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Standard Model parameters and heavy quarks on the lattice

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I review recent progresses in heavy quarks physics on the lattice. I focus on decay constants and form factors relevant for the extraction of CKM matrix elements from experimental data. $B-\bar{B}$ mixing is also discussed. In the last part of the paper I describe phenomenological applications of Heavy Quark Effective Theory (HQET) on the lattice, presenting in some detail the recent non-perturbative determination of the b-quark mass including $\mathrm{O}(1/m_{\mathrm{b}})$ corrections.

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1. Introduction

Next year CERN's Large Hadron Collider (LHC) will start delivering proton beams for physics collision. The LHCb experiment is designed to exploit the enormous LHC potential in the b-quark sector for measurements of the CKM parameters to such a high precision that possible contributions from TeV-scale New Physics to the mixing mechanism will become visible. To give an idea, the production of b hadrons at LHCb is expected with the annual yield of 10^{12} b-$\bar{\text{b}}$ pairs [1]. Possible future super-B factories would further extend the set of high precision b-physics measurements [2].

This programme can provide a stringent test of the Standard Model and potentially lead to the discovery of New-Physics only if at the same time a significant progress on the theory side is made. To get a flavor about the required precision it is useful to have a look at the experimental and theoretical situation for a few low-energy flavor-violating observables where non-Standard effects were expected to contribute.

Let us start with the inclusive radiative B-meson decay. The world average performed by the Heavy Flavor Averaging Group [3] for $E_\gamma > 1.6$ GeV yields the branching ratio

$$\mathscr{B}(\bar{B} \to X_s\gamma) = (3.55 \pm 0.24^{+0.09}_{-0.10} \pm 0.03) \times 10^{-4}\,, \tag{1.1}$$

to be compared with the Standard Model NNLO analysis of Ref. [4], which for the same cut on the photon energy gives

$$\mathscr{B}(\bar{B} \to X_s\gamma) = (3.15 \pm 0.23) \times 10^{-4}\,. \tag{1.2}$$

The values are consistent basically within one (combined) sigma (about 10%), which implies that the difference between the Standard Model (SM) and the experimental numbers can be of order 20%. Notice that this estimate does not depend on theoretical inputs from the lattice.

New Physics in principle can be found also in purely leptonic $B^\pm$ (or $D^\pm$) decays, which can be enhanced by charged Higgs exchange contributions in any model with two Higgs doublets [5]. Again, the average of the experimental numbers for $\mathscr{B}(B \to \tau\nu)$ from Belle and Babar [6, 7] is in good agreement with the SM theoretical computation although the total error is above 30% in the first case and around 20% in the second. On the theory side this is mainly due to the uncertainty on V_{ub} and on the decay constant F_{B}, which will be discussed in more detail in the next section. The experimental error on the other hand is expected to decrease in the future. At a Super-B factory with hundred times the luminosity of Belle the branching ratio would be measured with a precision of 3%.

The leptonic decays of the neutral $B_{(\text{s})}$ meson are very rare (the SM branching ratio is $\text{O}(10^{-9})$) and they haven't been observed so far. The most recent experimental upper bound on $\mathscr{B}(B_\text{s} \to \mu^+\mu^-)$ is 1×10^{-7} from CDF [8]. This decay is included among the LHCb physics goals with an SM expectation of 20 events per year [1]. There is quite some excitement around this channel as it can be significantly enhanced in various extensions of the Standard Model. For example in the Minimal Supersymmetric Standard Model with large $\tan\beta$ (where $\tan\beta$ is the ratio of the two neutral Higgs field vacuum expectation values) the enhancement can be up to three orders of magnitude compared to the SM. That is due to the appearance of flavor changing couplings of the neutral Higgs bosons generated by non-holomorphic terms after supersymmetry breaking [9].

Finally New Physics might contribute to $D - \bar{D}$ mixing, which has been recently observed by Babar [10]. It is hard to quantify the possible size of non-Standard effects here as the SM theoretical

predictions are very uncertain. It is also not clear whether useful quantities can be computed on the lattice to describe the process, which is affected by long-distance contributions that are not captured by the Operator Product Expansion (see [11] for a more exhaustive discussion).

