Fermionic basis for space of operators in the XXZ model

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In the recent study of correlation functions for the infinite XXZ spin chain, a new pair of anti-commuting operators $\mathbf{b}(\zeta), \mathbf{c}(\zeta)$ was introduced. They act on the space of *quasi-local* operators, which are local operators multiplied by the disorder operator $q^{2\alpha S(0)}$ where $S(0) = \frac{1}{2}\sum_{j=-\infty}^{0}\sigma_j^3$. For the inhomogeneous chain with the spectral parameters ξ_k, these operators have simple poles at $\zeta^2 = \xi_k^2$. The residues are denoted by $\mathbf{b}_k, \mathbf{c}_k$. At $q = i$, we show that the operators $\mathbf{b}_k, \mathbf{c}_k$ are cubic monomials in free fermions. In other words, the action of these operators is very simple in the fermion basis. We give an explicit construction of these fermions. Then, we show that the existence of the fermionic basis is a consequence of the Grassmann relation, the equivariance with respect to the action of the symmetric group and the reduction property, which are all valid for the operators $\mathbf{b}_k, \mathbf{c}_k$ in the case of generic q.

BETHE ANSATZ: 75 YEARS LATER
October 19-21, 2006
Brussels, Belgium

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1. Introduction

This is a continuation of our previous paper [1] in which we studied the infinite XXZ spin chain with the Hamiltonian

$$H_{XXZ} = \frac{1}{2} \sum_{k=-\infty}^{\infty} \left(\sigma_k^1 \sigma_{k+1}^1 + \sigma_k^2 \sigma_{k+1}^2 + \Delta \sigma_k^3 \sigma_{k+1}^3 \right), \quad \Delta = \frac{q+q^{-1}}{2}.$$

Before going into the details of the present paper, we review the main points of [1]. In that paper we gave an algebraic formula for the vacuum expectation values of quasi-local operators of the form

$$q^{\alpha \Sigma_{k=-\infty}^0 \sigma_k^3} \mathscr{O}. \tag{1.1}$$

Here, $\mathscr{O}$ is a local operator, and α is a disorder parameter. We denote by

$$S(j) = \frac{1}{2} \sum_{k=-\infty}^{j} \sigma_k^3$$

the total spin operator on the left half of the chain.

Let $\mathscr{W}_\alpha$ be the space of quasi-local operators of this form. The main ingredient in our formula was a pair of operators $\mathbf{b}(\zeta, \alpha), \mathbf{c}(\zeta, \alpha)$ acting on the spaces of quasi-local operators,

$$\mathbf{b}(\zeta, \alpha) : \mathscr{W}_\alpha \to \mathscr{W}_{\alpha+1}, \ \mathbf{c}(\zeta, \alpha) : \mathscr{W}_\alpha \to \mathscr{W}_{\alpha-1}.$$

It is convenient to introduce the space

$$\mathscr{W}_{[\alpha]} = \oplus_{k \in \mathbb{Z}} \mathscr{W}_{\alpha+k},$$

and use the notation $\mathbf{b}(\zeta), \mathbf{c}(\zeta) : \mathscr{W}_{[\alpha]} \to \mathscr{W}_{[\alpha]}$.

The vacuum expectation valued are given by the formula

$$\frac{\langle \text{vac} | q^{2\alpha S(0)} \mathscr{O} | \text{vac} \rangle}{\langle \text{vac} | q^{2\alpha S(0)} | \text{vac} \rangle} = \mathbf{tr}^\alpha \left(e^\Omega \left(q^{2\alpha S(0)} \mathscr{O} \right) \right),$$

where $\mathbf{tr}^\alpha$ stands for the weighted trace

$$\mathbf{tr}^\alpha(X) = \cdots \text{tr}_1^\alpha \, \text{tr}_2^\alpha \, \text{tr}_3^\alpha \cdots (X) \quad (X \in \mathscr{W}_\alpha),$$

$$\text{tr}^\alpha(x) = \frac{1}{q^{\frac{\alpha}{2}} + q^{-\frac{\alpha}{2}}} \text{tr} \left(q^{-\frac{1}{2}\alpha\sigma^3} x \right) \quad (x \in \text{End}(\mathbb{C}^2)),$$

and on each $\mathscr{W}_\alpha$, Ω admits an expression

$$\Omega = -\text{res}_{\zeta_1, \zeta_2 = 1} \left(\omega(\zeta_1/\zeta_2, \alpha) \mathbf{b}(\zeta_1) \mathbf{c}(\zeta_2) \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} \right). \tag{1.2}$$

Here $\omega(\zeta, \alpha)$ is a known scalar function (whose explicit formula can be found in (1.5), [1]).

The operators $\mathbf{b}(\zeta), \mathbf{c}(\zeta)$ are obtained from the operators $\mathbf{b}_{[k,l]}(\zeta), \mathbf{c}_{[k,l]}(\zeta)$ which act on the direct sum of the finite dimensional spaces

$$\mathscr{V}_{[\alpha],[k,l]} = \oplus_{j \in \mathbb{Z}} \mathscr{V}_{\alpha+j,[k,l]}, \ \mathscr{V}_{\alpha+j,[k,l]} \simeq \text{End}\left((\mathbb{C}^2)^{\otimes(l-k+1)} \right)$$

by taking an inductive limit when $k \to -\infty, l \to \infty$. Here the index $[k, l]$ indicates the finite interval $[k, l] \subset \mathbb{Z}$ of the spin chain.

We use $\mathbf{c}^+ = \mathbf{c}$ and $\mathbf{c}^- = \mathbf{b}$ in the sequel. On each finite interval $[k, l]$, the operator $\mathbf{c}^{\pm}_{[k,l]}(\zeta)$ has the form

$$\mathbf{c}^{\pm}_{[k,l]}(\zeta) = -\frac{1}{2} \sum_{j=1}^{l-k+1} \frac{\mathbf{c}^{\pm(j)}_{[k,l]}}{(\zeta-1)^j} .$$

The operators $\mathbf{b}^{(j)}_{[k,l]}, \mathbf{c}^{(j)}_{[k,l]}$ depend on k, l in such a way that the reduction properties hold:

$$\mathbf{c}^{\pm}_{[k,l]}(X_{[k,l-1]}) = \mathbf{c}^{\pm}_{[k,l-1]}(X_{[k,l-1]}) ,$$
$$\mathbf{c}^{\pm}_{[k,l]}(q^{\alpha \sigma^3_k} X_{[k+1,l]}) = q^{(\alpha \mp 1)\sigma^3_k} \mathbf{c}^{\pm}_{[k+1,l]}(X_{[k+1,l]}) .$$

Here $X_{[k,l]}$ denotes an element of $\mathscr{V}_{\alpha,[k,l]}$.

In [1], the operator $\mathbf{c}^{\pm}_{[k,l]}(\zeta)$ was constructed from the trace of a monodromy matrix whose auxiliary space consists of the tensor product of $\mathbb{C}^2$ and the q oscillator representations $W^{\pm}$. This is similar to the construction of Q operators in [2]. However our construction differs from that paper in two respects. We take the trace of the off-diagonal entry of the monodromy matrix which is triangular with respect to the auxiliary space $\mathbb{C}^2$; and we consider only the singular part in the expansion at $\zeta = 1$, since otherwise the operator is dependent on the way of triangularization. As a spin-off of our construction we obtained the Grassmann relation among the operators $\mathbf{c}^{\pm}(\zeta)$:

$$\mathbf{c}^{\varepsilon_1}(\zeta_1)\mathbf{c}^{\varepsilon_2}(\zeta_2) = -\mathbf{c}^{\varepsilon_2}(\zeta_2)\mathbf{c}^{\varepsilon_1}(\zeta_1) .$$

In this paper, we further investigate the structure of the operators $\mathbf{c}^{\pm}(\zeta)$. We consider the inhomogeneous lattice with the spectral parameters ξ_k ($k \in \mathbb{Z}$). There are two advantages in considering the inhomogeneous case. The singularities of the trace which lie at $\zeta^2 = 1$ in the homogeneous case split into simple poles at $\zeta^2 = \xi_k^2$ in the inhomogeneous case. Therefore the operator $\mathbf{c}^{\pm}_{[k,l]}(\zeta)$ is decomposed into a partial fraction

$$\mathbf{c}^{\pm}_{[k,l]}(\zeta) = -\frac{1}{2} \sum_{j=k}^{l} \frac{\xi_j \mathbf{c}^{\pm}_{j,[k,l]}}{\zeta - \xi_j} .$$

This is the first point. The reduction property and the Grassmann relation persist in the inhomogeneous case. Passing to the inductive limit, we obtain a set of operators $\mathbf{c}^{\pm}_j$ acting on $\mathscr{W}_{[\alpha]}$ and satisfying the Grassmann relation:

$$\mathbf{c}^{\varepsilon_1}_{j_1}\mathbf{c}^{\varepsilon_2}_{j_2} = -\mathbf{c}^{\varepsilon_2}_{j_2}\mathbf{c}^{\varepsilon_1}_{j_1} .$$

Moreover, in the inhomogeneous case, the operators $\mathbf{c}^{\pm}_j$ enjoy the equivariance with respect to the action of the infinite symmetric group inherited from the R matrix symmetry:

$$\mathbf{s}_i \cdot \mathbf{c}^{\pm}_j = \mathbf{c}^{\pm}_{s_i(j)} \cdot \mathbf{s}_i, \quad \mathbf{s}_i \in \mathfrak{S}_\infty .$$

This is the second point.

Although the operators $\mathbf{c}_j^{\pm}$ satisfy the Grassmann relation and annihilate some elements in $\mathscr{W}_{[\alpha]}$, they do not constitute the annihilation part of the free fermion algebra. We see this because they have a large kernel in common, which is spanned by the $\mathfrak{S}_{\infty}$ orbits of $q^{2\alpha S(j)}$ ($j \in \mathbb{Z}$). We call elements in the common kernel *vacuum states*. The existence of a large kernel is a simple consequence of the reduction property and the equivariance with respect to the $\mathfrak{S}_{\infty}$ action. In fact, the reduction property holds in a stronger sense:

$$\mathbf{c}_{l,[k,l]}^{\pm}\big(V_{\alpha,[k,l]}\big) \subset \mathscr{V}_{\alpha,[k,l-1]} \otimes \mathrm{id}.$$

Here id is the identity operator on the *l*-th tensor component. This implies that there exists an $\mathfrak{S}_{\infty}$-equivariant filtration of the space $\mathscr{W}_{[\alpha]}$,

$$F^0 \mathscr{W}_{[\alpha]} \subset \cdots \subset F^n \mathscr{W}_{[\alpha]} \subset F^{n+1} \mathscr{W}_{[\alpha]} \subset \cdots \mathscr{W}_{[\alpha]},$$
$$\mathbf{s}_i\big(F^n \mathscr{W}_{[\alpha]}\big) \subset F^n \mathscr{W}_{[\alpha]},$$

such that $F^0 \mathscr{W}_{[\alpha]}$ is the space of the vacuum states and the operator $\mathbf{c}_j^{\pm}$ decreases the 'particle' number *n*:

$$\mathbf{c}_j^{\pm}\big(F^n \mathscr{W}_{[\alpha]}\big) \subset F^{n-1} \mathscr{W}_{[\alpha]}.$$

In the case $q = i$, it is well-known that the Hamiltonian is diagonalized by the Jordan-Wigner transformation. The latter turns the tensor product $(\mathbb{C}^2)^{\otimes n}$ into the irreducible representation of the free fermion algebra with $2n$ generators

$$\psi_j^{\pm} = \sigma_j^{\pm} i^{\mp \sum_{k=1}^{j-1} \sigma_k^3} \quad (1 \le j \le n).$$

We introduce a fermion algebra with $4n$ generators $\widehat{\Psi}_j^{\pm}, \widehat{\Phi}_j^{\pm}$ ($1 \le j \le n$),

$$[\widehat{\Psi}_{j_1}^{\varepsilon_1}, \widehat{\Psi}_{j_2}^{\varepsilon_2}] = 0, \; [\widehat{\Phi}_{j_1}^{\varepsilon_1}, \widehat{\Phi}_{j_2}^{\varepsilon_2}] = 0, \; [\widehat{\Psi}_{j_1}^{\varepsilon_1}, \widehat{\Phi}_{j_2}^{\varepsilon_2}] = \delta_{\varepsilon_1, -\varepsilon_2} \delta_{j_1, j_2},$$

which act on $\mathrm{End}\big((\mathbb{C}^2)^{\otimes n}\big)$ irreducibly. They are linear combinations of the left and the right multiplications by $\psi_j^{\pm}$, equivariant relative to the $\mathfrak{S}_n$ action, and satisfy the reduction property which enables us to extend their definition to the infinite lattice. It is convenient to introduce the fermion number operators $N_k^{\pm} = \widehat{\Phi}_k^{\pm} \widehat{\Psi}_k^{\mp}$. The vacuum states $\mathbb{C} q^{2\alpha S(j)}$ ($j \in \mathbb{Z}$) are characterized by the fermion numbers

$$N_k^{\pm} = \begin{cases} 1 & \text{if } k \le j; \\ 0 & \text{if } k \ge j+1. \end{cases}$$

Finally, we modify these fermions to another set of free fermions $\chi_k^{\pm}, \chi_k^{*\pm}$ satisfying $N^{\pm} = \chi_k^{*\pm} \chi_k^{\mp}$, and obtain the simple formula

$$\mathbf{c}_k^{\pm} = \chi_k^{\pm} \chi_k^{\mp} \chi_k^{*\pm}.$$

Namely, the operator $\mathbf{c}_k^{\pm}$ is a fermion annihilation operator: if the fermion number of a state is $N_k^{\mp} = 1$, it is changed to $N_k^{\mp} = 0$; otherwise the state is annihilated.

Let us return to generic q. The properties of the operators $\mathbf{c}_k^{\pm}$ formulated above are enough to deduce the existence of a fermionic basis of the space $\mathcal{W}_{[\alpha]}$ on which they act exactly as in the $q = i$ case (see Theorem 4.1). This is our conclusion. Such a fermionic basis is by no means unique. We hope to discuss a more explicit construction in a future publication.

The plan of this paper is as follows. In Section 2 we recall the definition of $\mathbf{b}(\zeta)$ and $\mathbf{c}(\zeta)$ given in [1]. Appendix A is a supplement to this section, giving a proof of some operator identities. Section 3 with the detailed calculations in Appendix B gives the complete details of the $q = i$ case. Section 4 is devoted to the existence proof of the fermionic basis for generic q.

2. Operators b and c

In this section, we give the definition of the operators $\mathbf{b}(\zeta)$, $\mathbf{c}(\zeta)$ and discuss their basic properties: the Grassmann relation (2.9), the equivariance with respect to the symmetric group (2.10), and the reduction properties (2.11) and (2.12).

Throughout the paper, q denotes a non-zero complex number and V a two-dimensional vector space with fixed basis v_+, v_-.

2.1 Quantum spaces

We consider an inhomogeneous spin chain with spectral parameters ξ_j ($j \in \mathbb{Z}$). They are attached to the 'quantum' spaces V_j ($j \in \mathbb{Z}$), where V_j is a copy of V.

In this and the next section, we fix a positive integer n and consider a finite segment $[1, n]$ of the infinite spin chain. We work with the space

$$\mathcal{V} = \mathrm{End}(V_1 \otimes \cdots \otimes V_n)$$

consisting of operators which act non-trivially on the lattice sites $j = 1, \cdots, n$. We denote by $P_{i,j} \in \mathrm{End}(\mathcal{V})$ the transposition of the i-th and the j-th tensor components. We consider the transfer matrix acting on the space $\mathcal{V}$. It depends on a parameter α playing the role of a boundary condition. Later we shall extend $\mathcal{V}$ to an inductive limit, where the index j runs over $\mathbb{Z}$. The parameter α enters the construction of this inductive limit as well.

To make distinction, we call an element of $\mathcal{V}$ a *state*, and an element of $\mathrm{End}(\mathcal{V})$ an *operator*. When it is necessary, we extend the coefficient field of $\mathcal{V}$ or $\mathrm{End}(\mathcal{V})$ to the field of rational functions in the variables $q^{\alpha}, \xi_1, \ldots, \xi_n$. We denote by $r_{i,j}$ the operation of exchanging spectral parameters $\xi_i \leftrightarrow \xi_j$.

The total spin operator $S = \frac{1}{2} \sum_{j=1}^n \sigma_j^3$ belongs to $\mathcal{V}$, and its adjoint action

$$\mathbb{S}(X) = [S, X] \quad (X \in \mathcal{V})$$

belongs to $\mathrm{End}(\mathcal{V})$. The space $\mathcal{V}$ decomposes as $\mathcal{V} = \bigoplus_{s \in \mathbb{Z}} \mathcal{V}^{(s)}$ where $\mathcal{V}^{(s)} = \{ X \in \mathcal{V} \,|\, \mathbb{S}(X) = sX \}$.

The spin reversal $\mathbb{J}$ is an operator acting on $\mathcal{V}$

$$\mathbb{J}(X) = \prod_{j=1}^n \sigma_j^1 \cdot X \cdot \prod_{j=1}^n \sigma_j^1 \,.$$

When we extend the coefficient field, we use the operator $\mathbf{j}$ given by

$$\mathbf{j}(X) = \mathbb{J}(X)|_{q^{\alpha} \to q^{-\alpha}} \,.$$

2.2 *R* **matrix**

The space *V* is endowed with a structure of a two-dimensional evaluation module of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. We will need only the formula for the associated *R* matrix:

$$R(\zeta) = (q\zeta - q^{-1}\zeta^{-1}) \cdot \begin{pmatrix} 1 & & & \\ & \beta(\zeta) & \gamma(\zeta) & \\ & \gamma(\zeta) & \beta(\zeta) & \\ & & & 1 \end{pmatrix},$$

where

$$\beta(\zeta) = \frac{(1-\zeta^2)q}{1-q^2\zeta^2}, \quad \gamma(\zeta) = \frac{(1-q^2)\zeta}{1-q^2\zeta^2}.$$

The matrix $R(\zeta)$ is a Laurent polynomial, while $R(\zeta)^{-1}$ has poles at $\zeta^2 = q^{\pm 2}$.

