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An Introduction to Warped Dimensions and Holography

Tony Gherghetta*

School of Physics and Astronomy, University of Minnesota, Minneapolis, MN 55455, USA

E-mail: tgher@umn.edu

I briefly review the theoretical tools required to construct models in warped extra dimensions. This includes how to localise zero modes in the warped bulk and how to obtain the holographic interpretation using the AdS/CFT correspondence. The formalism is then applied to describe the Standard Model Yukawa coupling hierarchies both from the 5D gravity and 4D holographic point of view.

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Warped extra dimensions have provided a new framework for addressing the hierarchy problem in extensions of the Standard Model. Many different applications of this idea have been applied to areas ranging from Higgs physics, supersymmetry and even QCD. This will not be an extensive review of the different applications but instead will outline the basic theoretical tools necessary to construct models in a warped extra dimension and the corresponding 4D holographic interpretation. A simple application to Standard Model Yukawa couplings will be used to illustrate that all warped model constructions can be given a purely 4D description with the warped extra dimension merely being a calculational tool. Further details and references to the literature can be found in Ref. [1].

1. Bulk fields in a slice of AdS_5

Consider a 5D spacetime with the AdS_5 metric

$$ds^2 = e^{-2ky}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \equiv g_{MN}dx^M dx^N, \tag{1.1}$$

where k is the AdS curvature scale. The spacetime indices $M = (\mu, 5)$ where $\mu = 0,1,2,3$ and $\eta_{\mu\nu} = \mathrm{diag}(-+++)$ is the Minkowski metric. The fifth dimension y is compactified on a Z_2 orbifold with a UV (IR) brane located at the orbifold fixed points $y^* = 0(\pi R)$. Between these two three-branes the metric (1.1) is a solution to Einstein's equations provided the bulk cosmological constant and the brane tensions are appropriately tuned. This slice of AdS_5 is the Randall-Sundrum solution [2] (RS1) and is geometrically depicted in Fig.1.

In RS1 the Standard Model particle states are confined to the IR brane. The hierarchy problem is then solved by noting that generic mass scales M in the 5D theory are scaled down to $Me^{-\pi kR}$ on the IR brane at $y = \pi R$. However on the IR brane higher-dimension operators with dimension greater than four, such as those associated with proton decay, flavour changing neutral currents

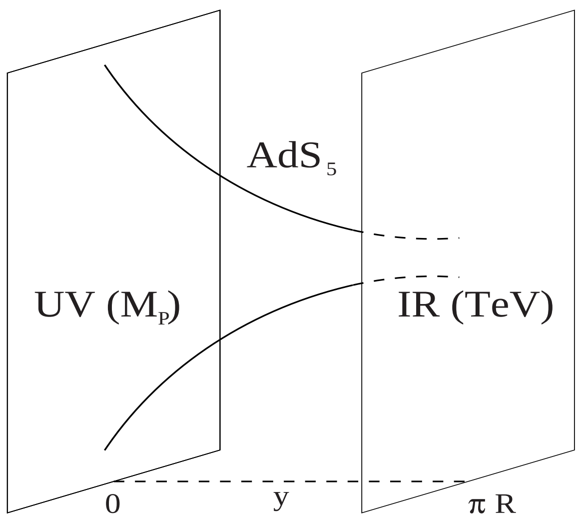

Figure 1: A slice of AdS_5: The Randall-Sundrum scenario.

(FCNC) and neutrino masses will now also be suppressed by the warped down scale

$$\frac{1}{M_5^2}\bar{\Psi}_i\Psi_j\bar{\Psi}_k\Psi_l \rightarrow \frac{1}{(M_5 e^{-\pi kR})^2}\bar{\Psi}_i\Psi_j\bar{\Psi}_k\Psi_l \,, \tag{1.2}$$

$$\frac{1}{M_5}\nu\nu HH \rightarrow \frac{1}{M_5 e^{-\pi kR}}\nu\nu HH \,, \tag{1.3}$$

where Ψ_i is a Standard Model fermion and ν is the neutrino. This leads to generic problems with proton decay and FCNC effects, and also neutrino masses are no longer consistent with experiment. Instead in the slice of AdS$_5$ with the Standard Model fields confined on the IR brane one has to resort to discrete symmetries to forbid the offending higher-dimension operators. Of course it is not adequate just to forbid the leading higher-dimension operators. Since the cutoff scale on the IR brane is low ($\mathscr{O}$(TeV)), successive higher-dimension operators must also be eliminated to very high order.

This feature of RS1 stems from the fact that all Standard Model particles are localised on the IR brane. However to address the hierarchy problem, only the Higgs field needs to be localised on the IR brane. The Standard Model fermions and gauge fields do not have a hierarchy problem and therefore can be placed anywhere in the bulk [3, 4, 5]. In this way the UV brane can be used to provide a sufficiently high scale to help suppress higher-dimension operators while still solving the hierarchy problem [4].