From the examples above I conclude that to keep the pace with experiments and help in the search of New Physics lattice computations must aim at high precision, typically between a few percent and 10% depending on the process and the corresponding non-perturbative parameters needed. In order to achieve such an accuracy the computations must start from first principles, which implies that the light fermions must be treated as dynamical degrees of freedom, and all the systematics associated with renormalization, extrapolation to the continuum limit and chiral extrapolation must be kept under control. In the rest of the review I will try to show that each of those effects can introduce an uncertainty of O(5%), and I will do that while presenting a selection of recent results for B and D meson decay constants, the $B_{\mathrm{B}_{(\mathrm{s})}}$ parameter and semi-leptonic form factors for heavy-light and heavy-heavy transitions.

In the last part I will describe an approach (not necessarily the only one) in which all these systematics can be addressed non-perturbatively. As an application I will present the (quenched) computation of the b-quark mass in HQET including $\mathrm{O}(1/m_{\mathrm{b}})$ effects [12].

2. $B_{(\mathrm{s})}$ and $D_{(\mathrm{s})}$ meson decay constants

In the Standard Model the purely leptonic decays of charged B and D mesons proceed via quark annihilation into a W boson. Taking as example the $B \to \tau\nu_\tau$ channel, the branching ratio can be parameterized as

$$\mathscr{B}(B^- \to \tau^- \bar{\nu}_\tau) \propto F_{\mathrm{B}}^2 |V_{\mathrm{ub}}|^2 \,, \tag{2.1}$$

which turns out to be $\mathrm{O}(10^{-4})$. The proportionality factor is a function of well-known masses, lifetimes and the Fermi constant. In eq. (2.1) F_{B} is the B meson decay constant, which is given by the matrix element of the heavy-light axial current between the vacuum and the B-meson state, while V_{ub} is the relevant element (actually the smallest and least known) of the CKM matrix. Similarly $F_{\mathrm{B_s}}$ is the non-perturbative matrix element necessary for the SM prediction of the $B_{\mathrm{s}} \to \mu^+\mu^-$ branching ratio discussed in the Introduction.

The B meson decay constant has been computed with three dynamical flavors by the HPQCD and the Fermilab, MILC Collaborations [13, 14, 15]. In both cases the rooted staggered quarks configurations generated by the MILC Collaboration with the AsqTad action have been employed [16]. The heavy b-quark is simulated by using NRQCD in [13] and the Fermilab action in [14, 15]. The results from [13, 14] for $\Phi_{\mathrm{q}} = F_{\mathrm{B_q}}\sqrt{m_{\mathrm{B_q}}}$ are shown in figure 1 as a function of the sea quark mass in units of the strange quark mass and for the unitary (light sea quark mass equal to the light valence quark mass) points only. The curves are the Staggered Chiral Perturbation Theory (SχPT) [17] fits. Although the same formulae have been used and the lattice resolutions are not too coarse, the results suggest quite different chiral behaviors (reflected in a 5% difference on the ratio $F_{\mathrm{B_s}}/F_{\mathrm{B}}$), probably due to residual cutoff effects. It is interesting to note the consistency of the Fermilab data with the curvature predicted from SχPT, notice however that there the coupling $g_{\mathrm{B^*B}\pi}$ appearing in the non-analytic terms has been set to $g_{\mathrm{D^*D}\pi}$ from the CLEO experiment before performing the fit.

The final result quoted in [13] is $F_{\mathrm{B}} = 216(9)(19)(4)(6)$ MeV, where the first error is statistical (including chiral extrapolations) and the others are estimates of the systematics. The largest

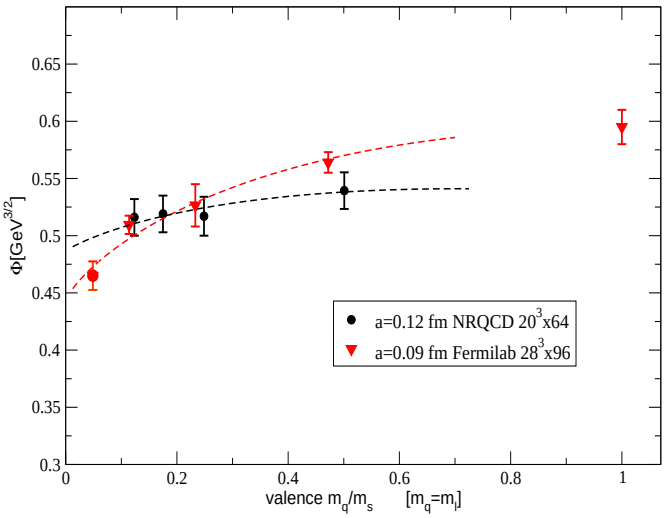

Figure 1: SχPT-guided extrapolations for $F_{\rm B}$ from [13, 14].