The *R* matrix gives an action of the symmetric group $\mathfrak{S}_n$ on $\mathscr{V}$, the simple reflection s_i being represented by

$$\mathbf{s}_i = r_{i,i+1}\check{\mathbb{R}}_{i,i+1}(\xi_i/\xi_{i+1}).$$

Here

$$\mathbb{R}_{i,i+1}(\xi_i/\xi_{i+1})(X) = \check{R}_{i,i+1}(\xi_i/\xi_{i+1})X\check{R}_{i,i+1}(\xi_i/\xi_{i+1})^{-1},$$
$$\check{R}_{i,i+1}(\zeta) = P_{i,i+1}R_{i,i+1}(\zeta).$$

We use the monodromy matrix

$$T_a(\zeta) = R_{a,n}(\zeta/\xi_n)\cdots R_{a,1}(\zeta/\xi_1).$$

Here and after, the suffix *a* indicates the two-dimensional auxiliary space $V_a \simeq V$.

The total spin operator which commutes with $R_{a,j}(\zeta)$ is given by $\frac{1}{2}\sigma_a^3 + S$,

$$[\tfrac{1}{2}\sigma_a^3 + S, R_{a,j}(\zeta)] = 0.$$

The transfer matrix $\mathbf{t}(\zeta, \alpha)$ acting on $\mathscr{V}$ is defined by using the inverse adjoint action of the monodromy matrix.

$$\mathbf{t}(\zeta, \alpha)(X) = \mathrm{tr}_a(q^{-\alpha\sigma_a^3}T_a(\zeta)^{-1}XT_a(\zeta)) \quad (X \in \mathscr{V}).$$

Here $\mathrm{tr}_a = \mathrm{tr}_{V_a}$ stands for the trace over V_a, which gives a functional

$$\mathrm{tr}_a(q^{-\alpha\sigma_a^3} \cdot) : \mathrm{End}(V_a) \to \mathbb{C}q^\alpha \oplus \mathbb{C}q^{-\alpha}.$$

We have

$$\mathbf{s}_i \cdot \mathbf{t}(\zeta, \alpha) = \mathbf{t}(\zeta, \alpha) \cdot \mathbf{s}_i, \ \mathbf{j} \cdot \mathbf{t}(\zeta, \alpha) = \mathbf{t}(\zeta, \alpha) \cdot \mathbf{j}.$$

2.3 Oscillator algebra

Following [2], we define another kind of monodromy matrices. They have the q oscillator algebra *Osc* as auxiliary spaces. The oscillator algebra is generated by $\mathbf{a}, \mathbf{a}^*, q^{\pm D}$ with the relations

$$q^D \mathbf{a} q^{-D} = q^{-1}\mathbf{a}, \; q^D \mathbf{a}^* q^{-D} = q\mathbf{a}^*, \; \mathbf{a}\mathbf{a}^* = 1 - q^{2D+2}, \; \mathbf{a}^*\mathbf{a} = 1 - q^{2D}.$$

We use the suffix A indicating the infinite dimensional auxiliary space Osc_A and its generators $\mathbf{a}_A$, $\mathbf{a}_A^*, D_A \in Osc_A$.

We shall use two representations $W^\pm$ of *Osc*,

$$W^+ = \oplus_{k \geq 0}\mathbb{C}|k\rangle, \; W^- = \oplus_{k \leq -1}\mathbb{C}|k\rangle.$$

The action is

$$q^D|k\rangle = q^k|k\rangle, \; \mathbf{a}|k\rangle = (1 - q^{2k})|k-1\rangle, \; \mathbf{a}^*|k\rangle = (1 - \delta_{k,-1})|k+1\rangle.$$

The L operator $L^\pm(\zeta)$ belongs to $Osc \otimes \text{End}(V)$. We have

$$L^+(\zeta) = i\zeta^{-1/2}q^{-1/4}(1 - \zeta\mathbf{a}^*\sigma^+ - \zeta\mathbf{a}\sigma^- - \zeta^2 q^{2D+2}\tau^-)q^{\sigma^3 D},$$
$$L^-(\zeta) = \sigma^1 L^+(\zeta)\sigma^1.$$

Here $\tau^\pm = \sigma^\pm\sigma^\mp$. The inverse of $L^+(\zeta)$ is given by

$$L^+(\zeta)^{-1} = \frac{1}{\zeta - \zeta^{-1}}\overline{L}^+(\zeta),$$
$$\overline{L}^+(\zeta) = i\zeta^{-1/2}q^{1/4} \cdot q^{-\sigma^3 D}(1 + \zeta\mathbf{a}^*\sigma^+ + \zeta\mathbf{a}\sigma^- - \zeta^2 q^{2D}\tau^+).$$

Apart from the power $\zeta^{-1/2}$, the L operator is a quadratic polynomial in ζ, and its inverse has a pole at $\zeta^2 = 1$.

The total spin operator which commutes with the L operator is

$$S^\pm = \mp D_A + S, \; [S^\pm, L_{A,j}^\pm(\zeta)] = 0.$$

This suggests a construction of an operator similar to $\mathbf{t}(\zeta, \alpha)$ using the trace on $W^\pm$, in which $q^{-\alpha\sigma^3}$ is replaced by $q^{\pm 2\alpha D}$. This construction leads to the Q operators acting on $\mathscr{V}$:

$$\mathbf{Q}^\pm(\zeta, \alpha)(X) = \pm(1 - q^{\pm 2(\alpha - \mathbb{S})})\zeta^{\pm(\alpha - \mathbb{S})}\text{Tr}_A^\pm\left(q^{\pm 2\alpha D_A}T_A^\pm(\zeta)^{-1}XT_A^\pm(\zeta)\right), \quad (2.1)$$
$$T_A^\pm(\zeta) = L_{A,n}^\pm(\zeta/\xi_n)\cdots L_{A,1}^\pm(\zeta/\xi_1).$$

Here the trace functional

$$\text{Tr}_A^\pm(q^{\pm 2\alpha D_A} \cdot) : Osc_A \to \mathbb{C}(q^{2\alpha})$$

is defined to be zero on each spin sector $Osc^{(s)} = \{x \in Osc \,|\, [D, x] = sx\}$ with $s \neq 0$, and on the spin zero sector $Osc^{(0)}$ it is defined by

$$\text{Tr}_A^\pm(q^{\pm 2\alpha D_A}q^{mD_A}) = \pm(1 - q^{\pm 2\alpha + m})^{-1}.$$

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Since $T_A^\pm(\zeta)^{-1}XT_A^\pm(\zeta)\big|_{\zeta=0} = q^{\mp2\mathbb{S}D_A}(X)$, we have $\zeta^{\mp(\alpha-\mathbb{S})}\mathbf{Q}^\pm(\zeta,\alpha)\big|_{\zeta=0} = \mathrm{id}_{\mathscr{V}}$. This explains the prefactor $\pm(1-q^{\pm2(\alpha-\mathbb{S})})$ in the definition of $\mathbf{Q}^\pm$.

The trace functional satisfies

$$\mathrm{Tr}^+(q^{2\alpha D}x) = -\mathrm{Tr}^-(q^{2\alpha D}x).$$

From this follows

$$\mathbf{j}\cdot\mathbf{Q}^+(\zeta,\alpha) = \mathbf{Q}^-(\zeta,\alpha)\cdot\mathbf{j}.$$

The Yang-Baxter relation for the *L* operator

$$R_{i,i+1}(\zeta_i/\zeta_{i+1})L_{A,i}(\zeta_i)L_{A,i+1}(\zeta_{i+1}) = L_{A,i+1}(\zeta_{i+1})L_{A,i}(\zeta_i)R_{i,i+1}(\zeta_i/\zeta_{i+1})$$

implies the equivariance.

$$\mathbf{s}_i\cdot\mathbf{Q}^\pm(\zeta,\alpha) = \mathbf{Q}^\pm(\zeta,\alpha)\cdot\mathbf{s}_i.$$

2.4 Baxter's TQ relation

The following proposition is due to [2]. We give here a proof, for the purpose of introducing further notation and formulas.

Proposition 2.1.

$$\mathbf{t}(\zeta,\alpha)\mathbf{Q}^\pm(\zeta,\alpha) = \mathbf{Q}^\pm(q^{-1}\zeta,\alpha) + \mathbf{Q}^\pm(q\zeta,\alpha). \tag{2.2}$$

Proof. Define the triangular transfer matrix

$$T_{\{A,a\}}^+(\zeta) = (G_{A,a}^+)^{-1}T_A^+(\zeta)T_a(\zeta)G_{A,a}^+, \tag{2.3}$$

$$G_{A,a}^+ = q^{-\sigma_a^3 D_A}(1+\mathbf{a}_A^*\sigma_a^+). \tag{2.4}$$

The triangularity is local:

$$L_{\{A,a\},j}^+(\zeta) = (G_{A,a}^+)^{-1}L_{A,j}^+(\zeta)R_{a,j}(\zeta)G_{A,a}^+ \tag{2.5}$$

$$= (q\zeta - q^{-1}\zeta^{-1})\cdot\begin{pmatrix} L_{A,j}^+(q^{-1}\zeta)q^{-\sigma_j^3/2} & 0 \\ \gamma(\zeta)L_{A,j}^+(q\zeta)\sigma_j^+ q^{-2D_A+1/2} & \beta(\zeta)L_{A,j}^+(q\zeta)q^{\sigma_j^3/2} \end{pmatrix}_a.$$

The inverse is given by

$$L_{\{A,a\},j}^+(\zeta)^{-1} = \frac{1}{q\zeta - q^{-1}\zeta^{-1}}$$

$$\times\begin{pmatrix} q^{\sigma_j^3/2}L_{A,j}^+(q^{-1}\zeta)^{-1} & 0 \\ -\gamma(q^{-1}\zeta)\sigma_j^+ q^{-2D_A-1/2}L_{A,j}^+(q^{-1}\zeta)^{-1} & \beta(\zeta)^{-1}q^{-\sigma_j^3/2}L_{A,j}^+(q\zeta)^{-1} \end{pmatrix}_a.$$

Using the commutativity

$$[q^{2\alpha D_A - \alpha\sigma_a^3}, G_{A,a}^+] = 0,$$

we obtain the TQ relation. The effect of the shift $\zeta^{\alpha-\mathbb{S}} \to (q^{\mp1}\zeta)^{\alpha-\mathbb{S}}$ is cancelled by $q^{\mp\sigma_j^3/2}$ in the diagonal elements of $L_{\{A,a\},j}^+(\zeta)^{\pm1}$ and $q^{-\alpha\sigma_a^3}$ in $\mathbf{t}(\zeta,\alpha)$. $\qquad\square$

In the TQ relation, the order of the product $\mathbf{t}(\zeta,\alpha)\mathbf{Q}^{\pm}(\zeta,\alpha)$ can be reversed. We can show

$$\mathbf{Q}^{\pm}(\zeta,\alpha)\mathbf{t}(\zeta,\alpha) = \mathbf{Q}^{\pm}(q\zeta,\alpha) + \mathbf{Q}^{\pm}(q^{-1}\zeta,\alpha).$$

The argument is quite parallel. Set[1]

$$T^{*+}_{\{a,A\}}(\zeta) = (F^{+}_{A,a})^{-1}T_a(\zeta)T^{-}_A(\zeta)F^{+}_{A,a},$$
$$F^{+}_{A,a} = 1 - \mathbf{a}_A\sigma^{+}_a, \quad [q^{-2\alpha D_A - \alpha\sigma^3_a}, F^{+}_{A,a}] = 0.$$

We have

$$L^{*+}_{\{a,A\},j}(\zeta) \overset{\text{def}}{=} (F^{+}_{A,a})^{-1}R_{a,j}(\zeta)L^{-}_{A,j}(\zeta)F^{+}_{A,a}$$
$$= (q\zeta - q^{-1}\zeta^{-1}) \cdot \begin{pmatrix} \beta(\zeta)L^{-}_{A,j}(q\zeta)q^{-\sigma^3_j/2} & 0 \\ \gamma(\zeta)\sigma^{+}_j L^{-}_{A,j}(q\zeta)q^{-\sigma^3_j/2} & L^{-}_{A,j}(q^{-1}\zeta)q^{\sigma^3_j/2} \end{pmatrix}_a$$

and

$$L^{*+}_{\{a,A\},j}(\zeta)^{-1}$$
$$= \frac{1}{q\zeta - q^{-1}\zeta^{-1}} \cdot \begin{pmatrix} \beta(\zeta)^{-1}q^{\sigma^3_j/2}L^{-}_{A,j}(q\zeta)^{-1} & 0 \\ -\gamma(q^{-1}\zeta)q^{-\sigma^3_j/2}L^{-}_{A,j}(q^{-1}\zeta)^{-1}\sigma^{+}_j & q^{-\sigma^3_j/2}L^{-}_{A,j}(q^{-1}\zeta)^{-1} \end{pmatrix}_a.$$

Using these quantities we arrive at

$$\mathbf{Q}^{-}(\zeta,\alpha)\mathbf{t}(\zeta,\alpha) = \mathbf{Q}^{-}(q\zeta,\alpha) + \mathbf{Q}^{-}(q^{-1}\zeta,\alpha).$$

Later on we will also use

$$L^{-}_{\{A,a\},j}(\zeta) = \sigma^1_a\sigma^1_j L^{+}_{\{A,a\},j}(\zeta)\sigma^1_a\sigma^1_j,$$
$$L^{*-}_{\{A,a\},j}(\zeta) = \sigma^1_a\sigma^1_j L^{*+}_{\{A,a\},j}(\zeta)\sigma^1_a\sigma^1_j.$$

2.5 Conjugate transfer matrices

We define conjugate transfer matrices by reversing the operations inside the trace.

$$\mathbf{t}^{*}(\zeta,\alpha)(X) = \text{tr}_a\left(T_a(\zeta)q^{\alpha\sigma^3_a}XT_a(\zeta)^{-1}\right),$$
$$\mathbf{Q}^{*\pm}(\zeta,\alpha)(X) = \pm(1 - q^{\pm 2(\alpha-\mathbb{S})})\zeta^{\pm(\alpha-\mathbb{S})}\text{Tr}^{\pm}_A\left(T^{\mp}_A(\zeta)q^{\pm 2\alpha D_A}XT^{\mp}_A(\zeta)^{-1}\right).$$

They satisfy the same TQ relations

$$\mathbf{t}^{*}(\zeta,\alpha)\mathbf{Q}^{*\pm}(\zeta,\alpha) = \mathbf{Q}^{*\pm}(q^{-1}\zeta,\alpha) + \mathbf{Q}^{*\pm}(q\zeta,\alpha) \tag{2.6}$$

and the equivariance with respect to $\mathbf{s}_i$ and $\mathbf{j}$.

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[1]Notice the signs $\pm$ in $T^{*+}_{\{a,A\}}$ and T^{-}_A; they are *not* misprints.

2.6 Reduction property

Now, we consider the extension of the interval from $[1,n]$ to $[1,n+1]$ or $[0,n]$. We exhibit the dependence on the interval of various quantities by a suffix, and write $\mathscr{V}_{[1,n]}$, $\mathbf{t}_{[1,n]}(\zeta,\alpha)$ and so on. The operator $\mathbf{t}(\zeta,\alpha)$ has the reduction property to the right:

$$\mathbf{t}_{[1,n+1]}(\zeta,\alpha)(X_{[1,n]}\otimes 1) = \mathbf{t}_{[1,n]}(\zeta,\alpha)(X_{[1,n]})\otimes 1.$$

This is tautological because of the inverse adjoint action: in the left hand side, the adjoint action $R^{-1}_{a,n+1}(\zeta,\alpha)XR_{a,n+1}(\zeta,\alpha)$ on the quantum space $n+1$ appears at the innermost place. On the other hand, the operator $\mathbf{t}^*(\zeta,\alpha)$ has the reduction property to the left:

$$\mathbf{t}^*_{[0,n]}(\zeta,\alpha)(q^{\alpha\sigma^3}\otimes X_{[1,n]}) = q^{\alpha\sigma^3}\otimes \mathbf{t}^*_{[1,n]}(\zeta,\alpha)(X_{[1,n]}).$$

This is also tautological because the operation appearing at the innermost place is $R_{a,0}(\zeta,\alpha)XR_{a,0}(\zeta,\alpha)^{-1}$, which commutes with the multiplication by $q^{\alpha(\sigma_a^3+\sigma_0^3)}$.

In [1] were constructed operators which satisfy reduction to both ends. In order to realize such operators, we have to modify the form of the reduction. We do not change the reduction to the right. We change the reduction to the left in the form

$$\mathbf{c}^{\pm}_{[0,n]}(q^{\alpha\sigma^3}\otimes X_{[1,n]}) = q^{(\alpha\mp 1)\sigma^3}\otimes \mathbf{c}^{\pm}_{[1,n]}(X_{[1,n]}).$$

For this purpose we consider the off-diagonal blocks in the monodromy matrices.

2.7 Operators in off-diagonal blocks

We define a q difference operator with respect to the spectral variable

$$\Delta_q(F(\zeta)) = F(q\zeta) - F(q^{-1}\zeta).$$

A function of this form is said to be *q-exact*.