Therefore let us introduce fermion Ψ, scalar Φ and vector A_M bulk fields in a slice of AdS$_5$ with 5D action given by

$$S_5 = -\int d^4x\, dy\, \sqrt{-g}\left[\frac{1}{2}M_5^3 R + \Lambda_5 + \frac{1}{4g_5^2}F_{MN}^2 + |D_M\Phi|^2 + i\bar{\Psi}\Gamma^M\nabla_M\Psi \right.$$
$$\left. + \; m_\phi^2|\Phi|^2 + im_\psi\bar{\Psi}\Psi\right] \,, \tag{1.4}$$

where M_5 is the 5D fundamental scale, Λ_5 is the bulk cosmological constant and g_5 is the 5D gauge coupling. In curved space the gamma matrices are $\Gamma_M = e_M^A\gamma_A$, where e_M^A is the funfbein defined by $g_{MN} = e_M^A e_N^B \eta_{AB}$ and $\gamma_A = (\gamma_\alpha, \gamma_5)$ are the usual gamma matrices in flat space. The curved space covariant derivative $\nabla_M = D_M + \omega_M$, where ω_M is the spin connection and D_M is the gauge covariant derivative for fermion and/or scalar fields charged under some gauge symmetry. The action (1.4) also includes a mass term m_ϕ for the bulk scalar and a mass term m_ψ for the bulk fermion which are consistent with gauge symmetries.

1.1 Scalar fields

Consider a bulk scalar field with mass squared $m_\phi^2 = ak^2$ where we have defined the bulk scalar mass in units of the curvature scale k and dimensionless coefficient a. The equation of motion derived from the scalar part of the variation of the action (1.4) is

$$\partial^2\Phi + e^{2ky}\partial_5(e^{-4ky}\partial_5\Phi) - m_\phi^2 e^{-2ky}\Phi = 0, \tag{1.5}$$

where $\partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu$. The zero mode solution of this equation is obtained by assuming a separation of variables

$$\Phi(x,y) = \frac{1}{\sqrt{\pi R}}\sum_n \Phi^{(n)}(x)\phi^{(n)}(y) \,, \tag{1.6}$$

where $\Phi^{(n)}$ are the Kaluza-Klein modes satisfying $\partial^2\Phi^{(n)}(x) = m_n^2\Phi^{(n)}(x)$ and $\phi^{(n)}(y)$ is the profile of the Kaluza-Klein mode in the bulk. The general solution for the zero mode ($m_0 = 0$) is given by

$$\phi^{(0)}(y) = c_1\, e^{(2-\alpha)ky} + c_2\, e^{(2+\alpha)ky}\,, \tag{1.7}$$

where $\alpha \equiv \sqrt{4+a}$ and c_1, c_2 are arbitrary constants. In general there is no zero mode solution for simple Neumann or Dirichlet boundary conditions. Instead in order to obtain a zero mode we need to modify the boundary action and include boundary mass terms [4]

$$S_{bdy} = -\int d^4x\, dy\, \sqrt{-g}\, 2\, b\, k\, [\delta(y) - \delta(y-\pi R)]\, |\Phi|^2\,, \tag{1.8}$$

where b is a dimensionless constant parametrising the boundary mass in units of k. The Neumann boundary conditions are now modified to

$$\left(\partial_5\phi^{(0)} - b\, k\, \phi^{(0)}\right)\Big|_{0,\pi R} = 0. \tag{1.9}$$

Imposing the modified Neumann boundary conditions at $y^* = 0, \pi R$ leads to a zero mode solution

$$\phi^{(0)}(y) \propto e^{bky}\,, \tag{1.10}$$

where $b = 2 \pm \alpha$. Assuming α to be real (which requires $a \geq -4$ in accord with the Breitenlohner-Freedman bound [6] for the stability of AdS space), the parameter b has a range $-\infty < b < \infty$. The localisation features of the zero mode follows from considering the kinetic term

$$-\int d^5x\, \sqrt{-g}\, g^{\mu\nu}\, \partial_\mu\Phi^*\partial_\nu\Phi + \ldots = -\int d^5x\, e^{2(b-1)ky}\, \eta^{\mu\nu}\, \partial_\mu\Phi^{(0)*}(x)\partial_\nu\Phi^{(0)}(x) + \ldots \tag{1.11}$$

Hence, with respect to the 5D flat metric the zero mode profile is given by

$$\widetilde{\phi}^{(0)}(y) \propto e^{(b-1)ky} = e^{(1\pm\sqrt{4+a})ky}\,. \tag{1.12}$$

We see that for $b < 1$ ($b > 1$) the zero mode is localised towards the UV (IR) brane and when $b = 1$ the zero mode is flat. Therefore using the one remaining free parameter b the scalar zero mode can be localised anywhere in the bulk.

The differential equation (1.5) is actually a classical Sturm-Liouville equation which can be written in the form

$$-\frac{d}{dy}\left(p(y)\frac{d\phi^{(n)}}{dy}\right) + q(y)\phi^{(n)} = \lambda_n w(y)\phi^{(n)}\,, \tag{1.13}$$

where $p(y) = e^{-4ky}$, $q(y) = m_\phi^2 e^{-4ky}$, $w(y) = e^{-2ky}$ and the eigenvalues $\lambda_n = m_n^2$. From general results in Sturm-Liouville theory we know that since $p(y)$ is differentiable, $q(y)$ and $w(y)$ are continuous, $p(y) > 0$ and $w(y) > 0$ over the interval $[0, \pi R]$, the eigenvalues λ_n are real and well-ordered i.e. $\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \to \infty$. Furthermore the eigenfunctions $\phi^{(n)}(y)$ form a complete set and satisfy the orthonormal relation

$$\frac{1}{\pi R}\int_0^{\pi R} dy\, w(y)\, \phi^{(n)}\phi^{(m)} = \delta_{nm}\,. \tag{1.14}$$