one in particular is due to the matching between the heavy-light current in QCD and in NRQCD. This matching involves power divergent mixings between dimension-three and dimension-four operators in the effective theory and the subtraction has been performed by considering the one-loop contribution only. The other systematics included are discretization effects and relativistic corrections. Most of these cancel in the ratio $F_{\rm B_s}/F_{\rm B}$, for which the value $1.20(3)(1)$ is obtained.

The Fermilab Collaboration in [14] preferred to quote numbers for the ratio only as at that time the computation of the relevant renormalization constants was not yet completed. The result is $F_{\rm B_s}/F_{\rm B} = 1.27(2)(6)$ where the second uncertainty is mainly due to the chiral extrapolation. An update including results from two additional lattice resolutions ($a = 0.12$ and 0.15 fm) and the use of the matching renormalization constants computed at one-loop in [18] has been presented at this conference [15]. The preliminary analysis yields $F_{\rm B} = 191(5)(8)$ MeV and $F_{\rm B_s}/F_{\rm B} = 1.30(3)(4)$, both in good agreement with the NQRCD results.

The ALPHA Collaboration has completed the non-perturbative computation of the renormalization constant of the static-light axial current with two dynamical flavors in the Schrödinger functional (SF) scheme [19]. The main result is the universal (i.e. regularization independent) factor $\Phi(\mu)/\Phi_{\rm RGI}$ relating a matrix element $\Phi(\mu)$ of the static-light axial current renormalized at the scale μ to its scheme-independent (Renormalization Group Invariant) version. The result is shown in figure 2. For $\mu \leq 2$ GeV perturbation theory fails in reproducing the correct result and there would be no way to detect it within perturbation theory only, as the convergence of the series appears to be very good in all the range plotted. At the most non-perturbative scale, where large volume matrix elements relevant for phenomenology are usually renormalized, the discrepancy reaches 5%. The regularization dependent constants needed to match the bare matrix elements to the ones renormalized at this scale have also been computed in [19] for different static actions (see [23] for their precise definition) and for the range of bare couplings relevant for simulations in large volume using Wilson-Clover fermions.

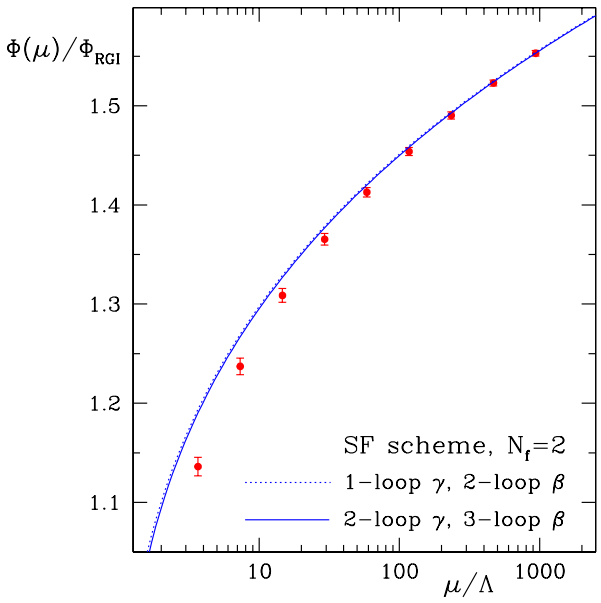

Figure 2: Non-perturbatively computed running matrix element of the static-light axial current in the SF scheme from [19]. The dotted and solid lines are obtained from perturbation theory using 1/2 and 2/3 loops expressions for the anomalous dimension of the current [20] and the β-function [21]. As an information, the Λ parameter from [22] is $\Lambda_{\rm SF} \simeq 100$ MeV.