Proposition 2.2. *Set*

$$\begin{aligned}
&\mathbf{k}^{\pm}(\zeta,\alpha)(X) \tag{2.7}\\
&= \pm(1-q^{\pm 2(\alpha-\mathbb{S})})\zeta^{\pm(\alpha-\mathbb{S})}\mathrm{Tr}_A^{\pm}\mathrm{tr}_a\left(q^{\pm 2(\alpha\mp 1)D_A}\sigma_a^{\pm}T_a(\zeta)^{-1}T_A^{\pm}(\zeta)^{-1}XT_A^{\pm}(\zeta)T_a(\zeta)\right)\\
&= \pm(1-q^{\pm 2(\alpha-\mathbb{S})})\zeta^{\pm(\alpha-\mathbb{S})}\mathrm{Tr}_A^{\pm}\mathrm{tr}_a\left(q^{\pm 2\alpha D_A}\sigma_a^{\pm}T_{\{A,a\}}^{\pm}(\zeta)^{-1}XT_{\{A,a\}}^{\pm}(\zeta)\right).
\end{aligned}$$

This operator satisfies the reduction property modulo a q-exact form:

$$\begin{aligned}
&\mathbf{k}^{\pm}_{[1,n]}(\zeta,\alpha)(q^{\alpha\sigma_1^3}X_{[2,n]}) = q^{(\alpha\mp 1)\sigma_1^3}\mathbf{k}^{\pm}_{[2,n]}(\zeta,\alpha)(X_{[2,n]}) \tag{2.8}\\
&+\sigma_1^{\pm}\Delta_q\left(\frac{q-q^{-1}}{\zeta/\xi_1-(\zeta/\xi_1)^{-1}}\mathbf{Q}^{\pm}_{[2,n]}(\zeta,\alpha\mp 1)(X_{[2,n]})\right).
\end{aligned}$$

Proof. We subdivide the quantum space as $\mathscr{V} = \mathscr{V}_1\otimes \mathscr{V}_{[2,n]}$, where $\mathscr{V}_1 = \mathrm{End}(V_1)$ and $\mathscr{V}_{[2,n]} = \mathrm{End}(V_2\otimes\cdots\otimes V_n)$. Let us calculate

$$\begin{aligned}
C &= (1-q^{2(\alpha-\mathbb{S})})^{-1}\zeta^{-(\alpha-\mathbb{S})}\mathbf{k}^+(\zeta,\alpha)(q^{\alpha\sigma_1^3}X_{[2,n]})\\
&= \mathrm{Tr}_A^+\mathrm{tr}_a\left(q^{2(\alpha-1)D_A}\sigma_a^+T_a(\zeta)^{-1}T_A^+(\zeta)^{-1}q^{\alpha\sigma_1^3}X_{[2,n]}T_A^+(\zeta)T_a(\zeta)\right)
\end{aligned}$$

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where $X_{[2,n]} \in \mathscr{V}_{[2,n]}$. Using the commutativity of the L operator and the total spin operator, we have

$$C = q^{(\alpha-1)(\sigma_1^3-1)} \mathrm{Tr}_A^+ \mathrm{tr}_a \big(\sigma_a^+ R_{a,1}(\zeta/\xi_1)^{-1} L_{A,1}^+(\zeta/\xi_1)^{-1} q^{\sigma_1^3} q^{(\alpha-1)(2D_A-\sigma_a^3)}$$
$$\times T_{a,[2,n]}(\zeta)^{-1} T_{A,[2,n]}^+(\zeta)^{-1} X_{[2,n]} T_{A,[2,n]}^+(\zeta) T_{a,[2,n]}(\zeta) L_{A,1}^+(\zeta/\xi_1) R_{a,1}(\zeta/\xi_1) \big)$$

Note that $\left(G_{A,a}^+ \right)^{-1} \sigma_a^+ G_{A,a}^+ = \sigma_a^+ q^{2D_A}$. Using (2.5) we rewrite as

$$C = q^{(\alpha-1)(\sigma_1^3-1)} \mathrm{Tr}_A^+ \mathrm{tr}_a \big(\sigma_a^+ q^{2D_A} L_{\{A,a\},1}^+(\zeta/\xi_1)^{-1} q^{\sigma_1^3} q^{(\alpha-1)(2D_A-\sigma_a^3)}$$
$$\times T_{\{A,a\},[2,n]}^+(\zeta)^{-1} X_{[2,n]} T_{\{A,a\},[2,n]}^+(\zeta) L_{\{A,a\},1}^+(\zeta/\xi_1) \big) \,.$$

Collecting the terms containing operators acting on $\mathscr{V}_1$, we obtain

$$C = q^{(\alpha-1)(\sigma_1^3-1)} \mathrm{Tr}_A^+ \mathrm{tr}_a \big(\{ L_{\{A,a\},1}^+(\zeta/\xi_1)^{t_1} \sigma_a^+ q^{2D_A} (L_{\{A,a\},1}^+(\zeta/\xi_1)^{-1} q^{\sigma_1^3})^{t_1} \}^{t_1}$$
$$\times q^{(\alpha-1)(2D_A-\sigma_a^3)} T_{\{A,a\},[2,n]}^+(\zeta)^{-1} X_{[2,n]} T_{\{A,a\},[2,n]}^+(\zeta) \big) \,.$$

The part $\{ \cdots \}^{t_1}$ reads as

$$\{ \cdots \}^{t_1} = q^{2D_A} \sigma_a^+ + \frac{q-q^{-1}}{q\zeta/\xi_1 - (q\zeta/\xi_1)^{-1}} \sigma_1^+ \tau_a^- - \frac{q-q^{-1}}{q^{-1}\zeta/\xi_1 - q(\zeta/\xi_1)^{-1}} \sigma_1^+ \tau_a^+ \,.$$

Using (2.1) we obtain (2.8). The term $q^{2D_A} \sigma_a^+$ creates the main term and the rest creates the q-exact term. The presence of the factor $(1 - q^{\alpha-\mathbb{S}})$ in $\mathbf{k}^+$ and $\mathbf{Q}^+$ does not effect the form of the equality because $\mathbb{S}_{[1,n]}(\sigma_1^+ X_{[2,n]}) = \sigma_1^+ (\mathbb{S}_{[2,n]} + 1)(X_{[2,n]})$. On the other hand, the factor $\zeta^{\alpha-\mathbb{S}}$ matters because it does not commute with the q difference operator. It adjusts the effect of $\tau_a^\pm \sigma_a^3 = \pm \tau_a^\pm$ in the factor $q^{(\alpha-1)(2D_A-\sigma_a^3)}$. $\qquad\square$

The operators $\mathbf{k}^\pm(\zeta,\alpha)$ enjoy the non-trivial reduction property to the left with a q-exact form as remainder term. In the next subsection we define the operators $\mathbf{c}_k^\pm(\zeta,\alpha)$ by taking the singular part at $\zeta = \xi_k$. The operator $\mathbf{Q}_{[2,n]}^\pm(\zeta,\alpha \mp 1)$ has no pole at $\zeta = q^{\pm 1}\xi_k$. Therefore, the q-exact term has no pole at $\zeta = \xi_k$. It implies that the operators $\mathbf{c}_k^\pm(\zeta,\alpha)$ satisfy the reduction property without remainder terms.

Next we define operators with non-trivial reduction to the right. We demand the tautological reduction to the left in the form

$$\mathbf{k}_{[1,n]}^{*\pm}(q^{(\alpha\pm1)\sigma_1^3} X_{[2,n]}) = q^{\alpha\sigma_1^3} \mathbf{k}_{[2,n]}^{*\pm}(X_{[2,n]}).$$

This is achieved by defining

$$\mathbf{k}^{*\pm}(\zeta,\alpha)(X) = \pm(1 - q^{\pm2(\alpha\pm1-\mathbb{S})}) \zeta^{\pm(\alpha\pm1-\mathbb{S})}$$
$$\times \mathrm{Tr}_A^\pm \mathrm{tr}_a \left(\sigma_a^\pm T_a(\zeta) T_A^\mp(\zeta) q^{\alpha(\pm2D_A+\sigma_a^3)\mp2S} X T_A^\mp(\zeta)^{-1} T_a(\zeta)^{-1} \right) \,.$$

In this definition, we make the shift $q^{\mp2S}X$ because we want to have the reduction to the left as above. As for the reduction to the right we have

Proposition 2.3.

$$\mathbf{k}_{[1,n]}^{*\pm}(\zeta,\alpha)(X_{[1,n-1]}) = \mathbf{k}_{[1,n-1]}^{*\pm}(\zeta,\alpha)(X_{[1,n-1]})$$
$$+\sigma_n^{\pm}\Delta_q\left(\frac{q-q^{-1}}{\zeta/\xi_n-(\zeta/\xi_n)^{-1}}\mathbf{Q}_{[1,n-1]}^{*\pm}(\zeta,\alpha)(q^{\mp2S_{[1,n-1]}}X_{[1,n-1]})\right).$$

Proof. Let us calculate

$$\mathrm{Tr}_A^+\mathrm{tr}_a\left(\sigma_a^+T_a(\zeta)T_A^-(\zeta)q^{\alpha(2D_A+\sigma_a^3)-2S_{[1,n]}}X_{[1,n-1]}T_A^-(\zeta)^{-1}T_a(\zeta)^{-1}\right)$$
$$=\mathrm{Tr}_A^+\mathrm{tr}_a\left(\sigma_a^+T_{\{a,A\}}^{*+}(\zeta)q^{\alpha(2D_A+\sigma_a^3)-2S_{[1,n]}}X_{[1,n-1]}T_{\{a,A\}}^{*+}(\zeta)^{-1}\right)$$
$$=\mathrm{Tr}_A^+\mathrm{tr}_a\Big(\Big\{(q^{-\sigma_n^3}L_{\{a,A\},n}^{*+}(\zeta/\xi_n)^{-1})^{t_n}\sigma_a^+L_{\{a,A\},n}^{*+}(\zeta/\xi_n)^{t_n}\Big\}^{t_n}$$
$$\times T_{\{a,A\},[1,n-1]}^{*+}(\zeta)q^{\alpha(2D_A+\sigma_a^3)-2S_{[1,n-1]}}X_{[1,n-1]}T_{\{a,A\},[1,n-1]}^{*+}(\zeta)^{-1}\Big).$$

The statement of the proposition follows from

$$\Big\{(q^{-\sigma_n^3}L_{\{a,A\},n}^{*+}(\zeta)^{-1})^{t_n}\sigma_a^+L_{\{a,A\},n}^{*+}(\zeta)^{t_n}\Big\}^{t_n}$$
$$=\sigma_a^+ + \frac{q-q^{-1}}{q\zeta-(q\zeta)^{-1}}\sigma_n^+\tau_a^+ - \frac{q-q^{-1}}{q^{-1}\zeta-q\zeta^{-1}}\sigma_n^+\tau_a^-. \qquad \square$$

2.8 Operators b and c

The operator $\mathbf{t}(\zeta,\alpha)$ does not have a pole at $\zeta^2=\xi_j^2$, but has a pole at $\zeta^2=q^{\pm2}\xi_j^2$. The operators $\mathbf{Q}^{\pm}(\zeta,\alpha)$ do not have a pole at $\zeta^2=q^{\pm2}\xi_j^2$, but have a pole at $\zeta^2=\xi_j^2$. These are consequences of the pole structures of $R_{a,j}(\zeta)^{\pm1}$ and $L_{A,j}(\zeta)^{\pm1}$. Consider the matrix $T_{\{A,a\}}^{\pm}(\zeta)^{-1}XT_{\{A,a\}}^{\pm}(\zeta)$, which is triangular in the auxiliary space V_a. The diagonal part does not have a pole at $\zeta^2=\xi_j^2$ because poles cancel in the adjoint action. This cancellation breaks down for the off-diagonal elements. The operators $\mathbf{c}^{\pm}(\zeta,\alpha)$ do have a simple pole at $\zeta^2=\xi_j^2$.

Set

$$\mathbf{c}^-(\zeta,\alpha) = \mathbf{b}(\zeta,\alpha) = (1-q^{2(\alpha-\mathbb{S})})^{-1}\mathrm{sing}\,\mathbf{k}^-(\zeta,\alpha),$$
$$\mathbf{c}^+(\zeta,\alpha) = \mathbf{c}(\zeta,\alpha) = q^{\alpha-\mathbb{S}}\mathrm{sing}\,\mathbf{k}^+(\zeta,\alpha).$$

The symbol sing means taking the singular part at $\zeta=\xi_i$ ($i=1,\dots,n$). Therefore, we have a partial fraction

$$\mathbf{c}^{\pm}(\zeta,\alpha) = -\frac{1}{2}\sum_{k=1}^n\frac{\xi_k}{\zeta-\xi_k}\mathbf{c}_k^{\pm}(\alpha).$$

The q dependent normalization factors in the definition do not alter the reduction properties. Taking the singular part in (2.8) we obtain

Proposition 2.4. *We have*

$$\mathbf{c}_{[1,n]}^{\pm}(\zeta,\alpha)(X_{[1,n-1]}) = \mathbf{c}_{[1,n-1]}^{\pm}(\zeta,\alpha)(X_{[1,n-1]}),$$
$$\mathbf{c}_{[1,n]}^{\pm}(\zeta,\alpha)(q^{\alpha\sigma_1^3}X_{[2,n]}) = q^{(\alpha\mp1)\sigma_1^3}\mathbf{c}_{[2,n]}^{\pm}(\zeta,\alpha)(X_{[2,n]}).$$

The normalization factors are so chosen that we have the Grassmann relation [2].

Proposition 2.5.

$$\mathbf{c}^{\varepsilon_1}(\zeta_1, \alpha - \varepsilon_2)\mathbf{c}^{\varepsilon_2}(\zeta_2, \alpha) = -\mathbf{c}^{\varepsilon_2}(\zeta_2, \alpha - \varepsilon_1)\mathbf{c}^{\varepsilon_1}(\zeta_1, \alpha).$$

The proof is given in Appendix B.

It follows from Proposition 2.4 that the residue $\mathbf{c}_n^{\pm}(\alpha)$ of $\mathbf{c}_{[1,n]}^{\pm}(\zeta, \alpha)$ taken at the right end of the interval $\zeta = \xi_n$ annihilate states of the form $X_{[1,n-1]}$. This operator is given explicitly as follows.

Proposition 2.6. *Let $X \in \mathcal{V}_{[1,n]}$. Then we have*

$$\mathbf{c}_n^+(\alpha)(X) = \frac{(1 - q^{2(\alpha-\mathbb{S})})(q\xi_n)^{\alpha-\mathbb{S}}}{\prod_{j=1}^{n-1} \beta(\xi_n/\xi_j)} \tilde{\mathbf{Q}}^+(\xi_n, \alpha)\left((\mathbf{a}, 1)_n X \begin{pmatrix} 1 \\ -\mathbf{a} \end{pmatrix}_n\right),$$

$$\mathbf{c}_n^-(\alpha)(X) = \frac{(q^2\xi_n)^{-(\alpha-\mathbb{S})}}{\prod_{j=1}^{n-1} \beta(\xi_n/\xi_j)} \tilde{\mathbf{Q}}^-(\xi_n, \alpha)\left((1, \mathbf{a})_n X \begin{pmatrix} -\mathbf{a} \\ 1 \end{pmatrix}_n\right)$$

where for $\tilde{X} \in Osc \otimes \mathcal{V}_{[1,n-1]}$ we set

$$\tilde{\mathbf{Q}}^{\pm}(\zeta, \alpha)(\tilde{X}) = q^{\mp S_{[1,n-1]}}\mathrm{Tr}_A^{\pm}\left(q^{\pm 2\alpha D_A}T_{A,[1,n-1]}^{\pm}(q\zeta)^{-1}\tilde{X}\, T_{A,[1,n-1]}^{\pm}(q^{-1}\zeta)\right)q^{\mp S_{[1,n-1]}}.$$

In particular, we have $\mathbf{c}_n^{\pm}(\alpha)(X) \in \mathcal{V}_{[1,n-1]}$.

Proof. Consider $\mathbf{k}^{\pm}(\zeta, \alpha)$ given by (2.7). Because of the triangular structure of $L_{\{A,a\},j}^{\pm}(\zeta)$, $\mathrm{tr}_a\left(\sigma_a^{\pm}T_{\{A,a\}}^{\pm}(\zeta)^{-1}(X)T_{\{A,a\}}^{\pm}(\zeta)\right)$ consists of $2n$ terms. The singular part of $\mathbf{k}^{\pm}(\zeta, \alpha)$ at $\zeta = \xi_n$ comes only from the factor $\gamma(q^{-1}\zeta/\xi_n)$ and $\beta(\zeta/\xi_n)^{-1}$ in $L_{\{A,a\},n}^{\pm}(\zeta/\xi_n)^{-1}$. Note also that $\beta(\zeta/\xi_n)$ is zero at $\zeta = \xi_n$. From these properties, it follows that only two terms have singularities at $\zeta = \xi_n$. Summing them up, we obtain the above expressions. $\qquad\square$

To simplify the notation, let us use the following convention. For each $\beta \in \alpha + \mathbb{Z}$ take a copy $\mathcal{V}_\beta \simeq \mathcal{V}$, and define an operator α acting on $\mathcal{V}_{[\alpha]} = \oplus_{\beta \in \alpha + \mathbb{Z}}\mathcal{V}_\beta$ by

$$\alpha|_{\mathcal{V}_\beta^{(s)}} = \beta\, \mathrm{id}_{\mathcal{V}_\beta^{(s)}}.$$

Let $I \subset \mathbb{Z}$ be a finite interval. We consider the quantum space $\mathcal{V}_{[\alpha],I}$ by changing $[1,n]$ to I. If $J = [j_1, j_2]$ ($i_1 \le j_1 \le j_2 \le i_2$) is a subinterval of $I = [i_1, i_2]$, then $\mathcal{V}_{[\alpha],J}$ is naturally embedded in $\mathcal{V}_{[\alpha],I}$:

$$\iota_{J,I} : \mathcal{V}_{\beta,J} \to \mathcal{V}_{\beta,I}, \quad X_{[j_1,j_2]} \mapsto q^{\sum_{j=i_1}^{j_1-1} \beta\sigma_j^3} X_{[j_1,j_2]}.$$

We often drop $\iota_{J,I}$ when the meaning is clear without writing it explicitly.

---

[2]One can choose the normalization so that the equivariance $\mathbf{c}^+ \leftrightarrow \mathbf{c}^-$ with respect to the spin reversal is valid. We do not exploit this possibility because we will not use this property, and also because it would require introduction of a square root.

In this notation, the basic properties of the operators $\mathbf{c}^+(\zeta) = \mathbf{c}(\zeta)$ and $\mathbf{c}^-(\zeta) = \mathbf{b}(\zeta)$ are summarized as follows.

$$\mathbf{c}^{\varepsilon_1}(\zeta_1)\mathbf{c}^{\varepsilon_2}(\zeta_2) = -\mathbf{c}^{\varepsilon_2}(\zeta_2)\mathbf{c}^{\varepsilon_1}(\zeta_1), \tag{2.9}$$

$$\mathbf{s}_i\mathbf{c}^{\pm}(\zeta) = \mathbf{c}^{\pm}(\zeta)\mathbf{s}_i, \tag{2.10}$$

$$\mathbf{c}^{\pm}_{[1,n]}(\zeta)(X_{[1,n-1]}) = \mathbf{c}^{\pm}_{[1,n-1]}(\zeta)(X_{[1,n-1]}), \tag{2.11}$$

$$\mathbf{c}^{\pm}_{[1,n]}(\zeta)(q^{\alpha\sigma^3_1}X_{[2,n]}) = q^{\alpha\sigma^3_1}\mathbf{c}^{\pm}_{[2,n]}(\zeta)(X_{[2,n]}). \tag{2.12}$$

3. Free Fermion Point

In this section, we discuss the free fermion case $q = i$. The main point is that the operators $\mathbf{b}_k$ and $\mathbf{c}_k$ are cubic in appropriate free fermions (see Theorem 3.6 below).