The general solution of the Kaluza-Klein modes corresponding to $m_n \neq 0$ is given by

$$\phi^{(n)}(y) = e^{2ky}\left[c_1 J_\alpha\left(\frac{m_n}{ke^{-ky}}\right) + c_2 Y_\alpha\left(\frac{m_n}{ke^{-ky}}\right)\right] , \tag{1.15}$$

where $c_{1,2}$ are arbitrary constants. The Kaluza-Klein masses (or eigenvalues) are determined by imposing the boundary conditions and in the limit $\pi kR \gg 1$ lead to the approximate values

$$m_n \approx \left(n + \frac{1}{2}\sqrt{4+a} - \frac{3}{4}\right)\pi k\, e^{-\pi kR} , \tag{1.16}$$

for $n = 1, 2, \ldots$. The fact that the Kaluza-Klein mass scale is associated with the IR scale ($ke^{-\pi kR}$) is consistent with the fact that the Kaluza-Klein modes are localised near the IR brane, and unlike the zero mode can not be arbitrarily localised in the bulk.

1.2 Fermions

Consider next bulk fermions in a slice of AdS$_5$ [7, 4]. In five dimensions an irreducible spinor representation has four components, so fermions are described by Dirac spinors Ψ. Under the Z_2 symmetry $y \to -y$ a fermion transforms (up to a phase $\pm$) as

$$\Psi(-y) = \pm\gamma_5 \Psi(y) , \tag{1.17}$$

so that $\bar{\Psi}\Psi$ is odd. Since only invariant (or even) terms under the Z_2 symmetry can be added to the bulk Lagrangian the corresponding mass parameter for a fermion must necessarily be odd and given by

$$m_\Psi = c\, k\, (\varepsilon(y) - \varepsilon(y - \pi R)) , \tag{1.18}$$

where c is a dimensionless mass parameter and $\varepsilon(y) = y/|y|$. The corresponding equation of motion for fermions resulting from the action (1.4) is

$$\begin{aligned} e^{ky}\gamma^\mu \partial_\mu \widehat{\Psi}_- + \partial_5 \widehat{\Psi}_+ + m_\Psi \widehat{\Psi}_+ &= 0 , \\ e^{ky}\gamma^\mu \partial_\mu \widehat{\Psi}_+ - \partial_5 \widehat{\Psi}_- + m_\Psi \widehat{\Psi}_- &= 0 , \end{aligned} \tag{1.19}$$

where $\widehat{\Psi} = e^{-2ky}\Psi$ and $\Psi_\pm$ are the components of the Dirac spinor $\Psi = \Psi_+ + \Psi_-$ with $\Psi_\pm = \pm\gamma_5\Psi_\pm$. Note that the equation of motion (1.19) is now a first order coupled equation between the components of the Dirac spinor Ψ.

The solutions of the bulk fermion equation of motion (1.19) are again obtained by separating the variables

$$\Psi_\pm(x,y) = \frac{1}{\sqrt{\pi R}} \sum_n \Psi_\pm^{(n)}(x)\, \psi_\pm^{(n)}(y) , \tag{1.20}$$

where $\Psi_\pm^{(n)}$ are the Kaluza-Klein modes satisfying $\eta^{\mu\nu}\gamma_\mu\partial_\nu\Psi_\pm^{(n)} = m_n\Psi_\pm^{(n)}$. The zero mode solutions can be obtained for $m_n = 0$ and the general solution of (1.19) is given by

$$\widehat{\psi}_\pm^{(0)}(y) = d_\pm\, e^{\mp cky} , \tag{1.21}$$

where $d_\pm$ are arbitrary constants. The Z_2 symmetry implies that one of the components $\psi_\pm$ must always be odd. If $\gamma_5 = \mathrm{diag}(1,-1)$, then (1.17) implies that $\psi_\mp$ is odd and there is no corresponding zero mode for this component of Ψ. In fact this is how 4D chirality is recovered from the vectorlike 5D bulk and is the result of compactifying on the Z_2 orbifold. For the remaining zero mode the boundary condition with the boundary mass term (1.18) is the modified Neumann condition

$$\left(\partial_5 \widehat{\psi}_\pm^{(0)} \pm c\, k\, \widehat{\psi}_\pm^{(0)}\right)\Big|_{0,\pi R} = 0\,. \tag{1.22}$$

Thus there will always be a zero mode since the boundary condition is trivially the same as the equation of motion. For concreteness let us choose ψ_- to be odd, then the only nonvanishing zero mode component of Ψ is

$$\psi_+^{(0)}(y) \propto e^{(2-c)ky}\,. \tag{1.23}$$

Again the localisation features of this mode are obtained by considering the kinetic term

$$\begin{aligned} &-\int d^5x\, \sqrt{-g}\; i\bar{\Psi}\Gamma^\mu \partial_\mu \Psi + \ldots \\ &= -\int d^5x\; e^{2(\frac{1}{2}-c)ky}\; i\bar{\Psi}_+^{(0)}(x)\gamma^\mu \partial_\mu \Psi_+^{(0)}(x) + \ldots\,. \end{aligned} \tag{1.24}$$

Hence with respect to the 5D flat metric the fermion zero mode profile is

$$\widetilde{\psi}_+^{(0)}(y) \propto e^{(\frac{1}{2}-c)ky}\,. \tag{1.25}$$

When $c > 1/2$ ($c < 1/2$) the fermion zero mode is localised towards the UV (IR) brane while the zero mode fermion is flat for $c = 1/2$. So, just like the scalar field zero mode, the fermion zero mode can be localised anywhere in the 5D bulk.