As a first application $F_{\rm B_s}^{\rm stat}$ has been computed on a $24^3 \times 32$ lattice with $a \simeq 0.08$ fm and (degenerate) sea quark masses close to the strange quark mass. The result $F_{\rm B_s}^{\rm stat} = 297(14)$ MeV is rather large compared for example to the quenched value $F_{\rm B_s} = 193(6)$ MeV obtained in [24] by linearly interpolating in the inverse meson mass between continuum results in the static approximation and in the relativistic theory with heavy quarks around the charm. Several effects may concur in producing the large $N_{\rm f} = 2$ number, for instance cutoff effects, $1/m_{\rm b}$ corrections or sea quark mass effects. While to estimate the latter it is necessary to repeat the computation at lighter sea quark masses, for the first two an impression can be gathered by comparing with the static result at a similar lattice spacing in the quenched approximation, which turns out to be $F_{\rm B_s}^{\rm stat}(N_{\rm f} = 0, a \simeq 0.08 \text{ fm}) = 247(5)$ MeV from [24]. This still leaves room for sizeable effects of the dynamical fermions, which I will consider again in the following when discussing the $D_{(\rm s)}$ meson decay constant. Remaining within the quenched approximation $F_{\rm B_s}$ has also been computed including $1/m_{\rm b}$ corrections explicitly in HQET [25]. The final result is nicely consistent with the one obtained by the interpolation discussed above, although with larger errors. The computation will be described in more detail in the last section.

Let us now consider the $D_{(\rm s)}$ system. The decay constants $F_{\rm D}$ and $F_{\rm D_s}$ can be used to extract the CKM matrix elements $V_{\rm cd}$ and $V_{\rm cs}$ from the CLEO data [26]. The most recent computation by the HPQCD Collaboration [27] includes the effects of 2+1 dynamical flavors implemented in the staggered AsqTad formalism by use of the fourth root of the quark determinant. For the valence fermions two different variants of the new Highly Improved Staggered Quark (HISQ) action [28] have been used for the light (including strange) and the charm quarks. Some simulations parameters are collected in table 1 while results are shown in figure 3, taken from [27].

V	a	am_c
$16^3 \times 48$	0.15 fm	0.85
$20^3 \times 64$	0.12 fm	$\simeq 0.65$
$24^3 \times 64$	0.12 fm	$\simeq 0.65$
$28^3 \times 96$	0.09 fm	$\simeq 0.43$

Table 1: Lattice volumes, lattice spacings a and values of the charm quark mass in units of a from [27].

The values of the charm quark mass in units of a are quite large in this study and they might cause some concern about the size of cutoff effects. In figure 3 however these appear to be roughly at the 10% level at the coarsest lattice resolution. The concern is then whether all the data are in the scaling region and a continuum limit extrapolation is justified or not. It would therefore be desirable to repeat the computation at the very fine resolution $a = 0.06$ fm where the MILC Collaboration is indeed producing configurations.

The final result $F_{\mathrm{D_s}} = 241(3)$ MeV, $F_{\mathrm{D_s}}/F_{\mathrm{D}} = 1.162(9)$ is obtained by performing a simultaneous chiral and continuum extrapolation of the data at different quark masses and lattice spacings. The overall error includes corrections due to the u/d quark mass difference and electromagnetic effects (see table 2 in [27] for the detailed error budget), which make the claimed precision clearly impressive. In my opinion such a precision calls for a complete clarification of the issues related to the use of the "fourth root trick" in dynamical simulations of staggered quarks. The discussion on the localization, the unitarity and the symmetry content of the "rooted" theory [29, 30, 31, 32] is still ongoing and a final conclusion in favor or disfavor of it hasn't been reached yet. Also, to be able to conclusively judge on the error budget it would be useful to have more details concerning the Bayesian fits performed, the precise functional forms used in the continuum/chiral extrapolations and also some algorithmic details. Simulations are indeed described for sea quark masses above one fifth of the strange quark mass only [16]. Some of these points will probably be clarified in the longer publication announced in [28].