3.1 Jordan-Wigner transformation in adjoint action

We continue to work with $\mathscr{V} = \mathrm{End}(V_1 \otimes \cdots \otimes V_n)$. Define the Jordan Wigner transformation

$$\psi_j^{\pm} = \sigma_j^{\pm} i^{\mp\Sigma_{l=1}^{j-1}\sigma_l^3} \in \mathscr{V}.$$

We define fermions[3] acting on $X \in \mathscr{V}^{(s)}$.

$$\Psi_j^{\pm}(X) = \psi_j^{\pm}X - (-1)^s X\psi_j^{\pm},$$

$$\Phi_j^{\pm}(\alpha)(X) = \frac{1}{1+i^{\mp 2(\alpha-s)}}(\psi_j^{\pm}X + i^{\mp 2\alpha}X\psi_j^{\pm}).$$

Define further the operator $\Psi_j^{\pm}$ (resp. $\Phi_j^{\pm}$) on $\mathscr{V}_{[\alpha]}$ whose restriction to $V_\beta^{(s)}$ (resp. $V_{\beta\pm 1}^{(s)}$) is given by

$$\Psi_j^{\pm} : \mathscr{V}_\beta^{(s)} \to \mathscr{V}_{\beta\mp 1}^{(s\pm 1)}, \quad \Phi_j^{\pm}(\beta) : \mathscr{V}_{\beta\pm 1}^{(s)} \to \mathscr{V}_\beta^{(s\pm 1)}.$$

They satisfy the canonical anti-commutation relations

$$[\Psi_j^{\varepsilon}, \Psi_{j'}^{\varepsilon'}]_+ = 0, \quad [\Phi_j^{\varepsilon}, \Phi_{j'}^{\varepsilon'}]_+ = 0, \quad [\Psi_j^{\varepsilon}, \Phi_{j'}^{\varepsilon'}]_+ = \delta_{\varepsilon+\varepsilon',0}\delta_{j,j'}.$$

When $q = i$, there is certain redundancy in considering $\mathscr{V}_\beta$ for all $\beta \in \mathbb{Z} + \alpha$, since we have the periodicity $\Phi_k^{\pm}(\alpha+2) = \Phi_k^{\pm}(\alpha)$, $\mathbf{t}(\zeta,\alpha+4) = \mathbf{t}(\zeta,\alpha)$. Nevertheless, we choose to retain this convention in order to keep contact with the generic case. Also $\mathbf{Q}^{\pm}(\zeta,\alpha)$ is not quite periodic due to the overall power $\zeta^{\pm(\alpha-\mathbb{S})}$.

Proposition 3.1. *The fermions $\Psi_{j,[1,n]}^{\pm}$ and $\Phi_{j,[1,n]}^{\pm}$ satisfy the reduction property to the both ends except for $j = 1, n$.*

$$\Psi_{j,[1,n]}^{\pm}(X_{[1,n-1]}) = (1-\delta_{j,n})\Psi_{j,[1,n-1]}^{\pm}(X_{[1,n-1]}), \tag{3.1}$$

$$\Psi_{j,[1,n]}^{\pm}(i^{\alpha\sigma^3_1}X_{[2,n]}) = i^{\alpha\sigma^3_1}\Psi_{j,[2,n]}^{\pm}(X_{[2,n]}) \ (j \neq 1), \tag{3.2}$$

$$\Phi_{j,[1,n]}^{\pm}(X_{[1,n-1]}) = \Phi_{j,[1,n-1]}^{\pm}(X_{[1,n-1]}) \ (j \neq n), \tag{3.3}$$

$$\Phi_{j,[1,n]}^{\pm}(i^{\alpha\sigma^3_1}X_{[2,n]}) = (1-\delta_{j,1})i^{\alpha\sigma^3_1}\Phi_{j,[2,n]}^{\pm}(X_{[2,n]}). \tag{3.4}$$

---

[3]For $X \in \mathscr{V}^{(s)}$, the operator $\Phi_j^{\pm}(\alpha)$ in this paper corresponds to $\Phi_{\alpha-s+1,j}^{\pm}$ in [1].

Proof. The reduction to the right (3.1), (3.3) except for $j = n$ is tautological. The case $j = n$ for (3.1) goes

$$\psi_n^+ X_{[1,n-1]} - (-1)^s X_{[1,n-1]} \psi_n^+ = i^{-2S_{[1,n-1]}} \sigma_n^+ X_{[1,n-1]} - (-1)^s X_{[1,n-1]} i^{-2S_{[1,n-1]}} \sigma_n^+ \,.$$

Since $(-1)^{S_{[1,n-1]}} X_{[1,n-1]} = (-1)^s X_{[1,n-1]} (-1)^{S_{[1,n-1]}}$, this is zero. The reduction to the left (3.2),(3.4) except for $j = 1$ follows from $\psi_{j,[1,n]}^{\pm} i^{(\alpha \pm 1)\sigma_1^3} = \psi_{j,[2,n]}^{\pm} i^{\alpha \sigma_1^3}$. The case $j = 1$ goes

$$\sigma_1^{\pm} i^{(\alpha \pm 1)\sigma_1^3} X_{[2,n]} + i^{\mp 2\alpha} i^{(\alpha \pm 1)\sigma_1^3} X_{[2,n]} \sigma_1^{\pm} = \sigma_1^+ i^{-1 \mp \alpha} X_{[2,n]} + i^{\mp 2\alpha} i^{1 \pm \alpha} X_{[2,n]} \sigma_1^{\pm} = 0 \,.$$

$\square$

The free fermions $\Psi_j^{\pm}, \Phi_j^{\pm}$ do not satisfy the equivariance with respect to the symmetric group action. In the next subsection, we transform them to those which enjoy the equivariance.

3.2 Equivariant fermions

Set

$$\widehat{\Psi}^{\pm}(\zeta) = \sum_{j=1}^{n} \Psi_j^{\pm} \frac{i^j \gamma(\zeta/\xi_j)}{\prod_{l=1}^{j} \beta(\zeta/\xi_l)}, \ \widehat{\Phi}^{\pm}(\zeta) = \sum_{j=1}^{n} \Phi_j^{\pm} \frac{i^j \gamma(\zeta/\xi_j)}{\prod_{l=1}^{j} \beta(\zeta/\xi_l)}.$$

Proposition 3.2. *The operators $\widehat{\Psi}^{\pm}(\zeta), \widehat{\Phi}^{\pm}(\zeta)$ commute with the action of the symmetric group* $\mathfrak{S}_n$,

$$\mathbf{s}_i \widehat{\Psi}^{\pm}(\zeta) = \widehat{\Psi}^{\pm}(\zeta)\mathbf{s}_i, \ \mathbf{s}_i \widehat{\Phi}^{\pm}(\zeta) = \widehat{\Phi}^{\pm}(\zeta)\mathbf{s}_i.$$

Proof. The statement follows from the identity

$$(\mathbf{s}_1 - 1)\left(\sigma_1^+ \frac{\gamma(\zeta/\xi_1)}{\beta(\zeta/\xi_1)} + \sigma_2^+ \sigma_1^3 \frac{\gamma(\zeta/\xi_2)}{\beta(\zeta/\xi_1)\beta(\zeta/\xi_2)}\right) = 0.$$

$\square$

Set

$$\widehat{\Psi}_k^{\pm} = -\operatorname{res}_{\zeta = \xi_k} \widehat{\Psi}^{\pm}(\zeta)\frac{d\zeta}{\zeta}, \quad \widehat{\Phi}_k^{\pm} = -i\widehat{\Phi}^{\pm}(i\xi_k). \tag{3.5}$$

Proposition 3.2 implies the equivariance

$$\mathbf{s}_i \widehat{\Psi}_k^{\pm} = \widehat{\Psi}_{s_i(k)}^{\pm} \mathbf{s}_i, \quad \mathbf{s}_i \widehat{\Phi}_k^{\pm} = \widehat{\Phi}_{s_i(k)}^{\pm} \mathbf{s}_i, \tag{3.6}$$

where $s_i = (i, i+1)$ is the transposition. They are upper or lower triangular linear combinations of $\Psi_j^{\pm}$ or $\Phi_j^{\pm}$.

$$\widehat{\Psi}_k^{\pm} = \sum_{j=k}^{n} \Psi_j^{\pm} \frac{i^{j-1}\gamma(\xi_k/\xi_j)}{\prod_{\substack{1 \le l \le j \\ l(\ne k)}} \beta(\xi_k/\xi_l)},$$

$$\widehat{\Phi}_k^{\pm} = \sum_{j=1}^{k} \Phi_j^{\pm} i^{-j+1} \gamma(\xi_k/\xi_j) \prod_{l=1}^{j-1} \beta(\xi_k/\xi_l).$$

Proposition 3.3. *The fermions $\widehat{\Psi}_k^\pm, \widehat{\Phi}_k^\pm$ satisfy the canonical anti-commutation relations.*

$$[\widehat{\Psi}_j^\varepsilon, \widehat{\Psi}_{j'}^{\varepsilon'}]_+ = 0, \ [\widehat{\Phi}_j^\varepsilon, \widehat{\Phi}_{j'}^{\varepsilon'}]_+ = 0, \ [\widehat{\Psi}_j^\varepsilon, \widehat{\Phi}_{j'}^{\varepsilon'}]_+ = \delta_{\varepsilon+\varepsilon',0}\delta_{j,j'}.$$

Proof. Define an $n \times n$ matrix $P = (P_{jk})$ by

$$P_{jk} = \begin{cases} \dfrac{2\xi_j/\xi_k}{1-\xi_j^2/\xi_k^2} \displaystyle\prod_{l=j+1}^{k-1} \dfrac{1+\xi_j^2/\xi_l^2}{1-\xi_j^2/\xi_l^2} & \text{if } j < k; \\ 1 & \text{if } j = k; \\ 0 & \text{if } j > k. \end{cases} \tag{3.7}$$

The inverse is of a similar form,

$$(P^{-1})_{jk} = \begin{cases} \dfrac{2\xi_k/\xi_j}{1-\xi_k^2/\xi_j^2} \displaystyle\prod_{l=j+1}^{k-1} \dfrac{1+\xi_k^2/\xi_l^2}{1-\xi_k^2/\xi_l^2} & \text{if } j < k; \\ 1 & \text{if } j = k; \\ 0 & \text{if } j > k. \end{cases}$$

The statement follows from the equalities

$$\widehat{\Psi}_k^\pm = \prod_{l=1}^{k-1} \frac{1+\xi_k^2/\xi_l^2}{1-\xi_k^2/\xi_l^2} \sum_{j=1}^n \Psi_j^\pm P_{kj}, \quad \widehat{\Phi}_k^\pm = \prod_{l=1}^{k-1} \frac{1-\xi_k^2/\xi_l^2}{1+\xi_k^2/\xi_l^2} \sum_{j=1}^n \Phi_j^\pm (P^{-1})_{jk}.$$

$\square$

Let $N_k^\pm = \widehat{\Phi}_k^\pm \widehat{\Psi}_k^\mp$ be the number operators corresponding to the fermion creation operators $\widehat{\Phi}_k^\pm$ and the fermion annihilation operators $\widehat{\Psi}_k^\mp$. They satisfy the following reduction properties.

$$N_{j,[1,n]}^\pm(X_{[1,n-1]}) = (1-\delta_{j,n})N_{j,[1,n-1]}^\pm(X_{[1,n-1]}),$$

$$N_{j,[1,n]}^\pm(i^{\alpha\sigma_1^3}X_{[2,n]}) = \begin{cases} i^{\alpha\sigma_1^3}X_{[2,n]} & \text{if } j = 1; \\ i^{\alpha\sigma_1^3}N_{j,[2,n]}^\pm(X_{[2,n]}) & \text{if } j \geq 2. \end{cases}$$

3.3 Cubic form in fermions

We introduce another set of free fermions in which the operators $\mathbf{b}_k, \mathbf{c}_k$ are written as cubic monomials. Set

$$U_k^\pm = \prod_{l \neq k} \frac{(1-\xi_k^2/\xi_l^2)^{N_l^\pm}}{(1+\xi_k^2/\xi_l^2)^{N_l^\mp}},$$

$$\chi_k^+ = \widehat{\Psi}_k^+ U_k^+ \xi_k^{\alpha-\mathbb{S}-1} i^\alpha,$$

$$\chi_k^- = \widehat{\Psi}_k^- U_k^- \xi_k^{-\alpha+\mathbb{S}-1}(-1)^{N_k^-} \frac{i^{-\mathbb{S}+1}}{1+i^{2(\alpha-\mathbb{S})}},$$

$$\chi_k^{*+} = i^{\mathbb{S}-1}(1+i^{2(\alpha-\mathbb{S})})(-1)^{N_k^-}\xi_k^{\alpha-\mathbb{S}+1}(U_k^-)^{-1}\widehat{\Phi}_k^+,$$

$$\chi_k^{*-} = i^{-\alpha}\xi_k^{-\alpha+\mathbb{S}+1}(U_k^+)^{-1}\widehat{\Phi}_k^-.$$

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Proposition 3.4. *The operators $\chi_k^{\pm}, \chi_k^{*\pm}$ satisfy the equivariance with respect to the symmetric group action*

$$\mathbf{s}_i \chi_k^{\pm} = \chi_{s_i(k)}^{\pm} \mathbf{s}_i, \ \mathbf{s}_i \chi_k^{*\pm} = \chi_{s_i(k)}^{*\pm} \mathbf{s}_i, \tag{3.8}$$

the canonical anti-commutation relations

$$[\chi_k^{\varepsilon}, \chi_{k'}^{\varepsilon'}]_+ = 0, [\chi_k^{*\varepsilon}, \chi_{k'}^{*\varepsilon'}]_+ = 0, [\chi_k^{\varepsilon}, \chi_{k'}^{*\varepsilon'}]_+ = \delta_{\varepsilon+\varepsilon',0}\delta_{k,k'}, \tag{3.9}$$

and the reduction relations

$$\chi_{k,[1,n+1]}(X_{[1,n]}) = \chi_{k,[1,n]}(X_{[1,n]}) \ (1 \leq k \leq n), \tag{3.10}$$

$$\chi_{k,[0,n]}(q^{\alpha\sigma_0^3}X_{[1,n]}) = q^{\alpha\sigma_0^3}\chi_{k,[1,n]}(X_{[1,n]}) \ (1 \leq k \leq n), \tag{3.11}$$

where $\chi = \chi^{\pm}, \chi^{*\pm}$.

Proof. The equivariance (3.8) follows from (3.6). The anti-commutation relation (3.9) follows from Proposition 3.3. The reduction (3.11) follows from Proposition 3.1. In particular, we use $\Psi_n^{\pm}(X_{[1,n-1]}) = 0$ and $\Phi_1^{\pm}(i^{\alpha\sigma_1^3}X_{[2,n]}) = 0$. $\qquad \square$

Set $N_k^{\pm}(\alpha) = N_k^{\pm}|_{\mathcal{V}_\alpha}$. The following formulas are proved in Appendix B.

Theorem 3.5. *The transfer matrix and the Q operators are diagonalized by the fermion operators* (3.5).

$$\mathbf{t}(\zeta,\alpha) = (i^{\alpha-\mathbb{S}} + i^{-\alpha+\mathbb{S}})\prod_{k=1}^{n}\left(\frac{1-\zeta^2/\xi_k^2}{1+\zeta^2/\xi_k^2}\right)^{N_k^+(\alpha+1)+N_k^-(\alpha+1)}, \tag{3.12}$$

$$\zeta^{\mp(\alpha-\mathbb{S})}\mathbf{Q}^{\pm}(\zeta,\alpha) = \prod_{k=1}^{n}\frac{(1+\zeta^2/\xi_k^2)^{N_k^{\pm}(\alpha+1)}}{(1-\zeta^2/\xi_k^2)^{N_k^{\mp}(\alpha+1)}}, \tag{3.13}$$

$$\mathbf{t}^*(\zeta,\alpha)(X) = i^{2S}(i^{\alpha-\mathbb{S}} + i^{-\alpha+\mathbb{S}})\prod_{k=1}^{n}\left(\frac{1-\zeta^2/\xi_k^2}{1+\zeta^2/\xi_k^2}\right)^{2-N_k^+(\alpha+1)-N_k^-(\alpha+1)}(i^{-2S}X), \tag{3.14}$$

$$\zeta^{\mp(\alpha-\mathbb{S})}\mathbf{Q}^{*\pm}(\zeta,\alpha)(X) = i^{2S}\prod_{k=1}^{n}\frac{(1+\zeta^2/\xi_k^2)^{1-N_k^{\mp}(\alpha+1)}}{(1-\zeta^2/\xi_k^2)^{1-N_k^{\pm}(\alpha+1)}}(i^{-2S}X). \tag{3.15}$$

Theorem 3.6. *The operator $\mathbf{c}_k^{\pm}$ is a cubic monomial in the free fermions.*

$$\mathbf{c}_k^{\pm} = \chi_k^{\pm}\chi_k^{\mp}\chi_k^{*\pm}.$$

Theorem 3.6 shows that, though $\mathbf{c}_k^{\pm}$ themselves are not fermions, they act in a very simple manner on a basis created by fermions. For later reference, let us record the formula for this basis. Set $\chi_k^{*0} = 1$, $\chi_k^{*\bar{0}} = \chi_k^{*+}\chi_k^{*-}$. Define a basis $\{\chi_p^{(\beta)}\}$ of the space $\mathcal{V}_\beta$ indexed by sequences $p = (p_1, \ldots, p_n)$ with $p_j \in \{+, -, 0, \bar{0}\}$ by

$$\chi_p^{(\beta)} = \varepsilon(p)\chi_n^{*p_n}\cdots\chi_1^{*p_1}(\mathrm{id}_{\beta+\sum_{j=1}^n s(p_j)}),$$

where $\mathrm{id}_\beta \in \mathcal{V}_\beta$ is the identity operator, $s(\pm) = \pm 1$, $s(0) = s(\bar{0}) = 0$,

$$\varepsilon(p) = (-1)^{\sharp\{(i,j)|i<j, \ p_i=-,p_j=+\}}, \tag{3.16}$$

and $\sharp(Z)$ the cardinality of a set Z.