The nonzero Kaluza-Klein fermion modes can be obtained by solving the coupled equations of motion for the Dirac components $\psi_\pm^{(n)}$. This solution can be included as part of one general expression for all types of bulk fields [4]

$$f^{(n)}(y) = \frac{e^{\frac{s}{2}ky}}{N_n}\left[J_\alpha\left(\frac{m_n}{ke^{-ky}}\right) + b_\alpha Y_\alpha\left(\frac{m_n}{ke^{-ky}}\right)\right]\,, \tag{1.26}$$

for $f^{(n)} = (\phi^{(n)}, \widehat{\psi}_\pm^{(n)}, A_\mu^{(n)})$ where

$$b_\alpha = -\frac{(-r+\frac{s}{2})J_\alpha(\frac{m_n}{k}) + \frac{m_n}{k}J'_\alpha(\frac{m_n}{k})}{(-r+\frac{s}{2})Y_\alpha(\frac{m_n}{k}) + \frac{m_n}{k}Y'_\alpha(\frac{m_n}{k})}\,, \tag{1.27}$$

and

$$N_n \simeq \frac{1}{\sqrt{\pi^2 R\, m_n\, e^{-\pi kR}}}\,, \tag{1.28}$$

with $s = (4,1,2)$, $r = (b, \mp c, 0)$ and $\alpha = (\sqrt{4+a}, |c \pm \frac{1}{2}|, 1)$. The graviton modes $h_{\mu\nu}^{(n)}$ are identical to the scalar modes $\phi^{(n)}$ except that $a = b = 0$. The Kaluza-Klein mass spectrum is approximately given by

$$m_n \simeq \left(n + \frac{1}{2}(\alpha - 1) \mp \frac{1}{4}\right)\pi k\, e^{-\pi kR}\,, \tag{1.29}$$

for even (odd) modes and $n = 1, 2, \ldots$. Note that the Kaluza-Klein modes for all types of bulk fields are always localised near the IR brane. Unlike the zero mode there is no freedom to delocalise the Kaluza-Klein (nonzero) modes away from the IR brane. Note that the zero mode of the bulk graviton is localised on the UV brane [8], with a profile given by (1.12) with $a = b = 0$, while the zero mode of the bulk gauge field is flat and not localised in the 5D bulk [9, 10].

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2. AdS/CFT and holography

Remarkably 5D warped models in a slice of AdS can be given a purely 4D description. This is essentially what makes warped extra dimensions more interesting than flat extra dimensions. This holographic correspondence between the 5D theory and the 4D theory originates from the AdS/CFT correspondence in string theory [11]. It conjectures that

$$\begin{array}{c}\text{type IIB string theory}\\ \text{on } \mathrm{AdS}_5\times S^5\end{array} \quad \overset{\text{DUAL}}{\Longleftrightarrow} \quad \mathcal{N}=4 \text{ SU(N) 4D gauge theory} \tag{2.1}$$

where $\mathcal{N}$ is the number of supersymmetry generators and

$$\frac{R_{AdS}^4}{l_s^4} = 4\pi g_{YM}^2 N\,, \tag{2.2}$$

with $R_{AdS}\equiv 1/k$, l_s is the string length and g_{YM} is the SU(N) Yang-Mills gauge coupling. The $\mathcal{N}=4$ gauge theory is actually a conformal field theory (CFT).

In the warped bulk we have only considered gravity and neglected any string corrections. This bulk gravity description is only valid provided $R_{AdS}\gg l_s$ and via (2.2) leads to the condition that $g_{YM}^2 N\gg 1$, which means that the 4D dual CFT is strongly coupled! Thus for our purposes the correspondence takes the form of a duality in which the weakly coupled 5D gravity description is dual to a strongly coupled 4D CFT.

An AdS/CFT dictionary that relates the two dual descriptions can be established. For every bulk field Φ there is an associated operator $\mathcal{O}$ of the CFT. In the AdS_5 metric (1.1) the boundary of AdS space is located at $y=-\infty$. The boundary value of the bulk field $\Phi(x^\mu, y=-\infty)\equiv\phi_0(x^\mu)$ acts as a source field for the CFT operator $\mathcal{O}$ [12, 13]. If a UV brane is then placed at $y=0$, the $-\infty<y<0$ part of AdS space is chopped off and the remaining $0<y<\infty$ part is reflected about $y=0$ with a Z_2 symmetry. The presence of the UV brane with an associated UV scale Λ_{UV} thus corresponds to an explicit breaking of the conformal invariance in the CFT at the UV scale (but only by nonrenormalisable terms) [14, 15, 16]. The fact that the CFT now has a finite UV cutoff means that the source field ϕ_0 becomes dynamical. A kinetic term for the source field will always be induced by the CFT but one can directly add an explicit kinetic term for the source field at the UV scale. Thus in the presence of a UV brane the AdS/CFT correspondence can be quantified in the following way