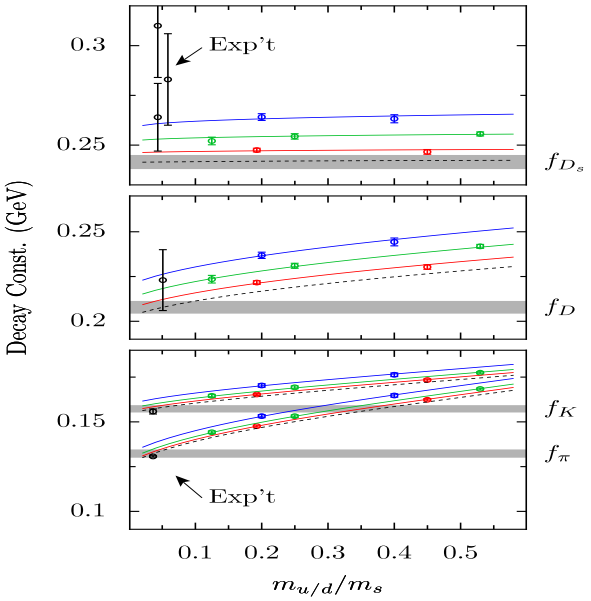

Figure 3: Results for the D, D_{s} (K and π) decay constants from [28] for three lattice resolutions (see table 1). The chiral fits are performed together with those of the corresponding meson masses. The continuum limit is given by the dashed lines and the final, chirally extrapolated, results are represented by the shaded bands.

The European Twisted Mass (ETM) Collaboration has presented at this conference an application in the charm sector of the twisted mass (tmQCD) formalism with two dynamical light flavors [33]. By working at maximal twist the quantities computed are automatically O(a) improved [34] and no renormalization constants have to be calculated to obtain the decay constants, as first pointed out in [35]. Configurations have been generated for two lattice volumes $24^3 \times 48$ and $32^3 \times 64$ with lattice spacings $a \simeq 0.09$ and 0.07 fm respectively. The sea quark masses are in the range $m_{\mathrm{s}}/6$ and $2m_{\mathrm{s}}/3$. The decay constants F_{D} and $F_{\mathrm{D_s}}$ have been obtained by interpolating to the proper value of the meson mass the results produced for heavy quarks around the charm. The interpolation in the case of the D meson at the coarser lattice resolution is shown in

figure 4, taken from [33]. In this case four points have been fitted with a three-parameters functional form inspired by HQET. The preliminary results quoted are $F_{\mathrm{D_s}} = 271(6)(4)(5)$ MeV and

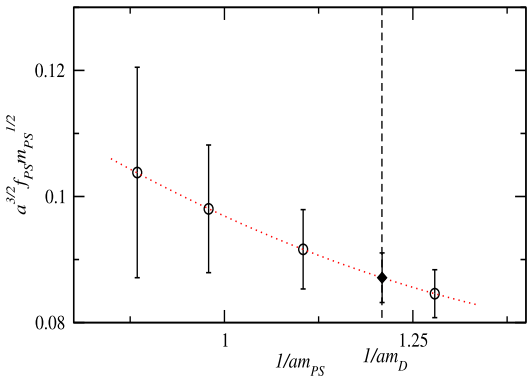

Figure 4: Scaling of $F_{\mathrm{PS}}\sqrt{m_{\mathrm{PS}}}$ as a function of the inverse pseudoscalar meson mass $1/m_{\mathrm{PS}}$. Plot from [33].

$F_{\mathrm{D_s}}/F_{\mathrm{D}} = 1.35(4)(1)(7)$ for $a \simeq 0.09$ fm. The second error comes from the uncertainty on the strange quark mass, while the third is due to the uncertainty on the lattice spacing in the case of $F_{\mathrm{D_s}}$ and to the chiral extrapolation in the case of the ratio $F_{\mathrm{D_s}}/F_{\mathrm{D}}$. The determinations at the finer lattice resolution provide consistent results though with larger errors. It is important to assess precisely the size of cutoff effects on the result above, as that is obtained interpolating in pseudoscalar meson masses which are very close to the cutoff scale (see figure 4).