Corollary 3.7. *The action of* $\mathbf{b}_k, \mathbf{c}_k$ *is given by*

$$\mathbf{b}_k(\chi_p^{(\beta)}) = \delta_{p_k,+}\varepsilon^+(p,k)\chi_{p[k]}^{(\beta+1)},$$

$$\mathbf{c}_k(\chi_p^{(\beta)}) = \delta_{p_k,-}\varepsilon^-(p,k)\chi_{p[k]}^{(\beta-1)},$$

where

$$\varepsilon^+(p,k) = (-1)^{\sharp(\{i|p_i=-\})+\sharp(\{i|i>k,\ p_i=+\})}, \tag{3.17}$$

$$\varepsilon^-(p,k) = (-1)^{\sharp(\{i|i>k,\ p_i=-\})}, \tag{3.18}$$

$$p[k]_j = \begin{cases} 0 & \text{if } j=k; \\ p_j & \text{if } j \neq k. \end{cases} \tag{3.19}$$

Example. It is simple to calculate

$$\chi_1^{*\bar{0}}(\mathrm{id}_\alpha) = \chi_{1,[1,1]}^{*+}\chi_{1,[1,1]}^{*-}(\mathrm{id}_\alpha) = \frac{i^\alpha + i^{-\alpha}}{i^\alpha - i^{-\alpha}} \cdot i^\alpha \sigma_1^3.$$

Using this and the reduction relations, we see that a successive application of $\chi_k^{*\bar{0}}$ to the identity operator produces the vacuum states,

$$\chi_{(\underbrace{\bar{0},\ldots,\bar{0}}_{k},\underbrace{0,\ldots,0}_{n-k})}^{(\alpha)} = \left(\frac{i^\alpha + i^{-\alpha}}{i^\alpha - i^{-\alpha}}\right)^k i^{\alpha \sum_{j=1}^k \sigma_j^3}.$$

4. Basis for generic q

In this section we introduce an inductive limit $\mathscr{W}_{[\alpha]}$ of the space $\mathscr{V}_{[\alpha],I}$ when the interval $I = [k,l]$ becomes infinite, $k \to -\infty, l \to \infty$. The operators $\mathbf{c}_j^\pm(\alpha) : \mathscr{W}_\alpha \to \mathscr{W}_{\alpha\mp 1}$ act on this space. Generalizing the result in the $q = i$ case, we prove the existence of a basis for generic q, on which the actions of $\mathbf{c}_j^\pm(\alpha)$ are simple.

4.1 Inductive limit

We work with an infinite lattice fully equipped with the spectral parameters. Let K be the field of rational functions in infinite variables ξ_j ($j \in \mathbb{Z}$) with coefficients in $\mathbb{C}(y)$ ($y = q^\alpha$). Actually, the field K is an inductive limit of the field of rational functions K_I ($I = [k,l]$) in the variables $\xi_k, \ldots, \xi_l$ when $k \to -\infty$ and $l \to \infty$. Let $\mathfrak{S}_\infty$ be the infinite symmetric group generated by the elements s_i, which is the transposition of i and $i+1$. On K there is an action of $\mathfrak{S}_\infty$ such that $s_i = r_{i,i+1}$. There is also an automorphism, $\tau(\xi_j) = \xi_{j+1}$, which corresponds to the shift of the lattice. Finally, there is an automorphism, $\kappa(y) = qy$, which corresponds to the shift of the disorder parameter α.

We define the vector space $\mathscr{W}_{[\alpha]}$ over K as the inductive limit of the vector spaces $\mathscr{V}_{[\alpha],I}$ where the inclusion maps are given by $\iota_{J,I}$ for intervals $J \subset I$. We denote the subspace $K \otimes \mathscr{V}_{[\alpha],I} \subset \mathscr{W}_{[\alpha]}$ by $\left(\mathscr{W}_{[\alpha]}\right)_I$. The total spin operator $\mathbb{S}$ is well-defined on $\mathscr{W}_{[\alpha]}$. We have the decomposition

$$\mathscr{W}_{[\alpha]} = \oplus_{s\in\mathbb{Z}}\mathscr{W}_{[\alpha]}^{(s)}, \quad \mathscr{W}_{[\alpha]}^{(s)} = \{X \in \mathscr{W}_{[\alpha]}|\mathbb{S}(X) = sX\}.$$

Note also that $\mathscr{W}_{[\alpha]}^{(s)} = \oplus_{\beta \in \alpha + \mathbb{Z}} \mathscr{W}_{\beta}^{(s)}$.

Schematically, an infinite tensor product of the form

$$\cdots \otimes y^{\sigma^3} \otimes y^{\sigma^3} \otimes \sigma^+ \otimes 1 \otimes \sigma^- \otimes y^{\sigma^3} \otimes 1 \otimes 1 \otimes \cdots$$

represents a vector in $\mathscr{W}_\alpha$. To be more precise, denote by $\mathscr{P}$ the set of all maps $p : \mathbb{Z} \to \{+, -, 0, \overline{0}\}$, $j \mapsto p_j$, such that

$$\sharp(\{j < 0 | p_j \neq \overline{0}\}) < \infty, \quad \sharp(\{j > 0 | p_j \neq 0\}) < \infty.$$

An element of $\mathscr{P}$ will be called *label*. We use labels to represent states in $\mathscr{W}_{\alpha+k}$:

$$v_p^{(\alpha+k)} = \otimes_{j \in \mathbb{Z}} v_{p_j}^{(\alpha+k)},$$

where $v_{\pm}^{(\alpha+k)} = \sigma^{\pm}$, $v_0^{(\alpha+k)} = 1$, $v_{\overline{0}}^{(\alpha+k)} = (q^k y)^{\sigma^3}$. The set of states $v_p^{(\beta)}$ is a basis of $\mathscr{W}_\beta$, but this is not the one we are looking for.

4.2 Actions on $\mathscr{W}_{[\alpha]}$

The automorphism κ induces an isomorphism κ of $\mathscr{W}_{[\alpha]}$ such that $\kappa : \mathscr{W}_\beta \to \mathscr{W}_{\beta+1}$. The automorphism τ induces an isomorphism τ of $\mathscr{W}_{[\alpha]}$ such that $\tau : \mathscr{W}_{\beta,[k,l]} \to \mathscr{W}_{\beta,[k+1,l+1]}$. The action of the symmetric group $\mathfrak{S}_n$ on $\mathscr{V}$ induces an action of the infinite symmetric group $\mathfrak{S}_\infty$ on $\mathscr{W}_{[\alpha]}$. We denote the action of $s_i \in \mathfrak{S}_\infty$ by $\mathbf{s}_i$. We have

$$\kappa \cdot \tau = \tau \cdot \kappa, \ \kappa \cdot \mathbf{s}_i = \mathbf{s}_i \cdot \kappa, \ \tau \cdot \mathbf{s}_i = \mathbf{s}_{i+1} \cdot \tau.$$

Let $\mathscr{A}$ be the Grassmann algebra over $\mathbb{C}$ generated by $c_i^{\pm}$ ($i \in \mathbb{Z}$). Because of the reduction property (2.11), (2.12), the action of the operators $\mathbf{c}_j^{\pm}(\alpha) : \mathscr{V}_\alpha \to \mathscr{V}_{\alpha \mp 1}$ induces an action $\mathbf{c}_j^{\pm} : \mathscr{W}_\alpha \to \mathscr{W}_{\alpha \mp 1}$. We extend this action to $\mathscr{W}_{[\alpha]}$ by requiring

$$\kappa \cdot \mathbf{c}_i^{\pm} = \mathbf{c}_i^{\pm} \cdot \kappa.$$

Then, because of (2.9), it gives an action of $\mathscr{A}$ on $\mathscr{W}_{[\alpha]}$. Because of (2.10) and by an obvious reason, we have the equivariance

$$\mathbf{s}_i \cdot \mathbf{c}_j^{\pm} = \mathbf{c}_{s_i(j)}^{\pm} \cdot \mathbf{s}_i, \tag{4.1}$$

$$\tau \cdot \mathbf{c}_j^{\pm} = \mathbf{c}_{j+1}^{\pm} \cdot \tau. \tag{4.2}$$

We observed that there is an action of the algebra $\widetilde{\mathscr{A}}$ over $\mathbb{C}$ generated by $\kappa^{\pm 1}$, $\tau^{\pm 1}$, $\mathbf{s}_i$ and $\mathbf{c}_j^{\pm}$. This action is $\mathbb{C}$-linear and satisfies the equivariance

$$\kappa(f \cdot X) = \kappa(f) \cdot \kappa(X),$$
$$\tau(f \cdot X) = \tau(f) \cdot \tau(X),$$
$$\mathbf{s}_i(f \cdot X) = r_{i,i+1}(f) \cdot \mathbf{s}_i(X),$$
$$\mathbf{c}_j^{\pm}(f \cdot X) = f \cdot \mathbf{c}_j^{\pm}(X)$$

for $f \in K$ and $X \in \mathscr{W}_{[\alpha]}$.

4.3 Space of labels

Let $\mathbb{C}[\mathbb{Z} \times \mathscr{P}]$ be the vector space with the basis $\mathbb{Z} \times \mathscr{P}$. Our strategy is first to define a representation of the algebra $\widetilde{\mathscr{A}}$ on $\mathbb{C}[\mathbb{Z} \times \mathscr{P}]$, and then to show the existence of an intertwiner

$$\mathscr{X} : \mathbb{C}[\mathbb{Z} \times \mathscr{P}] \to \mathscr{W}_{[\alpha]}.$$

The vectors $X_p^{(\alpha+k)} = \mathscr{X}(k, p) \in \mathscr{W}_{\alpha+k}$ ($k \in \mathbb{Z}, p \in \mathscr{P}$) constitute the wanted K basis of $\mathscr{W}_{[\alpha]}$.

There is an action of $\mathfrak{S}_\infty$ on the vector space $\mathbb{C}[\mathscr{P}]$ such that

$$s_i(p) = \begin{cases} -p & \text{if } (p_i, p_{i+1}) = (+,+) \text{ or } (-,-); \\ (\ldots, p_{i+1}, p_i \ldots) & \text{otherwise.} \end{cases}$$

This representation is isomorphic to a direct sum of the representations induced from the representation

$$\mathrm{id}_{(-\infty, k-1]} \otimes \mathrm{sgn}_{[k, l-1]} \otimes \mathrm{sgn}_{[l, m-1]} \otimes \mathrm{id}_{[m, \infty)}$$

of the parabolic subgroup

$$\mathfrak{S}_{(-\infty, k-1]} \times \mathfrak{S}_{[k, l-1]} \times \mathfrak{S}_{[l, m-1]} \times \mathfrak{S}_{[m, \infty)}$$

for all possible choices of $k \le l \le m$. Here $\mathfrak{S}_{[k, l-1]}$ denotes the symmetric group of degree $l - k$ acting on the interval $[k, l-1]$, and $\mathrm{sgn}_{[k, l-1]}$ is its sign representation. Set

$$\left(p^{(k,l,m)} \right)_j = \begin{cases} \bar{0} & \text{if } j \le k-1; \\ + & \text{if } k \le j \le l-1; \\ - & \text{if } l \le j \le m-1; \\ 0 & \text{if } m \le j. \end{cases}$$

We identify the element $p^{(k,l,m)} \in \mathscr{P}$ with the cyclic vector of the induced representation. The above action on $\mathbb{C}[\mathscr{P}]$ is lifted to $K[\mathscr{P}]$ by requiring the equivariance $s_i(f \cdot p) = r_{i,i+1}(f) \cdot s_i(p)$. Similarly, we have a natural action $\tau(p)_j = p_{j-1}$ of the shift operator. Next we define an action of the Grassmann algebra $\mathscr{A}$ on $\mathbb{C}[\mathscr{P}]$. For $p \in \mathscr{P}$ set

$$c_k^\pm(p) = \delta_{p_k, \mp} \varepsilon^\mp(p, k) p[k]. \tag{4.3}$$

Here we used (3.18)–(3.17) for $p \in \mathscr{P}$. One can check the equivariance of this action:

$$s_i c_j^\pm = c_{s_i(j)}^\pm s_i,$$
$$\tau c_j^\pm = c_{j+1}^\pm \tau.$$

Finally, we define the action of $\widetilde{\mathscr{A}}$ on $\mathbb{C}[\mathbb{Z} \times \mathscr{P}]$. The action of κ is such that $\kappa(k, p) = (k+1, p)$. The actions of the other operators are trivial on the $\mathbb{Z}$ component.

For $p \in \mathscr{P}$ define $n^\pm(p) = \sharp(\{j | p_j = \pm\})$, and set

$$\mathscr{P}_{l, n-l} = \{p \in \mathscr{P} | n^+(p) = l, \ n^-(p) = n - l\}, \ \mathscr{P}_n = \cup_{l=0}^n \mathscr{P}_{l, n-l}.$$

Note that $c_k^+ \mathscr{P}_{l,n-l} \subset \mathbb{C}[\mathscr{P}_{l,n-l-1}]$ and $c_k^- \mathscr{P}_{l,n-l} \subset \mathbb{C}[\mathscr{P}_{l-1,n-l}]$. We also define

$$\mathscr{P}_{[k,l]} = \{p \in \mathscr{P} \,|\, p_j = \overline{0}\ (j \le k-1),\ p_j = 0\ (j \ge l+1)\}.$$

Our goal is to show

Theorem 4.1. *We assume that q is generic. There exists an intertwiner of the $\widetilde{\mathscr{A}}$ modules*

$$\mathscr{X} : \mathbb{C}[\mathbb{Z} \times \mathscr{P}] \to \mathscr{W}_{[\alpha]}.$$

A proof is given in the subsequent subsections. The uniqueness of the intertwiner is not true. We need more structures, e.g., creation operators, in order to single out a unique basis.

Set $X_p^{(\alpha+k)} = \mathscr{X}(k,p) \in \mathscr{W}_{\alpha+k}$. The intertwining property means

$$X_p^{(\beta+1)} = \kappa(X_p^{(\beta)}), \tag{4.4}$$

$$X_{\tau(p)}^{(\beta)} = \tau(X_p^{(\beta)}), \tag{4.5}$$

$$X_{s_i(p)}^{(\beta)} = \mathbf{s}_i(X_p^{(\beta)}), \tag{4.6}$$

$$X_{c_j^\pm(p)}^{(\beta)} = \mathbf{c}_j^\pm(X_p^{(\beta)}). \tag{4.7}$$

If we define $X_p^{(\alpha)}$ all other $X_p^{(\beta)}$ are defined by (4.4). Our description below exploits this without mentioning it any further.

Before giving a proof we prepare several statements on the operators $\mathbf{c}_j^\pm$.

Proposition 4.2.

$$\mathbf{c}_j^\pm(\alpha)\left((\mathscr{W}_\alpha)_{[k,l]}\right) = 0 \ \ \text{unless } k \le j \le l, \tag{4.8}$$

$$\mathbf{c}_l^\pm(\alpha)\left((\mathscr{W}_\alpha)_{[k,l]}\right) \subset (\mathscr{W}_{\alpha\mp1})_{[k,l-1]}. \tag{4.9}$$

The property (4.8) follows immediately from the definition. The property (4.9) follows from Proposition 2.6.

4.4 Vacuum vectors

Let us consider the common kernel of $\mathbf{b}_k(\alpha), \mathbf{c}_k(\alpha)$ $(k \in \mathbb{Z})$. We call vectors in the kernel *vacuum vectors*. Set $p_{\text{vac}} = p^{(1,1,1)} \in \mathscr{P}_0$. The annihilation property (4.8) implies

$$\mathbf{c}_k^\pm(\alpha)v_{p_{\text{vac}}}^{(\alpha)} = 0. \tag{4.10}$$

We define

$$X_{p_{\text{vac}}}^{(\alpha)} = v_{p_{\text{vac}}}^{(\alpha)}.$$

Let $\mathscr{W}_{\alpha,\text{vac}}$ be the subspace of $\mathscr{W}_\alpha$ spanned by the states in the orbit of $X_{p_{\text{vac}}}^{(\alpha)}$ by $\mathbf{s}_i$ and τ^n $(i,n \in \mathbb{Z})$.

Proposition 4.3. *For generic q, the common kernel of the operators $\mathbf{b}_k(\alpha), \mathbf{c}_k(\alpha)$ $(k \in \mathbb{Z})$ is equal to $\mathscr{W}_{\alpha,\text{vac}}$:*

$$\mathscr{W}_{\alpha,\text{vac}} = \bigcap_{k \in \mathbb{Z}} (\text{Ker}\,\mathbf{b}_k(\alpha) \cap \text{Ker}\,\mathbf{c}_k(\alpha)).$$

Proof. It is enough to show the equality

$$\mathscr{W}_{\alpha,\text{vac}} \cap (\mathscr{W}_\alpha)_{[k,l]} = \left(\bigcap_{k \in \mathbb{Z}} (\text{Ker}\,\mathbf{b}_k(\alpha) \cap \text{Ker}\,\mathbf{c}_k(\alpha)) \right) \cap (\mathscr{W}_\alpha)_{[k,l]}. \tag{4.11}$$

The inclusion $\subset$ is a consequence of (4.10) and the equivariance (4.1), (4.2). We know that for $q = i$

$$\dim \mathscr{W}_{\alpha,\text{vac}} \cap (\mathscr{W}_\alpha)_{[k,l]} = 2^{l-k+1}, \tag{4.12}$$

and (4.11) holds. Moreover, for generic q, the space $\mathscr{W}_{\alpha,\text{vac}} \cap (\mathscr{W}_\alpha)_{[k,l]}$ is spanned by the 2^{l-k+1} elements in the orbit of $X^{(\alpha)}_{p_{\text{vac}}}$. By specialization of the parameter q, the dimension of the kernel does not decrease, while the dimension of the linear span does not increase. Therefore (4.12) is valid for generic q. The equality (4.11) follows from this. □

4.5 Filtration of $\mathscr{W}_{[\alpha]}$

Proposition 2.6 suggests a kind of particle structure in the space $\mathscr{W}_{[\alpha]}$ wherein the operators $\mathbf{b}_k(\alpha)$ and $\mathbf{c}_k(\alpha)$ act as annihilation operators. Starting from the subspace $\mathscr{W}_{\alpha,\text{vac}}$ we define a filtration of $\mathscr{W}_\alpha$:

$$0 = F^{-1}\mathscr{W}_\alpha \subset F^0\mathscr{W}_\alpha = \mathscr{W}_{\alpha,\text{vac}} \subset \cdots \subset F^n\mathscr{W}_\alpha \subset \cdots \subset \mathscr{W}_\alpha,$$
$$F^n\mathscr{W}_\alpha = \sum_{k \in \mathbb{Z}} \mathfrak{S}_\infty (\mathscr{W}_\alpha)_{[k,k+n-1]}.$$

We have

$$\mathbf{c}_k^\pm(\alpha) : F^n\mathscr{W}_\alpha \to F^{n-1}\mathscr{W}_{\alpha \mp 1}.$$

Set $\text{Gr}_F^n\mathscr{W}_\alpha = F^n\mathscr{W}_\alpha / F^{n-1}\mathscr{W}_\alpha$. We denote the induced mappings by the same symbols: $\mathbf{c}_k^\pm(\alpha) : \text{Gr}_F^n\mathscr{W}_\alpha \to \text{Gr}_F^{n-1}\mathscr{W}_{\alpha \mp 1}$. We will construct $X^{(\alpha)} : \mathbb{C}[\mathscr{P}_n] \to F^n\mathscr{W}_\alpha$ so that the intertwining properties (4.4)–(4.7) are satisfied and the induced mapping

$$X^{(\alpha)} : K[\mathscr{P}_n] \to \text{Gr}_F^n\mathscr{W}_\alpha$$

is an isomorphism. Our proof will show that

$$X^{(\alpha)} : K[\mathscr{P}_{[k,l]}] \to (\mathscr{W}_\alpha)_{[k,l]}$$

is an isomorphism.