$$\int \mathcal{D}\phi_0\, e^{-S_{UV}[\phi_0]}\int_{\Lambda_{UV}} \mathcal{D}\phi_{CFT}\, e^{-S_{CFT}[\phi_{CFT}]-\int d^4x\,\phi_0\mathcal{O}} \\ = \int \mathcal{D}\phi_0\, e^{-S_{UV}[\phi_0]}\int_{\phi_0}\mathcal{D}\phi\, e^{-S_{bulk}[\phi]}\,, \tag{2.3}$$

where S_{UV} is the UV Lagrangian for the source field ϕ_0. It is understood that now the source field $\phi_0=\Phi(x,y=0)$. Moving away from the UV brane at $y=0$ in the bulk corresponds in the 4D dual to running down from the UV scale to lower energy scales. Since the bulk is AdS the 4D dual gauge theory quickly becomes conformal at energies below the UV scale.

The presence of the IR brane at $y=\pi R$ corresponds to a spontaneous breaking of the conformal invariance in the CFT at the IR scale $\Lambda_{IR}=\Lambda_{UV}e^{-\pi kR}$ [14, 15]. The conformal symmetry is nonlinearly realised and particle bound states of the CFT can now appear. Thus, the dual interpretation of a slice of AdS not only contains a 4D dual CFT with a UV cutoff, but also a dynamical source field ϕ_0 with UV Lagrangian $S_{UV}[\phi_0]$. In particular note that the source field is an elementary (point-like) state up to the UV scale, while particles in the CFT sector are only effectively point-like below the IR scale but are composite above the IR scale.

2.1 Holography of scalar fields

As a simple application of the AdS/CFT correspondence in a slice of AdS$_5$ we shall present the dual 4D Lagrangian corresponding to a bulk scalar field Φ with boundary mass terms. It is customary to use conformal coordinates $z = (e^{ky} - 1)/k$, with $A(z) = (1 + kz)^{-1}$, $q = p/(kA(z))$ and $\Phi(p, z)$ is the 4D Fourier transform of $\Phi(x, z)$. Imposing the IR boundary condition (1.9) for the bulk scalar solution leads to the on-shell action

$$\begin{aligned} S_{eff} &= \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\left[A^3(z)\Phi(p,z)\left(\Phi'(-p,z) - b\,k\,A(z)\Phi(-p,z)\right)\right]\Big|_{z=z_0} \\ &= \frac{k}{2}\int \frac{d^4p}{(2\pi)^4} F(q_0,q_1)\Phi(p)\Phi(-p)\,, \end{aligned} \tag{2.4}$$

where

$$F(q_0,q_1) = \mp\, iq_0\left[J_{\nu\mp1}(iq_0) - Y_{\nu\mp1}(iq_0)\,\frac{J_\nu(iq_1)}{Y_\nu(iq_1)}\right]\left[J_\nu(iq_0) - Y_\nu(iq_0)\,\frac{J_\nu(iq_1)}{Y_\nu(iq_1)}\right]\,, \tag{2.5}$$

and $\nu \equiv \nu_\pm = \alpha \pm 1$.

The dual theory two-point function of the operator $\mathscr{O}$ sourced by the bulk field Φ is contained in the self-energy $\Sigma(p)$ obtained by

$$\begin{aligned} \Sigma(p) &= \int d^4x\, e^{-ip\cdot x}\frac{\delta^2 S_{eff}}{\delta(A_0^2\Phi(x,z_0))\delta(A_0^2\Phi(0,z_0))}\,, \\ &= \frac{k}{g_\phi^2}\frac{q_0(I_\nu(q_0)K_\nu(q_1) - I_\nu(q_1)K_\nu(q_0))}{I_{\nu\mp1}(q_0)K_\nu(q_1) + I_\nu(q_1)K_{\nu\mp1}(q_0)}\,, \end{aligned} \tag{2.6}$$

where $A_{0,1} = A(z_{0,1})$ for $z_{0,1}$ the location of the UV (IR) brane. A coefficient $1/g_\phi^2$ has also been factored out in front of the scalar kinetic term in (1.4), so that g_ϕ is a 5D expansion parameter with $\dim[1/g_\phi^2] = 1$. The behaviour of $\Sigma(p)$ can be studied for various momentum limits in order to obtain information about the dual 4D theory. The analytic terms in $\Sigma(p)$ are normally subtracted away by adding appropriate counterterms. However with a finite UV cutoff (corresponding to the scale of the UV brane) these terms are now interpreted as kinetic (and higher derivative terms) of the source field ϕ_0, so that the source becomes dynamical in the holographic dual theory. The source field can now mix with the CFT bound states and therefore the self-energy $\Sigma(p)$ must be resummed and the modified mass spectrum is obtained by inverting the whole quadratic term $S_{UV} + S_{eff}$. In the case with no UV boundary action S_{UV} this means that the zeroes of (2.6) are identical with the Kaluza-Klein mass spectrum (1.16) corresponding to (modified) Neumann conditions for the source field.