Finally, in [36], the QCDSF Collaboration calculated the decay constants of heavy-light pseudoscalar mesons on a very fine lattice ($a \simeq 0.04$ fm, $V = 40^3 \times 80$) using non-perturbatively O(a) improved Wilson fermions in the quenched approximation. The result for $F_{\mathrm{D_s}}$ is presented in figure 5 together with those obtained by the ALPHA collaboration using the same action but in a larger range of lattice resolutions [37, 38]. The agreement between the results is quite satisfactory and suggests the possibility of a joint continuum extrapolation (excluding for example the point at the coarsest lattice spacing). The computation of the D meson decay constant requires a chiral extrap-

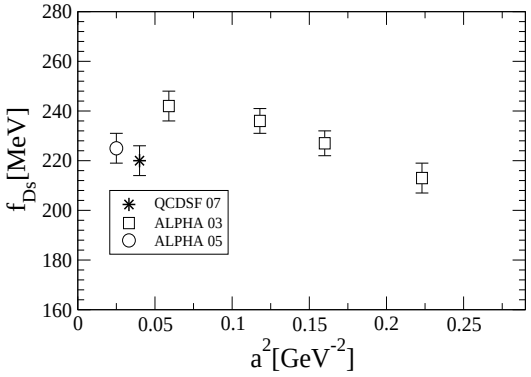

Figure 5: The D_{s} meson decay constant versus a^2 with non-perturbatively O(a) improved Wilson fermions in quenched QCD. Results from [36] (star) and [37, 38] (squares and circle).

olation, which in [36] has been performed by linearly extrapolating data corresponding to "pion" masses above 500 MeV. An uncertainty associated with this chiral extrapolation is not estimated

for the final error budget and the values $F_{D_s} = 220(6)(5)(11)$ MeV (the third error is ascribed to a 10% ambiguity in the lattice spacing) and $F_{D_s}/F_D = 1.068(18)(20)$ are eventually obtained.

The decay constants of B-mesons are also computed in [36]. In this case however bare quark masses m_q with $am_q \simeq 0.7$ need to be considered and the residual, O(a^2), cutoff effects on F_{B_s} are estimated by the authors of [36] to be 12%. This sets the limits of the approach. In addition, for such masses roundoff effects on the quark propagator at large time separations should be carefully checked as well [39].

The different determinations of the $D_{(s)}$ decay constant show statistically significant quenching effects. For F_{D_s}, which has been computed by most of the collaborations, the results discussed are collected in figure 6. There the errors have been conservatively added linearly. The figure also shows the tension, which is emerging with the latest experimental measurement ($F_{D_s} = 275(10)(5)$ MeV and $F_{D_s}/F_D = 1.24(10)(3)$) from CLEO-c [26]. The lattice determinations are indeed systematically below it and in some cases the discrepancy is above two standard deviations. As discussed,

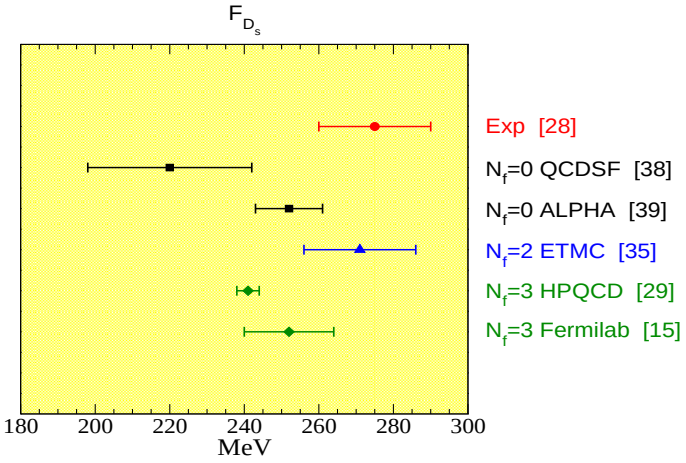

Figure 6: Recent determinations of the D_s meson decay constant compared to the experimental result.

for the $B_{(s)}$ system quenching effects appear even larger. This shouldn't be puzzling as for B-physics effective theories, rather than relativistic QCD, are simulated. The inclusion of dynamical fermions can therefore have different effects in the two cases.

3. $B_{(s)} - \bar{B}_{(s)}$ mixing

The weak interactions induce mixings among flavor eigenstates. At low energies and for B-mesons the process is described by the $\Delta B = 2$ Weak Effective Hamiltonian. In particular the matrix elements of four-fermion operators $O^{\Delta B=2}$ (corresponding to the box diagrams) among meson ($B_{(s)}$ and $\bar{B}_{(s)}$) states need to be computed. The mixing is expressed through the oscillation frequency $\Delta m_{(s)}$

$$m_{B_q} \Delta m_q \propto |V_{tq}^* V_{tb}|^2 \langle \bar{B}_q | O_{VV+AA} | B_q \rangle \,, \tag{3.1}$$

where the proportionality factor is given by the Wilson coefficients (functions of m_t/m_W and G_F). It is customary to introduce the $B_{B_{(s)}}$ parameter by dividing out the result in the vacuum-saturation

approximation

$$\frac{3}{8}\frac{\langle \overline{B}_q|O_{VV+AA}|B_q\rangle}{F_{B_q}^2 m_{B_q}^2} = B_{B_q} \ . \tag{3.2}$$