Proof of Theorem 4.1. We proceed by induction on n. Suppose that we have constructed $X^{(\alpha)}$ on $\cup_{j=0}^{n-1}\mathscr{P}_j$ and the intertwining properties are satisfied. We construct $X^{(\alpha)}$ on $\mathscr{P}_n$. Set $p^{(l,n-l)} = p^{(1,l+1,n+1)} \in \mathscr{P}_n$. The subspace $\mathbb{C}[\mathscr{P}_{l,n-l}]$ is generated from $p^{(l,n-l)}$ by the action of $\mathbb{C}[\mathfrak{S}_\infty]$ and the shift operators τ^n ($n \in \mathbb{Z}$). We construct $X^{(\alpha)}_{p^{(l,n-l)}} \in (\mathscr{W}_\alpha)_{[1,n]}$ which satisfies

$$\mathbf{s}_i X^{(\alpha)}_{p^{(l,n-l)}} = \begin{cases} X^{(\alpha)}_{p^{(l,n-l)}} & \text{if } i < 0 \text{ or } i \geq n+1; \\ -X^{(\alpha)}_{p^{(l,n-l)}} & \text{if } 1 \leq i \leq l-1 \text{ or } l+1 \leq i \leq n-1. \end{cases} \tag{4.13}$$

$$\mathbf{c}_k^\pm(\alpha) X^{(\alpha)}_{p^{(l,n-l)}} = X^{(\alpha \mp 1)}_{\mathbf{c}_k^\pm p^{(l,n-l)}}. \tag{4.14}$$

Then, we can induce $X^{(\alpha)}$ from $p^{(l,n-l)}$ to $\mathbb{C}[\mathscr{P}_{l,n-l}]$ by using the action of $\mathfrak{S}_\infty$ and the shift operators. By construction the intertwining properties (4.5) and (4.6) are satisfied. Because of (4.1) and (4.2), the last one (4.7) follows from (4.14).

Set $\overline{\mathscr{P}}_{l,n-l} = \mathscr{P}_{[1,n]} \cap \mathscr{P}_{l,n-l}$, $\overline{\mathscr{P}}_n = \cup_{l=0}^n \overline{\mathscr{P}}_{l,n-l}$, and $\mathscr{P}^{(k)} = \{p \in \mathscr{P} | p_k = 0\}$. The intertwining property (4.7) with (4.3) implies that if $p, p' \in \overline{\mathscr{P}}_{l,n-l}$, we have

$$\mathbf{c}_1^{-p_1}(\alpha + p_2 + \cdots + p_n) \cdots \mathbf{c}_n^{-p_n}(\alpha) X_{p'}^{(\alpha)} = \delta_{p,p'} \varepsilon(p) X_{p_{\mathrm{vac}}}^{(\alpha-2l+n)},$$

where $\varepsilon(p)$ is given in (3.16). Our immediate goal is to construct a family of states $Y_p^{(\alpha)}$ ($p \in \overline{\mathscr{P}}_{l,n-l} \cup (-\overline{\mathscr{P}}_{l,n-l})$) satisfying

$$Y_p^{(\alpha)} = -Y_{-p}^{(\alpha)}, \tag{4.15}$$

$$Y_{s_i(p)}^{(\alpha)} = \mathbf{s}_i(Y_p^{(\alpha)}), \tag{4.16}$$

$$\mathbf{c}_1^{-p_1}(\alpha + p_2 + \cdots + p_n) \cdots \mathbf{c}_n^{-p_n}(\alpha) Y_{p'}^{(\alpha)} = \delta_{p,p'} \varepsilon(p) X_{p_{\mathrm{vac}}}^{(\alpha-2l+n)}. \tag{4.17}$$

For each $0 \le l \le n$ take an $\mathfrak{S}_{[1,n]}$ invariant subspace $\mathscr{U}_{l,n-l} \subset F^n \mathscr{W}_\alpha \cap \mathscr{W}_\alpha^{(2l-n)}$ of dimension

$$\sharp(\overline{\mathscr{P}}_{l,n-l}) = \binom{n}{l}$$

such that $\oplus_{l=0}^n \mathscr{U}_{l,n-l} \to \mathrm{Gr}_F^n \mathscr{W}_\alpha$ is an isomorphism. The non-uniqueness of such spaces is the reason for the non-uniqueness of the basis. We have no a priori reason to choose one.

Fix an arbitrary basis of $\mathscr{U}_{l,n-l}$, $\{v_p | p \in \overline{\mathscr{P}}_{l,n-l}\}$. Consider the matrix $M = (M_{p,p'})_{p,p' \in \overline{\mathscr{P}}_{l,n-l}}$ given by

$$\mathbf{c}_1^{-p_1}(\alpha + p_2 + \cdots + p_n) \cdots \mathbf{c}_n^{-p_n}(\alpha) v_{p'} = M_{p,p'} X_{p_{\mathrm{vac}}}^{(\alpha-2l+n)}.$$

From Proposition 2.6 we see that the left hand side is proportional to $X_{p_{\mathrm{vac}}}^{(\alpha-2l+n)}$. The matrix M is invertible for generic q because it is so for the fermion case $q = i$. Therefore, there exists a unique set of elements $Y_p^{(\alpha)} \in \mathscr{U}_{l,n-l}$ ($p \in \overline{\mathscr{P}}_{l,n-l} \cap (-\overline{\mathscr{P}}_{l,n-l})$) where $Y_{-p}^{(\alpha)} = -Y_p^{(\alpha)}$ satisfying (4.17) for $p, p' \in \overline{\mathscr{P}}_{l,n-l}$. The equivariance (4.16) follows from (4.1), the Grassmann relation and the uniqueness of the solution to the linear equation (4.17).

We will modify $Y_p^{(\alpha)}$ to $X_p^{(\alpha)}$ by adding lower order terms in the filtration. Take $p \in \overline{\mathscr{P}}_{l,n-l}$. For each k, consider the state

$$\mathbf{c}_k^\pm(\alpha) Y_p^{(\alpha)} \in \mathbf{s}_k \cdots \mathbf{s}_{n-1} \left((\mathscr{W}_\alpha)_{[1,n-1]} \right).$$

By the induction hypothesis, it can be written as

$$\mathbf{c}_k^\pm(\alpha) Y_p^{(\alpha)} = \sum_{p' \in \mathscr{P}^{(k)} \cap \mathscr{P}_{[1,n]}} f_{k,p,p'}^\pm X_{p'}^{(\alpha \mp 1)}.$$

If $p' \in \mathscr{P}_{n-1}$ appears in the sum, we have $p_k = \mp$, $p' = p[k]$ and $f_{k,p,p'}^\pm = \varepsilon^\mp(p,k)$. For, if $p_k = \pm$, or $p_k = \mp$ but p' is an element of $\mathscr{P}_{n-1}$ other than $p[k]$, $Y_p^{(\alpha)}$ can be brought to non-zero multiple

of $X_{p_{\text{vac}}}^{(\alpha-2l+n)}$ by a sequence of operators other than $\mathbf{c}_1^{-p_1}(\alpha+p_2+\cdots+p_n)\cdots\mathbf{c}_n^{-p_n}(\alpha)$. This is a contradiction to (4.17). The equality $f_{k,p,p[k]}^{\pm}=\varepsilon^{\mp}(p,k)$ follows from

$$\mathbf{c}_1^{-p_1}(\alpha+p_2+\cdots+p_n)\cdots\mathbf{c}_n^{-p_n}(\alpha)Y_p^{(\alpha)}$$
$$=(-1)^{n-k}f_{k,p,p[k]}^{\pm}\mathbf{c}_1^{-p_1}(\alpha+p_2+\cdots+p_n)\cdots\overset{k}{\vee}\cdots\mathbf{c}_n^{-p_n}(\alpha)X_{p[k]}^{(\alpha\mp1)}.$$

Note that $\varepsilon^{\mp}(p,k)p[k]=c_k^{\pm}(p)$.

If $p'\notin\mathscr{P}_{n-1}$ appears in the sum, the total spin conservation requires

$$p'\in\begin{cases}\mathscr{P}_{l-i,n-l-1-i}\cap\mathscr{P}^{(k)}\cap\mathscr{P}_{[1,n]} & \text{for }\mathbf{c}_k^{+}(\alpha);\\ \mathscr{P}_{l-1-i,n-l-i}\cap\mathscr{P}^{(k)}\cap\mathscr{P}_{[1,n]} & \text{for }\mathbf{c}_k^{-}(\alpha).\end{cases} \tag{4.18}$$

Here $i\geq1$. Therefore, we have

$$\mathbf{c}_k^{\pm}(\alpha)Y_p^{(\alpha)}=X_{\mathbf{c}_k^{\pm}p}^{(\alpha\mp1)}+\sum_{p'}f_{k,p,p'}^{\pm}X_{p'}^{(\alpha\mp1)}$$

where the sum over p' is restricted as (4.18).

Set

$$X_p^{(\alpha)}=Y_p^{(\alpha)}-\sum_{\tilde{p}\in\mathscr{P}_{[1,n]}\cap(\mathscr{P}_{n-2}\cup\mathscr{P}_{n-4}\cup\cdots)}g_{\tilde{p}}X_{\tilde{p}}^{(\alpha)}.$$

We do not sum over $\tilde{p}\in\mathscr{P}_0$. We require $\mathbf{c}_k^{\pm}(X_p^{(\alpha)})=X_{\mathbf{c}_k^{\pm}p}^{(\alpha\mp1)}$. This is equivalent to

$$g_{\tilde{p}}=f_{k,p,\tilde{p}[k]}^{-\tilde{p}_k}\varepsilon^{\tilde{p}_k}(\tilde{p},k)$$

for all $1\leq k\leq n$ such that $\tilde{p}_k=\pm$. The Grassmann relation implies that the right hand side is independent of k: suppose that $\tilde{p}_k,\tilde{p}_l\neq0,\bar{0}$. From

$$\mathbf{c}_l^{-\tilde{p}_l}(\alpha+\tilde{p}_k)\mathbf{c}_k^{-\tilde{p}_k}(\alpha)Y_p^{(\alpha)}=-\mathbf{c}_k^{-\tilde{p}_k}(\alpha+\tilde{p}_l)\mathbf{c}_l^{-\tilde{p}_l}(\alpha)Y_p^{(\alpha)}$$

we obtain $f_{k,p,\tilde{p}[k]}^{-\tilde{p}_k}\varepsilon^{\tilde{p}_l}(\tilde{p}[k],l)=-f_{l,p,\tilde{p}[l]}^{-\tilde{p}_l}\varepsilon^{\tilde{p}_k}(\tilde{p}[l],k)$. From

$$\mathbf{c}_l^{-\tilde{p}_l}(\alpha+\tilde{p}_k)\mathbf{c}_k^{-\tilde{p}_k}(\alpha)X_{\tilde{p}}^{(\alpha)}=-\mathbf{c}_k^{-\tilde{p}_k}(\alpha+\tilde{p}_l)\mathbf{c}_l^{-\tilde{p}_l}(\alpha)X_{\tilde{p}}^{(\alpha)}$$

we obtain $\varepsilon^{\tilde{p}_k}(\tilde{p},k)\varepsilon^{\tilde{p}_l}(\tilde{p}[k],l)=-\varepsilon^{\tilde{p}_l}(\tilde{p},l)\varepsilon^{\tilde{p}_k}(\tilde{p}[l],k)$.

Since $Y_p^{(\alpha)}$ is equivariant by (4.16), and the construction of $X_p^{(\alpha)}$ is equivariant, we have

$$\mathbf{s}_i(X_p^{(\alpha)})=X_{s_i(p)}^{(\alpha)}\ (1\leq i\leq n-1).$$

$\square$

A. Anti-commutativity of $\mathbf{c}(\zeta,\alpha)$

In this section we sketch the derivation for the anti-commutativity property of the operators $\mathbf{c}(\zeta,\alpha)$. Anti-commutativity of $\mathbf{c}(\zeta,\alpha)$ with $\mathbf{b}(\zeta,\alpha)$ requires a different reasoning which is given in [1]. We do not repeat it here.

We need information about the R matrix. First assume that q is not a root of unity. Define $R_{A,B}(\zeta) \in \mathrm{End}(W_A^+ \otimes W_B^+)$ by

$$R_{A,B}(\zeta) = P_{A,B} h(\zeta, u_{A,B}) \zeta^{D_A + D_B},$$
$$u_{A,B} = \mathbf{a}_A^* q^{-2D_A} \mathbf{a}_B,$$

where $P_{A,B}$ denotes the permutation, and $h(\zeta,u)$ is the unique formal power series in u satisfying

$$h(\zeta, q^2 u)(1 + \zeta^{-1} u) = h(\zeta, u)(1 + \zeta u), \tag{A.1}$$
$$h(\zeta, 0) = 1.$$

When there exists a positive integer r such that $q^{2r} = 1$ and $q^{2j} \neq 1$ $(1 \le j \le r - 1)$, the representation W_A^+ has an r-dimensional quotient $W_{A,0}^+$ generated by $|0\rangle$. In this case $u_{A,B}^r = 0$. Define $R_{A,B}(\zeta) \in \mathrm{End}(W_{A,0}^+ \otimes W_{B,0}^+)$ by the same formula, where $h(\zeta, u)$ is the unique element in the truncated polynomial ring $\mathbb{C}[u]/(u^r)$ with the above properties.

Lemma A.1. *The operator $R_{A,B}(\zeta)$ satisfies the intertwining property*

$$R_{A,B}(\zeta_1/\zeta_2) L_{A,j}^+(\zeta_1) L_{B,j}^+(\zeta_2) = L_{B,j}^+(\zeta_2) L_{A,j}^+(\zeta_1) R_{A,B}(\zeta_1/\zeta_2).$$

Proof. If $uv = vu + (1 - q^2)w$ and $uw = q^{-2} wu$, then

$$f(u) v = v f(u) + \frac{f(u) - f(q^2 u)}{u} w$$

holds for any $f(u)$. Taking $u = u_{A,B}$ and $v = \mathbf{a}_B^* q^{2D_A}$, $w = \mathbf{a}_A^*$ or $v = \mathbf{a}_A$, $w = -\mathbf{a}_B q^{-2D_A}$, we deduce from (A.1) the identities

$$h(\zeta, u)(\zeta \mathbf{a}_A^* + \mathbf{a}_B^* q^{2D_A}) = (\zeta^{-1} \mathbf{a}_A^* + \mathbf{a}_B^* q^{2D_A}) h(\zeta, u),$$
$$h(\zeta, u)(\mathbf{a}_A + \zeta^{-1} \mathbf{a}_B q^{-2D_A}) = (\mathbf{a}_A + \zeta \mathbf{a}_B q^{-2D_A}) h(\zeta, u).$$

The intertwining relations follow from these. $\square$

The R matrix which intertwines the fused L operators

$$R_{\{A,a\},\{B,b\}}(\zeta_1/\zeta_2) L_{\{A,a\},j}^+(\zeta_1) L_{\{B,b\},j}^+(\zeta_2) \tag{A.2}$$
$$= L_{\{B,b\},j}^+(\zeta_2) L_{\{A,a\},j}^+(\zeta_1) R_{\{A,a\},\{B,b\}}(\zeta_1/\zeta_2)$$

has the triangular form

$$R_{\{A,a\},\{B,b\}}(\zeta) = \begin{pmatrix} R_{11} & 0 & 0 & 0 \\ R_{21} & R_{22} & 0 & 0 \\ R_{31} & 0 & R_{33} & 0 \\ R_{41} & R_{42} & R_{43} & R_{44} \end{pmatrix}.$$

Its non-zero entries read as follows.

$$R_{11} = q^{D_B} R_{A,B}(\zeta) q^{-D_A},$$

$$R_{21} = -q^{-1}\zeta^2 q^{D_A-D_B} R_{A,B}(q^{-1}\zeta)\mathbf{a}_B$$

$$R_{22} = q(1-q^{-2}\zeta^2)q^{D_A} R_{A,B}(q^{-2}\zeta)q^{D_B},$$

$$R_{31} = -\frac{\zeta}{1-q^2\zeta^2}q^{D_B-D_A} R_{A,B}(\zeta q)\mathbf{a}_B,$$

$$R_{41} = \frac{q\zeta^3}{1-q^2\zeta^2}q^{-D_B} R_{A,B}(\zeta)q^{-D_A}\mathbf{a}_B^2,$$

$$R_{42} = -q\zeta\cdot R_{A,B}(\zeta q^{-1})q^{-D_A+D_B}\mathbf{a}_B,$$

$$R_{33} = -\frac{q\zeta^2}{1-q^2\zeta^2}q^{-D_A} R_{A,B}(q^2\zeta)q^{-D_B},$$

$$R_{43} = \frac{q^2\zeta^4}{1-q^2\zeta^2}q^{D_A-D_B} R_{A,B}(\zeta q)q^{-2D_B}\mathbf{a}_B,$$

$$R_{44} = -\zeta^2 q^{D_A} R_{A,B}(\zeta)q^{-D_B}.$$

The anti-commutativity of $\mathbf{c}(\zeta,\alpha)$ is an immediate consequence of the following Lemma.