To obtain the holographic interpretation of the bulk scalar field, recall that the scalar zero mode can be localised anywhere in the bulk with $-\infty < b < \infty$ where $b \equiv b_\pm = 2 \pm \alpha$ and $-\infty < b_- < 2$

and $2 < b_+ < \infty$. Since $b_\pm = 1 \pm \nu_\pm$ we have $-1 < \nu_- < \infty$ and $1 < \nu_+ < \infty$. The ν_- branch corresponds to $b_- < 2$, while the ν_+ branch corresponds to $b_+ > 2$. Hence the $\nu_-(\nu_+)$ branch contains zero modes which are localised on the UV (IR) brane.

Consider first the ν_- branch. In the limit $A_0 \to \infty$ and $A_1 \to 0$ one obtains

$$\Sigma(p) \simeq -\frac{2k}{g_\phi^2}\left[\frac{1}{\nu}\left(\frac{q_0}{2}\right)^2 + \left(\frac{q_0}{2}\right)^{2\nu+2}\frac{\Gamma(-\nu)}{\Gamma(\nu+1)} + \dots\right], \tag{2.7}$$

where the expansion is valid for noninteger ν. The expansion for integer ν will contain logarithmns. Only the leading analytic term has been written in (2.7). The nonanalytic term is the pure CFT contribution to the correlator $\langle\mathcal{O}\mathcal{O}\rangle$. If A_0 is finite then the analytic term in (2.7) becomes the kinetic term for the source field ϕ_0. Placing the UV brane at $z_0 = 0$ with $A_0 = 1$ leads to the dual Lagrangian below the cutoff scale $\Lambda \sim k$

$$\mathcal{L}_{4D} = -Z_0(\partial\phi_0)^2 + \frac{\omega}{\Lambda^{\nu_-}}\,\phi_0\mathcal{O} + \mathcal{L}_{CFT}\,, \tag{2.8}$$

where Z_0, ω are dimensionless couplings. This Lagrangian describes a massless dynamical source field ϕ_0 interacting with the CFT via the mixing term $\phi_0\mathcal{O}$. This means that the mass eigenstate in the dual theory will be a mixture of the source field and CFT particle states. The coupling of the mixing term is irrelevant for $\nu_- > 0$ ($b_- < 1$), marginal if $\nu_- = 0$ ($b_- = 1$) and relevant for $\nu_- < 0$ ($b_- > 1$). This suggests the following dual interpretation of the massless bulk zero mode. When the coupling is irrelevant ($\nu_- > 0$), corresponding to a UV brane localised bulk zero mode, the mixing can be neglected at low energies, and hence to a very good approximation the bulk zero mode is dual to the massless 4D source field ϕ_0. However for relevant ($-1 < \nu_- < 0$) or marginal couplings ($\nu_- = 0$) the mixing can no longer be neglected. In this case the bulk zero mode is no longer UV-brane localised, and the dual interpretation of the bulk zero mode is a part elementary, part composite mixture of the source field with massive CFT particle states.

Finally consider the case $\nu = \nu_+ > 1$. In the limit $A_0 \to \infty$ and $A_1 \to 0$ we obtain for noninteger ν

$$\Sigma(p) \simeq -\frac{2k}{g_\phi^2}\left[(\nu-1) + \left(\frac{q_0}{2}\right)^2\frac{1}{(\nu-2)} + \left(\frac{q_0}{2}\right)^{2\nu-2}\frac{\Gamma(2-\nu)}{\Gamma(\nu-1)}\right], \tag{2.9}$$

where only the leading analytic terms have been written. The nonanalytic term is again the pure CFT contribution to the correlator $\langle\mathcal{O}\mathcal{O}\rangle$. At low energies $q_1 \ll 1$ one obtains

$$\Sigma(p)_{IR} \simeq -\frac{2k}{g_\phi^2}\left[(\nu-1) + \left(\frac{q_0}{2}\right)^2\frac{1}{(\nu-2)} - \nu(\nu-1)^2\frac{A_1^{2\nu}}{A_0^{2\nu}}\left(\frac{2}{q_0}\right)^2\right], \tag{2.10}$$

where the large-A_0 limit was taken first. We now see that at low energies the nonanalytic term has a pole at $p^2 = 0$ with the correlator

$$\langle\mathcal{O}\mathcal{O}\rangle = \frac{8k^3}{g_\phi^2}\nu_+(\nu_+-1)^2 e^{-2\nu_+\pi kR}\frac{1}{p^2}\,, \tag{2.11}$$

where $A_0 = 1$ and $A_1 = e^{-\pi kR}$. This pole indicates that the CFT has a massless scalar mode at low energies! What about the massless source field? As can be seen from (2.9) and (2.10) the leading analytic piece is a constant term which corresponds to a mass term for the source field [16]. This leads to the dual Lagrangian below the cutoff scale $\Lambda \sim k$

$$\mathscr{L}_{4D} = -\widetilde{Z}_0(\partial\phi_0)^2 + m_0^2\phi_0^2 + \frac{\chi}{\Lambda^{\nu_+-2}}\phi_0\mathscr{O} + \mathscr{L}_{CFT}\ , \tag{2.12}$$

where $\widetilde{Z}_0, \chi$ are dimensionless parameters and m_0 is a mass parameter of order the curvature scale k. The bare parameters $\widetilde{Z}_0$ and m_0 can be determined from (2.9). Thus, the holographic interpretation is perfectly consistent. There is a massless bound state in the CFT and the source field ϕ_0 receives a mass of order the curvature scale and decouples. In the bulk the zero mode is always localised towards the IR brane. Indeed for $\nu_+ > 2$ the coupling between the source field and the CFT is irrelevant and therefore the mixing from the source sector is negligible. Hence to a good approximation the mass eigenstate is predominantly the massless CFT bound state. When $1 \leq \nu_+ \leq 2$ the mixing can no longer be neglected and the mass eigenstate is again part elementary and part composite.