Oscillations of B mesons are comparatively "slow" and have been observed since UA1, the PDG [40] average for Δm is $0.507(5)$ ps^{-1}. On the contrary $B_s - \bar{B}_s$ mixing is very fast and Δm_s has been measured only recently by CDF [41], with the result $\Delta m_s = 17.77(10)(7)$ ps^{-1}. Notice that the accuracy of both measurements is at the percent level, which will be very difficult to match from the theoretical side. However, by combining these experimental determinations with the lattice computations of the $B_{B_{(s)}}$ parameters the Standard Model values for V_{td} and V_{ts} (or ratios thereof) could be extracted.

This year three Collaborations have reported results on the B-parameters with three dynamical flavors. The HPQCD Collaboration in [42] has computed B_{B_s} and also the matrix elements for $\Delta\Gamma_s$ on the MILC staggered AsqTad configurations at $a \simeq 0.12$ fm, in a volume $20^3 \times 64$ and for sea quark masses equal to one half and one quarter of the strange quark mass. The b-quark is treated using NRQCD. The results show very little dependence on the light quark masses within the errors and the final estimate is $F_{B_s}\sqrt{B_{B_s}^{RGI}} = 281(21)$ MeV and $B_{B_s}(m_b) = 0.76(11)$ using two-loop formulae for the conversion to the $\overline{MS}$ scheme. In the computation the operators in QCD are related to their NRQCD counterparts including $O(1/m_b)$ corrections, which bring in operators of dimension seven. These operators require a power divergent subtraction, which in [42] is performed at the one-loop level. This means that the subtracted operator is still power divergent. With the one-loop value for the coefficient the subtraction itself is about 10% of the final result on B_{B_s} and it gives the largest contribution to the systematical error (see table 2 in [42]). It is clear that the situation becomes worse as finer lattice resolutions are considered, as the subtraction grows linearly with $1/a$. A computation of the subtraction coefficient to higher orders in perturbation theory could at least help in reducing the systematic uncertainty associated to the matching. However, as pointed out in [43], part of this systematic cancels in the ratio $\xi = \frac{F_{B_s}\sqrt{B}_{B_s}}{F_B\sqrt{B}_B}$, which can be used to extract $\frac{|V_{td}|}{|V_{ts}|}$ from $\frac{\Delta m_d}{\Delta m_s}$. This quantity is now being computed by the HPQCD Collaboration which has presented a study using several time sources with smearing to reduce the statistical and fitting errors [43].

The Fermilab-MILC Collaborations reported about the work in progress on the computation of the ratio ξ [44] employing the Fermilab formalism for heavy quarks and again the MILC configurations generated at $a \simeq 0.12$ fm. Matching and renormalization (also including $O(1/m_b)$) are implemented in one-loop perturbation theory. The preliminary results are shown in figure 7 (statistical errors only). The light sea quark mass dependence seems rather small compared to the statistical error, whereas the dependence on the light valence quark mass is noticeable within statistics. To finalize the results the SχPT formulae for the relevant hadronic matrix elements are being determined in order to be able to simultaneously fit the results for different quark masses and lattice spacings. Indeed the Collaborations plan to repeat the computation on a finer and a coarser lattice.