Lemma A.2. *We have*

$$\mathbf{k}^+(\zeta_1,\alpha-1)\mathbf{k}^+(\zeta_2,\alpha)+\mathbf{k}^+(\zeta_2,\alpha-1)\mathbf{k}^+(\zeta_1,\alpha)$$
$$=\Delta_{q,\zeta_1}F(\zeta_1,\zeta_2)+\Delta_{q,\zeta_2}F(\zeta_2,\zeta_1),$$

where

$$F(\zeta_1,\zeta_2) = q^{\alpha-1}\frac{1}{1-\zeta_1^2/\zeta_2^2}\mathbf{Q}^{(1)}(\zeta_1,\alpha-1)\mathbf{k}^{(1)}(\zeta_2,\alpha)$$

$$+q^{\alpha-1}\frac{\zeta_1/\zeta_2}{1-\zeta_1^2/\zeta_2^2}\mathbf{Q}^{(2)}(\zeta_1,\alpha-1)\mathbf{k}^{(0)}(\zeta_2,\alpha-1)$$

$$-q^{2(\alpha-1)}\frac{\zeta_1/\zeta_2}{1-\zeta_1^2/\zeta_2^2}\mathbf{Q}^{(2)}(\zeta_1,\alpha-1)\Delta_{q,\zeta_2}\mathbf{Q}^{(0)}(\zeta_2,\alpha-1),$$

and

$$\mathbf{Q}^{(k)}(\zeta,\alpha)(X) = (1-q^{2(\alpha-\mathbb{S})})\zeta^{\alpha-\mathbb{S}}\mathrm{Tr}_A^+\left(q^{2\alpha D_A}\mathbf{a}_A^k T_A^+(\zeta)^{-1} X T_A^+(\zeta)\right),$$

$$\mathbf{k}^{(k)}(\zeta,\alpha)(X) = (1-q^{2(\alpha-\mathbb{S})})\zeta^{\alpha-\mathbb{S}}\mathrm{Tr}_A^+\mathrm{tr}_a\left(q^{2\alpha D_A}(\mathbf{a}_A^*)^{1-k}\sigma_a^+ T_{\{a,A\}}^+(\zeta)^{-1} X T_{\{a,A\}}^+(\zeta)\right).$$

Proof. Set

$$(T_{\{A,a\}}^+(\zeta))^{-1}X T_{\{A,a\}}^+(\zeta) = \begin{pmatrix} \mathscr{A}(\zeta)(X) & 0 \\ \mathscr{C}(\zeta)(X) & \mathscr{D}(\zeta)(X) \end{pmatrix}.$$

For brevity we write $\zeta = \zeta_1/\zeta_2$, $R_{ij} = R_{ij}(\zeta)$, $\mathscr{A}_1 = \mathscr{A}(\zeta_1)$, and so forth. The intertwining relation (A.2) contains the following relations.

$$R_{21}\mathscr{A}_2\mathscr{A}_1 + R_{22}\mathscr{C}_2\mathscr{A}_1 = \mathscr{A}_1\mathscr{C}_2 R_{11} + \mathscr{A}_1\mathscr{D}_2 R_{21},$$

$$R_{42}\mathscr{D}_2\mathscr{A}_1 + R_{44}\mathscr{D}_2\mathscr{C}_1 = \mathscr{C}_1\mathscr{D}_2 R_{22} + \mathscr{D}_1\mathscr{D}_2 R_{42},$$

$$R_{41}\mathscr{A}_2\mathscr{A}_1 + R_{42}\mathscr{C}_2\mathscr{A}_1 + R_{43}\mathscr{A}_2\mathscr{C}_1 + R_{44}\mathscr{C}_2\mathscr{C}_1$$
$$= \mathscr{C}_1\mathscr{C}_2 R_{11} + \mathscr{C}_1\mathscr{D}_2 R_{21} + \mathscr{D}_1\mathscr{C}_2 R_{31} + \mathscr{D}_1\mathscr{D}_2 R_{41}.$$

Eliminating $\mathscr{C}_2\mathscr{A}_1$ and $\mathscr{C}_1\mathscr{D}_2$, we obtain

$$\mathscr{C}_1\mathscr{C}_2 - R_{44}\mathscr{C}_2\mathscr{C}_1 R_{11}^{-1} = X\mathscr{A}_1\mathscr{A}_2 - \mathscr{D}_1\mathscr{D}_2 X \tag{A.3}$$
$$+ R_{43}\mathscr{A}_2\mathscr{C}_1 R_{11}^{-1} - R_{44}\mathscr{D}_2\mathscr{C}_1 R_{22}^{-1} R_{21} R_{11}^{-1} + R_{42}R_{22}^{-1}\mathscr{A}_1\mathscr{C}_2 - \mathscr{D}_1\mathscr{C}_2 R_{31} R_{11}^{-1},$$

where $X = (R_{41} - R_{42}R_{22}^{-1}R_{21})R_{11}^{-1}$. From the explicit form of the matrix elements we calculate

$$X = \left(\frac{q\zeta}{1-q^2\zeta^2} - \frac{q^{-1}\zeta}{1-q^{-2}\zeta^2} \right) a_A^2 q^{-2D_B},$$

$$R_{11}^{-1} q^{-2D_A} R_{43} = \frac{q^4\zeta^4}{1-q^2\zeta^2}(1+q^{-1}\zeta^{-1}u_{A,B})a_B q^{-2D_B},$$

$$R_{22}^{-1} R_{21} R_{11}^{-1} q^{-2D_A} R_{44} = \frac{q^{-2}\zeta^4}{1-q^{-2}\zeta^2}(1+q\zeta^{-1}u_{A,B})a_B q^{-2D_B},$$

$$q^{-2D_A} R_{42} R_{22}^{-1} = -\frac{q^2}{1-q^{-2}\zeta^2}(1+q^{-1}\zeta u_{A,B})a_A q^{-2D_A},$$

$$R_{31} R_{11}^{-1} = -\frac{1}{1-q^2\zeta^2}(1+q\zeta u_{A,B})a_A.$$

Multiply both sides of (A.3) by

$$(1-q^{2(\alpha-\mathbb{S}-1)})(1-q^{2(\alpha-\mathbb{S}+1)})\zeta_1^{\alpha-\mathbb{S}-1}\zeta_2^{\alpha-\mathbb{S}+1}q^{2(\alpha-1)D_A+2\alpha D_B}$$

and take the trace. Direct calculation leads to the assertion. $\qquad\square$

B. Proof of Theorem 3.5,3.6

In this section, we give a derivation of Theorems 3.5,3.6.

B.1 Preliminaries

Throughout this section, we fix n and work with the interval $[1,n]$. To perform the calculation we find it technically easier to pass from $\mathscr{V} = \text{End}(V^{\otimes n})$ to the $2n$ fold tensor product of V. Let $\iota : \mathscr{V} \xrightarrow{\sim} V^{\otimes 2n}$ denote the isomorphism of vector spaces given by

$$\iota\left(E_{\varepsilon_1,\varepsilon_1'} \otimes \cdots \otimes E_{\varepsilon_n,\varepsilon_n'}\right) = \prod_{j=1}^n \varepsilon_j' \cdot v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_n} \otimes v_{-\varepsilon_n'} \otimes \cdots \otimes v_{-\varepsilon_1'},$$

where $E_{\varepsilon,\varepsilon'} = \left(\delta_{\varepsilon\mu}\delta_{\varepsilon'\nu}\right)_{\mu,\nu}$ stands for the matrix unit. In what follows, we set

$$\bar{k} = 2n+1-k, \quad \xi_{\bar{k}} = q^{-1}\xi_k \quad (k = 1, \cdots, n).$$

Under the isomorphism ι, the left and the right multiplication by an element Z_k ($Z \in \text{End}(V)$) are translated respectively into

$$\iota(Z_k X) = Z_k \cdot \iota(X), \qquad \iota(XZ_k) = \left(\sigma^2 \, {}^tZ \, \sigma^2\right)_{\bar{k}} \cdot \iota(X).$$

In particular we have $\iota\left(\mathbb{S}(X)\right) = \widetilde{S} \cdot \iota(X)$ and $\iota(XS) = -\overline{S} \cdot \iota(X)$, where

$$\widetilde{S} = \frac{1}{2}\sum_{j=1}^{2n} \sigma_k^3, \quad \overline{S} = \frac{1}{2}\sum_{k=1}^n \sigma_{\bar{k}}^3.$$

We shall also make use of the relations

$$\iota\left(X\,L_{A,j}^{\pm}(\zeta)\right) = (q\zeta - q^{-1}\zeta^{-1})L_{A,\bar{j}}^{\pm}(q\zeta)^{-1}\cdot\iota(X),$$

$$\iota\left(XL_{A,j}^{\pm}(\zeta)^{-1}\right) = (\zeta - \zeta^{-1})^{-1}L_{A,\bar{j}}^{\pm}(q^{-1}\zeta)\cdot\iota(X),$$

which hold provided the entries of X commute with *Osc*.

We begin by rewriting Q operators. Introduce operators

$$\tilde{\psi}_j^{\pm} = \sigma_j^{\pm}q^{\pm\Sigma_{l=j+1}^{2n}\sigma_l^3}\quad(j=1,\cdots,2n),$$

$$\hat{\psi}_j^{\pm} = \tilde{\psi}_j^{\pm}q^{\mp 2\bar{S}},\quad\hat{\psi}_{\bar{j}}^{\pm} = \tilde{\psi}_{\bar{j}}^{\pm}q^{\pm2(\tilde{S}-\bar{S})},\quad(j=1,\cdots,n),$$

and set

$$\mathscr{L}_{A,j}(\zeta) = 1 + \zeta\mathbf{a}^*\tilde{\psi}_j^+ + \zeta\mathbf{a}\,\tilde{\psi}_j^- - \zeta^2q^{2D}\tilde{\psi}_j^+\tilde{\psi}_j^-,$$

$$\widehat{\mathscr{L}}_{A,j}(\zeta) = 1 - \zeta\mathbf{a}^*\hat{\psi}_j^+ - \zeta\mathbf{a}\,\hat{\psi}_j^- - \zeta^2q^{2D+2}\hat{\psi}_j^-\hat{\psi}_j^+.$$

Lemma B.1. *For $X\in\mathscr{V}^{(s)}$, the following hold.*

$$\iota\left(\zeta^{-\alpha+\mathbb{S}}\mathbf{Q}^+(\zeta,\alpha)(X)\right)\tag{B.1}$$

$$= \frac{1-q^{2(\alpha-s)}}{\prod_{l=1}^n(1-\zeta^2/\xi_l^2)}\mathrm{Tr}_A^+\left(q^{2(\alpha-s)D}\mathscr{L}_{A,1}(\zeta/\xi_1)\cdots\mathscr{L}_{A,2n}(\zeta/\xi_{2n})\right)\cdot\iota(X),$$

$$\iota\left(\zeta^{\alpha-\mathbb{S}}\mathbf{Q}^{*-}(\zeta,\alpha)(X)\right)\tag{B.2}$$

$$= -\frac{1-q^{-2(\alpha-s)}}{\prod_{l=1}^n(1-\zeta^2/\xi_l^2)}\mathrm{Tr}_A^-\left(\widehat{\mathscr{L}}_{A,n}(\zeta/\xi_n)\cdots\widehat{\mathscr{L}}_{A,1}(\zeta/\xi_1)\right.$$

$$\left.\times q^{-2\alpha D_A}\widehat{\mathscr{L}}_{A,\bar{1}}(\zeta/q\xi_1)\cdots\widehat{\mathscr{L}}_{A,\bar{n}}(\zeta/q\xi_n)q^{2sD_A}\right)\cdot\iota(X).$$

Proof. This can be shown by direct calculation noting that

$$i\zeta^{-1/2}q^{1/4}\mathscr{L}_{A,j}(\zeta) = q^{\Sigma_{l=j}^{2n}\sigma_l^3D}\cdot(\zeta-\zeta^{-1})L_{A,j}^+(\zeta)^{-1}\cdot q^{-\Sigma_{l=j+1}^{2n}\sigma_l^3D},$$

$$i\zeta^{-1/2}q^{-1/4}\widehat{\mathscr{L}}_{A,j}(\zeta)$$

$$= \begin{cases} q^{\Sigma_{l=j+1}^n\sigma_l^3\cdot D_A}\cdot L_{A,j}^+(\zeta)\cdot q^{-\Sigma_{l=j}^n\sigma_l^3\cdot D_A} & (1\le j\le n),\\ q^{(\Sigma_{l=j+1}^{2n}\sigma_l^3+\Sigma_{l=1}^n\sigma_l^3)D_A}\cdot L_{A,j}^+(\zeta)\cdot q^{-(\Sigma_{l=j}^{2n}\sigma_l^3+\Sigma_{l=1}^n\sigma_l^3)D_A} & (n+1\le j\le 2n). \end{cases}$$

$\square$

This rewriting is useful at $q=i$, when $\tilde{\psi}_j^{\pm}$, $\hat{\psi}_j^{\pm}$ become the Jordan-Wigner fermions on $V^{\otimes 2n}$. In the rest of this section we shall consider only this case.

B.2 Diagonalization of Q operators

Let us calculate the trace (B.1). First note the following simple fact.

Lemma B.2. *If q is a primitive r-th root of 1, the representation W^+ of Osc has an r'-dimensional quotient W_0^+ generated by $|0\rangle$, where*

$$r' = \begin{cases} r & \text{(r odd)},\\ r/2 & \text{(r even)}. \end{cases}$$

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We have the relation

$$\mathrm{Tr}^+\left(y^{2D_A}x\right) = \frac{1}{1-y^{2r'}}\mathrm{tr}_{W_0^+}\left(y^{2D_A}x\right) \qquad (x \in Osc).$$

When $q = i$, W_0^+ is two-dimensional with basis $|0\rangle, |1\rangle$. The q-oscillators are represented in this basis as

$$a \mapsto \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad a^* \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad i^{2D_A} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

while

$$\mathscr{L}_{A,k}(\zeta) \mapsto \begin{pmatrix} 1 - \zeta^2\,\tilde\psi_k^+\,\tilde\psi_k^- & 2\zeta\,\tilde\psi_k^- \\ \zeta\,\tilde\psi_k^+ & 1 + \zeta^2\,\tilde\psi_k^+\,\tilde\psi_k^- \end{pmatrix}.$$

Lemma B.3. *Let* η_j, η_j^* *($j = 1, \cdots, N$) be generators of a Grassmann algebra, and set*

$$L_j = \begin{pmatrix} 1 + \eta_j^*\eta_j/2 & \eta_j^* \\ \eta_j & 1 - \eta_j^*\eta_j/2 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}.$$

Then

$$\mathrm{tr}(L_N \cdots L_1 H) = (1+t)\exp\left(-\frac{t}{1+t}\sum_{j,k=1}^N \eta_k^*\eta_j + \frac{1}{2}\sum_{j=1}^N \eta_j^*\eta_j + \sum_{j<k}\eta_k^*\eta_j\right). \tag{B.3}$$

Proof. First consider the case $t = 0$. Extracting the factor $e^{\frac{1}{2}\sum_{j=1}^N \eta_j^*\eta_j}$, we are to show that

$$\mathrm{tr}\left(L_N' \cdots L_1' \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = e^{\sum_{j<k}\eta_k^*\eta_j} \tag{B.4}$$

where $L_j' = \begin{pmatrix} 1 & \eta_j^* \\ \eta_j & 1 - \eta_j^*\eta_j \end{pmatrix}$. Denote the left hand side of (B.4) by x_N. Using

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}L_N' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & \eta_N^* \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \eta_N^* \\ 0 & 1 \end{pmatrix}L_j'\begin{pmatrix} 1 & -\eta_N^* \\ 0 & 1 \end{pmatrix} = e^{\eta_N^*\eta_j}L_j',$$

we find a recurrence relation $x_N = e^{\eta_N^*\sum_{j=1}^{N-1}\eta_j}x_{N-1}$. Eq. (B.4) follows from this.

In general, (B.3) is linear in t and the coefficient of t is obtained from $t = 0$ by exchanging the roles of η_j^* and η_j. Combining them we obtain

$$\exp\left(\frac{1}{2}\sum_{j=1}^N \eta_j^*\eta_j + \sum_{j<k}\eta_k^*\eta_j\right) + t\exp\left(\frac{1}{2}\sum_{j=1}^N \eta_j\eta_j^* + \sum_{j<k}\eta_k\eta_j^*\right)$$

$$= \left(1 + t(1 - \sum_{j,k=1}^N \eta_k^*\eta_j)\right)\exp\left(\frac{1}{2}\sum_{j=1}^N \eta_j^*\eta_j + \sum_{j<k}\eta_k^*\eta_j\right)$$

$$= (1+t)\exp\left(-\frac{t}{1+t}\sum_{j,k=1}^N \eta_k^*\eta_j\right)\exp\left(\frac{1}{2}\sum_{j=1}^N \eta_j^*\eta_j + \sum_{j<k}\eta_k^*\eta_j\right).$$

Lemma is proved.   □

Now set

$$\phi_k^\pm = -i\tilde{\psi}_k^\pm + \tilde{\psi}_{\bar{k}}^\pm,$$

$$\phi_k^{*\pm}(\alpha) = \frac{1}{i^{\pm\alpha} - i^{\mp\alpha}} \left(i^{\pm\alpha+1} \tilde{\psi}_k^\pm - i^{\mp\alpha} \tilde{\psi}_{\bar{k}}^\pm \right).$$

They are related to the fermions $\Psi_k^\pm, \Phi_k^\pm(\alpha)$ via

$$\phi_k^\pm \iota(X) = (-1)^s \iota \left(\Psi_k^\pm(X) \right),$$

$$\phi_k^{*\pm}(\alpha)\iota(X) = (-1)^{s+1} \iota \left(\Phi_k^\pm(\alpha+s-1)(X) \right),$$

where $X \in \mathscr{V}^{(s)}$.