3. Application: Yukawa couplings

3.1 5D gravity description

The solution to the hierarchy problem only requires the Higgs field to be localised on the IR brane. This means that fermions and gauge fields can propagate in the bulk. The gauge field zero mode is flat and therefore couples with equal strength to both the UV and IR brane. However the fermions can be localised anywhere in the bulk, and therefore Yukawa coupling hierarchies can be naturally generated by separating the fermions from the Higgs on the IR brane. Each Standard Model fermion is identified with the zero mode of a corresponding 5D Dirac spinor Ψ. For example, the left-handed electron doublet e_L is identified with the zero mode of Ψ_{eL+}, which is the even component of the 5D Dirac spinor $\Psi_{eL} = \Psi_{eL+} + \Psi_{eL-}$. The odd component Ψ_{eL-} does not have a zero mode, but at the massive level it pairs up with the massive modes of Ψ_{eL+} to form a vectorlike Dirac mass. This embedding of 4D fermions into 5D fermions is repeated for each Standard Model fermion. The Standard Model Yukawa interactions, such as $\bar{\Psi}_{eL}\Psi_{eR}H$, are then promoted to 5D interactions in the warped bulk. This gives

$$\begin{aligned}&\int d^4x \int dy\ \sqrt{-g}\ \lambda_{ij}^{(5)} \left[\bar{\Psi}_{iL}(x,y)\Psi_{jR}(x,y) + h.c.\right] H(x)\delta(y-\pi R)\\ &\equiv \int d^4x\ \lambda_{ij}\ (\bar{\Psi}_{iL+}^{(0)}(x)\Psi_{jR+}^{(0)}(x)H(x) + h.c. + \ldots)\ ,\end{aligned} \tag{3.1}$$

where i,j are flavour indices, $\lambda_{ij}^{(5)}$ is the (dimensionful) 5D Yukawa coupling and λ_{ij} is the (dimensionless) 4D Yukawa coupling. Given that the zero mode profile is

$$\widetilde{\psi}_{iL+,R+}^{(0)}(y) \propto e^{(\frac{1}{2}-c_{iL,R})ky}\ , \tag{3.2}$$

this leads to an exponential hierarchy in the 4D Yukawa coupling [4]

$$\lambda_{ij} \simeq \lambda_{ij}^{(5)} k\sqrt{(c_{iL}-1/2)(c_{iR}-1/2)}\ e^{(1-c_{iL}-c_{jR})\pi kR}\ , \tag{3.3}$$

for $c_{iL,R} > 1/2$. Assuming $c_{iL} = c_{jR}$ for simplicity then the electron Yukawa coupling $\lambda_e \sim 10^{-6}$ is obtained for $c_e \simeq 0.64$. Instead when $c_{iL,R} < 1/2$, both fermions are localised near the IR brane giving

$$\lambda_{ij} \simeq \lambda_{ij}^{(5)} k\sqrt{(1/2-c_{iL})(1/2-c_{iR})}\ , \tag{3.4}$$

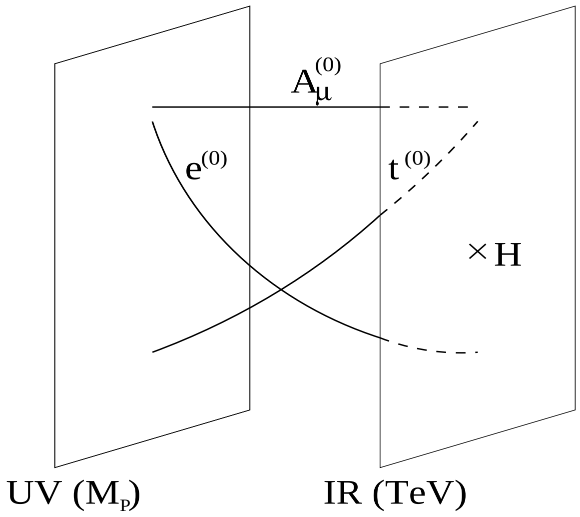

Figure 2: The Standard Model in the warped five-dimensional bulk.

with no exponential suppression. Hence the top Yukawa coupling $\lambda_t \sim 1$ is obtained for $c_t \simeq -0.5$. The remaining fermion Yukawa couplings, c_f then range from $c_t < c_f < c_e$ [4, 17]. Thus, we see that for bulk mass parameters c of $\mathscr{O}(1)$ the fermion mass hierarchy is explained. The fermion mass problem is now reduced to determining the c parameters in the 5D theory. This requires a UV completion of the 5D warped bulk model with fermions, such as string theory.

The geometric picture that emerges is a Standard Model in the warped bulk as depicted in Figure 2. The fermions are localised to varying degrees in the bulk with the electron zero mode, being the lightest fermion, furthest away from the Higgs on the TeV brane while the top, being the heaviest, is closest to the Higgs.

