{b} + \frac{1}{2}\widetilde{b}^2 \,,\; b = \gamma'(2) \,. \tag{7.7}$$

The weak and strong coupling asymptotics of the solution of this equation is given below

$$\widetilde{b} = -\frac{\pi^2}{6}z + \frac{\pi^4}{72}z^2 - \frac{\pi^6}{432}z^3 + \ldots \,,\; \lim_{z\to\infty} \widetilde{\Delta} = \frac{\sqrt{3}}{2\pi} z^{-1/2} \,, \tag{7.8}$$

which is in a good agreement with the above results for Δ and b.

8. Discussion

It is important, that in QCD the gluons and quarks are reggeized. For solving the unitarization problem for the BFKL Pomeron one should use the effective action for interactions of Reggeons and particles in the quasi-multi-Regge kinematics. The Reggeon calculus in the form of a 2+1 field theory can be derived from QCD. In the framework of this approach the t -channel unitarity is automatically fulfilled. The s-channel unitarity is incorporated in the Reggeon theory through the bootstrap equations (see [1]) and various relations among the effective vertices. The next-to-leading correction to the eigenvalue of the BFKL kernel in $N = 4$ SUSY does not contain the non-analytic terms. It is a sum of the most complicated functions which could appear in this order. Using the hypothesis of the maximal transcedentality for the universal anomalous dimension of the twist-2 operators this quantity was calculated up to the third order. We suggested a resummation procedure and verified the strong coupling predictions obtained from the AdS/CFT correspondence. In particular, the analytic properties of the ES equation for $\gamma(j)$ at $j \to \infty$ were investigated. It was shown, that the solution of the BES equation reproduces the string predictions for the anomalous dimension at large coupling constants. We calculated also the intercept of the BFKL Pomeron in $N = 4$ SUSY in the same limit.

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