Lemma B.4. *We have*

$$\frac{1-i^{2\alpha}}{\prod_{l=1}^n (1-\zeta^2/\xi_l^2)} \mathrm{Tr}_A^+ \left(i^{2\alpha D_A} \mathscr{L}_{A,1}(\zeta/\xi_1) \cdots \mathscr{L}_{A,2n}(\zeta/\xi_{2n}) \right) =$$

$$\exp \Big[\sum_{j,k=1}^n \left(\log(1+\zeta^2 M)_{jk} \phi_j^{*+}(\alpha+1)\phi_k^- - \log(1-\zeta^2 M)_{jk} \phi_j^{*-}(\alpha+1)\phi_k^+ \right) \Big],$$

where M is an upper triangular matrix with entries

$$M_{jk} = \begin{cases} 2\xi_j^{-1}\xi_k^{-1} & (j<k), \\ \xi_j^{-2} & (j=k), \\ 0 & (j>k). \end{cases}$$

Proof. We compute the trace first in the normal-ordered form, where normal ordering means that we bring all $\tilde{\psi}_k^+$ to the left and $\tilde{\psi}_k^-$ to the right. Taking $N = 2n$, $t = i^{2\alpha}$ and

$$\eta_k^* = \frac{2\zeta}{\xi_k} \tilde{\psi}_k^-, \quad \eta_k = \frac{\zeta}{\xi_k} \tilde{\psi}_k^+,$$

we apply formula (B.3) under the normal ordering symbol : :. The result is

$$(1-i^{2\alpha})\mathrm{Tr}_A^+ \left(i^{2\alpha D_A} \mathscr{L}_{A,1}(\zeta/\xi_1) \cdots \mathscr{L}_{A,2n}(\zeta/\xi_{2n}) \right)$$

$$=: \exp \sum_{j,k=1}^n \zeta^2 M_{jk} \left(\phi_j^{*+}(\alpha+1)\phi_k^- - \phi_k^+ \phi_j^{*-}(\alpha+1) \right) : .$$

Due to the formula

$$: \exp \Big(\sum_{j,k=1}^{2n} A_{jk} \tilde{\psi}_j^+ \tilde{\psi}_k^- \Big): = \exp \Big(\sum_{j,k=1}^{2n} \left(\log(I+A) \right)_{jk} \tilde{\psi}_j^+ \tilde{\psi}_k^- \Big),$$

the right hand side is rewritten as

$$\exp \left(\sum_{j,k=1}^n \left(\log(1+\zeta^2 M)_{jk} \phi_j^{*+}(\alpha+1)\phi_k^- + \log(1-\zeta^2 M)_{jk} \phi_k^+ \phi_j^{*-}(\alpha+1) \right) \right)$$

$$= \det(1-\zeta^2 M)$$

$$\times \exp \left(\sum_{j,k=1}^n \left(\log(1+\zeta^2 M)_{jk} \phi_j^{*+}(\alpha+1)\phi_k^- - \log(1-\zeta^2 M)_{jk} \phi_j^{*-}(\alpha+1)\phi_k^+ \right) \right).$$

Lemma follows from this. $\square$

Lemma B.5. *Set* $\mathscr{T} = \widehat{\mathscr{L}}_{A,n}(\zeta/\xi_n)\cdots\widehat{\mathscr{L}}_{A,1}(\zeta/\xi_1)$ *and* $\overline{\mathscr{T}} = \widehat{\mathscr{L}}_{A,\bar{1}}(\zeta/i\xi_1)\cdots\widehat{\mathscr{L}}_{A,\bar{n}}(\zeta/i\xi_n)$. *Then we have on* $\iota(\mathscr{V}^{(s)})$

$$-\frac{1 - i^{-2(\alpha-s)}}{\prod_{l=1}^{n}(1 - \zeta^2/\xi_l^2)} i^{2\overline{S}} \cdot \mathrm{Tr}_A^- \left(\mathscr{T} \, i^{-2\alpha D_A} \, \overline{\mathscr{T}} \, i^{2sD_A} \right) \cdot i^{-2\overline{S}}$$

$$= \exp\Big[\sum_{j,k=1}^{n} \big(\log(1 + \zeta^2 M)_{jk}\phi_k^-\,\phi_j^{*+}(\alpha-s+1) - \log(1 - \zeta^2 M)_{jk}\phi_k^+\,\phi_j^{*-}(\alpha-s+1) \big) \Big].$$

Proof. Let $\mathscr{T}_{even}$ (resp. $\mathscr{T}_{odd}$) be the sum of terms in $\mathscr{T}$ containing an even (resp. odd) number of fermions, and similarly for $\overline{\mathscr{T}}$. Since the total spin is preserved, we have

$$\mathrm{Tr}_A^- \left(\mathscr{T} \, i^{-2\alpha D_A} \, \overline{\mathscr{T}} \, i^{2sD_A} \right) = \mathrm{Tr}_A^- \left(\mathscr{T}_{even} i^{-2\alpha D_A} \overline{\mathscr{T}}_{even} i^{2sD_A} \right) + \mathrm{Tr}_A^- \left(\mathscr{T}_{odd} i^{-2\alpha D_A} \overline{\mathscr{T}}_{odd} i^{2sD_A} \right)$$

$$= \mathrm{Tr}_A^- \left((\overline{\mathscr{T}}_{even} - \overline{\mathscr{T}}_{odd}) i^{2sD_A} \mathscr{T} i^{-2\alpha D_A} \right)$$

$$= i^{2\overline{S}} \cdot \mathrm{Tr}_A^- \left(\overline{\mathscr{T}} i^{2sD_A} \mathscr{T} i^{-2\alpha D_A} \right) \cdot i^{-2\overline{S}}.$$

By using $\widehat{\mathscr{L}}_{A,\bar{j}}(\zeta/i\xi_j) = i^{-\sigma_j^3} \widehat{\mathscr{L}}_{A,\bar{j}}(i\zeta/\xi_j) i^{\sigma_j^3}$, the right hand side can be rewritten further as

$$i^{2s\overline{S}}\mathrm{Tr}_A^- \left(i^{-2(\alpha-s)D_A} \widehat{\mathscr{L}}_{A,\bar{1}}(i\zeta/\xi_1)\cdots\widehat{\mathscr{L}}_{A,\bar{n}}(i\zeta/\xi_n)\widehat{\mathscr{L}}_{A,n}(\zeta/\xi_n)\cdots\widehat{\mathscr{L}}_{A,1}(\zeta/\xi_1) \right) i^{-2s\overline{S}}.$$

The action of Osc on W_0^+ factors through that of the quotient algebra Osc_0 by the relation $i^{4D} = 1$. Let θ be the anti-automorphism of the latter given by $\theta(a) = -a$, $\theta(a^*) = -a^*$, $\theta(i^D) = i^{1-D}$. Denote also by θ the anti-algebra map of Clifford algebras sending $\tilde{\psi}_j^\pm$ to $\hat{\psi}_j^\pm$ ($j = 1,\cdots,2n$). It is easy to check that

$$\theta(\mathscr{L}_{A,j}(\zeta)) = \widehat{\mathscr{L}}_{A,j}(\zeta),$$

$$\theta\big((1 - i^{2\alpha})\mathrm{Tr}_A^+(i^{2\alpha D_A}x)\big) = -(1 - i^{-2\alpha})\mathrm{Tr}_A^-(i^{-2\alpha D_A}\theta(x)),$$

where $x \in Osc_0$. Using these we obtain

$$-\frac{1 - i^{-2(\alpha-s)}}{\prod_{l=1}^{n}(1 - \zeta^2/\xi_l^2)} i^{2\overline{S}} \cdot \mathrm{Tr}_A^- \left(\mathscr{T} \, i^{-2\alpha D_A} \, \overline{\mathscr{T}} \, i^{2sD_A} \right) \cdot i^{-2\overline{S}}$$

$$= i^{2(s+1)\overline{S}}\theta\Big(\frac{1 - i^{2(\alpha-s)}}{\prod_{l=1}^{n}(1 - \zeta^2/\xi_l^2)} \mathrm{Tr}_A^+ \big(i^{2(\alpha-s)D_A} \mathscr{L}_{A,1}(\zeta/\xi_1)\cdots\mathscr{L}_{A,2n}(\zeta/\xi_{2n}) \big) \Big) i^{-2(s+1)\overline{S}}.$$

The assertion follows from Lemma B.4 and the relation

$$i^{2(s+1)\overline{S}}\theta(\phi_j^{*\pm}(\alpha)\phi_k^\mp)i^{-2(s+1)\overline{S}}\big|_{\iota(\mathscr{V}^{(s)})} = \phi_k^\mp\phi_j^{*\pm}(\alpha)\big|_{\iota(\mathscr{V}^{(s)})}$$

$\square$

Proof of Theorem 3.5. We apply Lemma B.4 to Lemma B.1, replacing α by $\alpha - s$ and noting that $\phi_j^{*\pm}(\alpha)\phi_k^\mp\iota(X) = \iota\big(\Phi_j^\pm(\alpha+s)\Psi_k^\mp(X)\big)$ for $X \in \mathscr{V}^{(s)}$. We find

$$\zeta^{-\alpha+\mathbb{S}}\mathbf{Q}^+(\zeta,\alpha)$$

$$= \exp\Bigg(\sum_{j,k=1}^{n} \big(\log(1 + \zeta^2 M)_{jk}\Phi_j^+(\alpha+1)\Psi_k^- - \log(1 - \zeta^2 M)_{jk}\Phi_j^-(\alpha+1)\Psi_k^+ \big) \Bigg).$$

This expression can be further simplified by diagonalizing the matrix M as

$$PMP^{-1} = \text{diag}(\xi_1^{-2}, \cdots, \xi_n^{-2}),$$

where P is the matrix given in (3.7). We thus obtain the result

$$
\begin{aligned}
&\zeta^{-\alpha+\mathbb{S}}\mathbf{Q}^+(\zeta,\alpha) \\
&= \exp\left(\sum_{k=1}^n \left(\log(1+\zeta^2/\xi_k^2)\widehat{\Phi}_k^+(\alpha+1)\widehat{\Psi}_k^- - \log(1-\zeta^2/\xi_k^2)\widehat{\Phi}_k^-(\alpha+1)\widehat{\Psi}_k^+\right)\right),
\end{aligned}
$$

which is equivalent to (3.13).

Similarly, (3.15) can be shown by Lemma B.1 and Lemma B.5.

Finally, formulas (3.12),(3.14) follow from (3.13), (3.15) and the TQ relations (2.2), (2.6). $\quad\square$

B.3 A factorization

Let us proceed to the calculation of $\mathbf{b}(\zeta), \mathbf{c}(\zeta)$. A simplifying feature about the free fermion point is a factorization property.

Proposition B.6. *The following factorization takes place:*

$$\mathbf{k}^\pm(\zeta,\alpha) = \widehat{\Psi}^\pm(\zeta)\mathbf{Q}^\pm(-i\zeta,\alpha\mp 1)i^{\pm\alpha}, \tag{B.5}$$

$$\mathbf{k}^{*\pm}(\zeta,\alpha) = -i\widehat{\Phi}^\pm(i\zeta)(i^{\alpha-\mathbb{S}}+i^{-\alpha+\mathbb{S}})\mathbf{Q}^\pm(\zeta,\alpha). \tag{B.6}$$

Proof. First consider the action of $\mathbf{k}^+(\zeta,\alpha)$ on $X \in \mathscr{V}^{(s)}$. Inserting (2.5) in the definition (2.3) and taking the trace over V_a, we obtain $2n$ terms:

$$
\begin{aligned}
&\text{tr}_a\left(\sigma_a^+ T_{\{A,a\}}^+(\zeta)^{-1} X\, T_{\{A,a\}}^+(\zeta)\right) \\
&= -\sum_{j=1}^n \gamma(\zeta/i\xi_j)\prod_{l=1}^{j-1}\beta(\zeta/\xi_l)^{-1}i^{-\sigma_1^3/2}L_{A,1}^+(i\zeta/\xi_1)^{-1}\cdots\sigma_j^+ i^{-2D_A-1/2}L_{A,j}^+(\zeta/i\xi_j)^{-1} \\
&\quad\times\cdots i^{\sigma_n^3/2}L_{A,n}^+(\zeta/i\xi_n)^{-1}\cdot XL_{A,n}^+(\zeta/i\xi_n)i^{-\sigma_n^3/2}\cdots L_{A,1}^+(\zeta/i\xi_1)i^{-\sigma_1^3/2} \\
&\quad + \sum_{j=1}^n \gamma(\zeta/\xi_j)\prod_{l=1}^j\beta(\zeta/\xi_l)^{-1}i^{-\sigma_1^3/2}L_{A,1}^+(i\zeta/\xi_1)^{-1}\cdots i^{-\sigma_n^3/2}L_{A,n}^+(i\zeta/\xi_n)^{-1} \\
&\quad\times XL_{A,n}^+(i\zeta/\xi_n)i^{-\sigma_n^3/2}\cdots i^{-\sigma_j^3}L_{A,j}^+(\zeta/i\xi_j)\sigma_j^+ i^{-2D_A+1/2}\cdots L_{A,1}^+(\zeta/i\xi_1)i^{-\sigma_1^3/2},
\end{aligned}
$$

where in the second sum we have used

$$L_{A,j}^+(i\zeta/\xi_j)\sigma_j^+ = i^{-\sigma_j^3}L_{A,j}^+(\zeta/i\xi_j)\sigma_j^+.$$

Rewriting this expression in terms of fermions, we obtain

$$
\begin{aligned}
&\iota\left(\text{tr}_a\left(\sigma_a^+ T_{\{A,a\}}^+(\zeta)^{-1} X\, T_{\{A,a\}}^+(\zeta)\right)\right) \\
&= \sum_{j=1}^n \frac{2i\zeta/\xi_j}{1-\zeta^2/\xi_j^2}\frac{1}{\prod_{l=1}^{j-1}(1-\zeta^2/\xi_l^2)\prod_{l=j}^n(1+\zeta^2/\xi_l^2)}\cdot i^{-(2D+1)\widetilde{S}} \\
&\quad\times\left(\mathscr{L}_{A,1}(i\zeta/\xi_1)\cdots\mathscr{L}_{A,j-1}(i\zeta/\xi_{j-1})\widetilde{\psi}_j^+ \mathscr{L}_{A,j}(\zeta/i\xi_j)\cdots\mathscr{L}_{A,n}(\zeta/i\xi_n)\mathscr{L}_{A,\bar{n}}(\zeta/\xi_n)\cdots\mathscr{L}_{A,\bar{1}}(\zeta/\xi_1)\right. \\
&\quad\left.+ i\mathscr{L}_{A,1}(i\zeta/\xi_1)\cdots\mathscr{L}_{A,n}(i\zeta/\xi_n)\mathscr{L}_{A,\bar{n}}(-\zeta/\xi_n)\cdots\widetilde{\psi}_{\bar{j}}^+\mathscr{L}_{A,\bar{j}}(\zeta/\xi_j)\cdots\mathscr{L}_{A,\bar{1}}(\zeta/\xi_1)\right)\cdot\iota(X).
\end{aligned}
$$

Since

$$\mathscr{L}_{A,k}(\zeta)\tilde{\psi}_j^{\pm} = \tilde{\psi}_j^{\pm}\mathscr{L}_{A,k}(-\zeta) \quad (j \neq k),$$

we can bring $\tilde{\psi}_j^+$ to the leftmost place. The result factorizes into a linear form in the fermion

$$\sum_{j=1}^{n} \frac{2\zeta/\xi_j}{1-\zeta^2/\xi_j^2} \prod_{l=1}^{j-1} \frac{1+\zeta^2/\xi_l^2}{1-\zeta^2/\xi_l^2} \left(i\tilde{\psi}_j^+ - \tilde{\psi}_{\bar{j}}^+\right) \tag{B.7}$$

followed by the product

$$\prod_{l=1}^{n} \frac{1}{1+\zeta^2/\xi_j^2} \cdot i^{-(2D_A+1)(\widetilde{S}+1)} \mathscr{L}_{A,1}(-i\zeta/\xi_1)\cdots\mathscr{L}_{A,2n}(-i\zeta/\xi_{2n}) \cdot \iota(X). \tag{B.8}$$

Multiply both sides by $\zeta^{\alpha-\mathbb{S}}(1-i^{2(\alpha-\mathbb{S})})i^{2\alpha D_A}$ and take the trace Tr_A^+. In view of the relations

$$i\tilde{\psi}_j^{\pm} \cdot \iota(X) = (-1)^{s+1} \cdot \iota\left(\psi_j^{\pm} X\right), \quad -\tilde{\psi}_{\bar{j}}^{\pm} \cdot \iota(X) = \iota\left(X\psi_j^{\pm}\right),$$

the piece (B.7) yields $(-1)^{s+1}\widehat{\Psi}^+(\zeta)$. On the other hand, due to (B.1), trace of (B.8) (taken together with the prefactor) gives rise to $(-1)^{s+1}i^{\alpha}\mathbf{Q}^+(-i\zeta, \alpha-1)(X)$. Combining these we obtain (B.5).

Similarly, using

$$\left(i^{s-\alpha}\tilde{\psi}_j^- - i^{-s+\alpha+1}\tilde{\psi}_{\bar{j}}^-\right)\iota(X) = (i^{-\alpha-s+1} - i^{\alpha+s-1})\iota\left(\Phi_j^-(\alpha)(X)\right),$$

we compute

$$\mathbf{k}^{*-}(\zeta,\alpha)(X) = -i\widehat{\Phi}^-(i\zeta,\alpha)(1+i^{2(\alpha-\mathbb{S})})\prod_{l=1}^{n} \frac{1+\zeta^2/\xi_l^2}{1-\zeta^2/\xi_l^2} \cdot i^{-2S}\mathbf{Q}^{*-}(i\zeta,\alpha)(i^{2S}X).$$

On the other hand, (3.13) and (3.15) imply

$$\prod_{l=1}^{n} \frac{1+\zeta^2/\xi_l^2}{1-\zeta^2/\xi_l^2} \cdot i^{-2S}\mathbf{Q}^{*-}(i\zeta,\alpha)(i^{2S}X) = i^{-\alpha+\mathbb{S}}\mathbf{Q}^-(\zeta,\alpha)(X).$$

The assertion follows from these. $\qquad\square$

Proof of Theorem 3.6. From Proposition B.6 and Theorem 3.5, it is simple to calculate the residues of $\mathbf{k}^{\pm}(\zeta,\alpha)$. We have

$$\mathrm{res}_{\zeta=\xi_k}\mathbf{k}^{\pm}(\zeta,\alpha)\frac{d\zeta}{\zeta} = -\frac{1}{2}\widehat{\Psi}_k^{\pm} \cdot U_k^{\pm}(\alpha)(1-N_k^{\pm}(\alpha)) \cdot i^{\pm\mathbb{S}+1}\xi_k^{\pm(\alpha-\mathbb{S})-1}.$$

After simplification using $1-N_k^{\pm} = \chi_k^{\mp}\chi_k^{*\pm}$, we obtain the desired expression for $\mathbf{c}_k^{\pm}$. $\qquad\square$

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