3.1.1 4D holographic description

The Yukawa coupling hierarchies can also be understood from the dual 4D theory. Consider first an electron (or light fermion) with $c > 1/2$. In the dual 4D theory the electron is predominantly an elementary field. The dual 4D Lagrangian is obtained from analysing $\Sigma(p)$ for fermions, where the CFT induces a kinetic term for the source field $\psi_L^{(0)}$. It is similar to the dual scalar field Lagrangian (2.8) and given by [18]

$$\mathscr{L}_{4D} = \mathscr{L}_{CFT} + Z_0 \bar{\psi}_L^{(0)} i\gamma^\mu \partial_\mu \psi_L^{(0)} + \frac{\omega}{\Lambda^{|c+\frac{1}{2}|-1}} (\bar{\psi}_L^{(0)} \mathscr{O}_R + h.c.) \,, \tag{3.5}$$

where Z_0, ω are dimensionless couplings and dim $\mathscr{O}_R = 3/2 + |c+1/2|$. The source field $\psi_L^{(0)}$ pertains to the left-handed electron e_L and a similar Lagrangian is written for the right-handed electron e_R. At energy scales $\mu < k$ we define the dimensionless coupling $\xi(\mu) = \omega/\sqrt{Z(\mu)}(\mu/\Lambda)^\gamma$ with $\gamma = |c+1/2| - 1$. Since $c > 1/2$ the coupling ξ decreases in the IR and at the TeV scale ($ke^{-\pi kR}$) is given by

$$\xi(\text{TeV}) \sim \sqrt{c - \frac{1}{2}} \frac{4\pi}{\sqrt{N}} \left(\frac{ke^{-\pi kR}}{k} \right)^{c-\frac{1}{2}} = \sqrt{c - \frac{1}{2}} \frac{4\pi}{\sqrt{N}} e^{-(c-\frac{1}{2})\pi kR} \,. \tag{3.6}$$

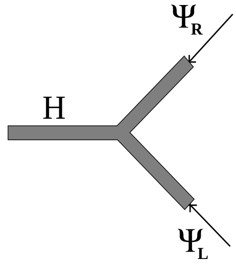

Figure 3: The three-point Yukawa coupling vertex in the dual theory when the fermions are predominantly elementary source fields.

The actual physical Yukawa coupling λ follows from the three-point vertex between the physical states. Since both e_L and e_R are predominantly elementary they can only couple to the composite Higgs via the mixing term in (3.5). This is depicted in Fig. 3. In a large-N gauge theory the matrix element $\langle 0|\mathscr{O}_{L,R}|\Psi_{L,R}\rangle \sim \sqrt{N}/(4\pi)$, and the vertex between three composite states $\Gamma_3 \sim 4\pi/\sqrt{N}$ [19]. Thus if each of the elementary fields e_L and e_R mixes in the same way with the CFT so that $c_{eL} = c_{eR} \equiv c$ then

$$\lambda \propto \langle 0|\mathscr{O}_{L,R}|\Psi_{L,R}\rangle^2 \, \Gamma_3 \, \xi^2(\text{TeV}) = \frac{4\pi}{\sqrt{N}}(c-1/2)e^{-2(c-\frac{1}{2})\pi kR} \, . \tag{3.7}$$

This agrees precisely with the bulk calculation (3.3) where $\lambda_{ij}^{(5)} k \sim 4\pi/\sqrt{N}$.

Similarly we can also obtain the Yukawa coupling for the top quark with $c \le -1/2$ in the dual theory. With this value of c the top quark is mostly a CFT bound state in the dual theory and we can neglect the mixing coupling with the CFT. As in the scalar field example this follows from the fact that the two point function $\langle \mathscr{O}_R \bar{\mathscr{O}}_R\rangle$ now has a massless pole. The CFT will again generate a mass term for the massless source field, so that the only massless state in the dual theory is the CFT bound state. The dual Lagrangian is given by [18]

$$\begin{aligned}\mathscr{L}_{4D} &= \mathscr{L}_{CFT} + Z_0 \, \bar{\psi}_L^{(0)} i\gamma^\mu \partial_\mu \psi_L^{(0)} + \tilde{Z}_0 \, \bar{\chi}_R i\gamma^\mu \partial_\mu \chi_R \\ &\quad + d \, k \, (\bar{\chi}_R \psi_L^{(0)} + h.c.) + \frac{\omega}{\Lambda^{|c+\frac{1}{2}|-1}} \, (\bar{\psi}_L^{(0)} \mathscr{O}_R + h.c.) \, ,\end{aligned} \tag{3.8}$$

where $Z_0, \tilde{Z}_0, d, \omega$ are dimensionless constants. The fermion $\psi_L^{(0)}$ pertains to t_L and a similar Lagrangian is written for t_R. Just as in the scalar case (2.12), this dual Lagrangian is inferred from the behaviour of $\Sigma(p)$ for fermions. The CFT again induces a kinetic term for the source field $\psi_L^{(0)}$ but also generates a Dirac mass term of order the curvature scale k with a new elementary degree of freedom χ_R. Hence the elementary source field decouples from the low energy spectrum and the mixing term is no longer relevant for the Yukawa coupling. Instead the physical Yukawa coupling will arise from a vertex amongst three composite states so that $\lambda_t \sim \Gamma_3 \sim 4\pi/\sqrt{N} \sim \lambda^{(5)}k$, and consequently there is no exponential suppression in the Yukawa coupling. This is again consistent with the bulk calculation.

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