The RBC and UKQCD Collaborations have implemented HQET at the leading order (static approximation) combined with light domain wall fermions for a computation of the mixing parameters with 2+1 dynamical flavors [45]. The lattice used has a linear extent $L \simeq 2$ fm with $a \simeq 0.12$ fm and $L_s = 16$, which for the residual mass from the five-dimensional Ward identity gives $am_{res} = 0.00308(4)$. Three values of the light sea quark mass have been considered, such that

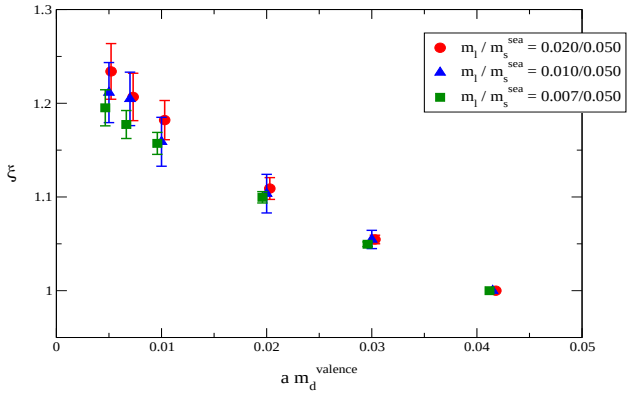

Figure 7: ξ as a function of the valence d quark mass and for different values of the sea quark mass $m_{\rm l}$. Figure from [44].

the lowest pion mass reached is 400 MeV, while for the static quark the APE and HYP2 [23] discretizations have been used. The preliminary results $F_{\rm B_s}^{\rm stat} = 220(32)$ MeV, $F_{\rm B_s}^{\rm stat}/F_{\rm B}^{\rm stat} = 1.10(^{+11}_{-5})$, $B_{\rm B_s}^{\rm stat}(m_{\rm b}) = 0.79(4)$ and $B_{\rm B}^{\rm stat}(m_{\rm b}) = 0.74(10)$ (in the $\overline{\rm MS}$ scheme) have been obtained by using one-loop mean-field improved estimates of the matching and renormalization factors [46] and by linearly extrapolating the data to the physical point. Large differences between the APE and the HYP2 results have been observed for example for the quantity $F_{\rm B_s}^{\rm stat}\sqrt{B_{\rm B_s}^{\rm stat}}$ where the discrepancy between the central values is 30%. Notice however that even if a chirally invariant light action is used, non-perturbative effects in the renormalization constant of the static-light axial current can be large (see figure 2) and in addition static light correlations functions are not automatically O(a) improved, therefore large O(a) contributions may still affect the results.

The non-perturbative renormalization programme for the parity-odd static-light four-fermion operators in the SF scheme has been completed by the ALPHA Collaboration for the quenched case and for two dynamical flavors. In all effective theories the operator $O^{\rm QCD}_{\rm VV+AA}$ is expanded as

$$O^{\rm QCD}_{\rm VV+AA}(m_{\rm b}) = C_{\rm L}(\mu, m_{\rm b}) O^{\rm eff}_{\rm VV+AA}(\mu) + C_{\rm S}(\mu, m_{\rm b}) O^{\rm eff}_{\rm SS+PP}(\mu) + O(1/m_{\rm b})\,, \qquad (3.3)$$

in other words, already at leading order, and in the continuum, the mixing between the two renormalized operators $O^{\rm eff}_{\rm VV+AA}(\mu)$ and $O^{\rm eff}_{\rm SS+PP}(\mu)$ has to be considered. On top of that the bare lattice operators may mix with operators of the same dimension under renormalization. In particular if chiral symmetry is broken by the lattice regularization (like with Wilson fermions) the bare operators $O^{\rm eff}_{\rm VV+AA}$, $O^{\rm eff}_{\rm VV-AA}$, $O^{\rm eff}_{\rm SS+PP}$ and $O^{\rm eff}_{\rm SS-PP}$ mix among themselves. In the static approximation, it has been shown in [47, 48] by using symmetry arguments that all the chirality breaking mixings can be ruled out if one works with Wilson-tmQCD at maximal twist.[1] The renormalization constants needed, in a mass independent scheme, can then be obtained by renormalizing the parity-odd operators $O^{\rm stat}_{\rm VA+AV}$ and $O^{\rm stat}_{\rm SP+SP}$ in the standard Wilson case [48] where indeed $O^{\rm stat}_{\rm VA+AV}$ and $O^{\rm stat}_{\rm SP+SP}$ do not mix with operators of different chirality.

[1] The transformation $\mathscr{P}_{\pi/2}$ introduced in [47] is not completely well defined. The conclusion is anyway unaffected as the absence of mixings in a mass independent scheme can be proven by using the transformations $\mathscr{P}'_{\pi/2}$ and Ex_5 only.

The non-perturbative universal factor relating a matrix element renormalized at the scale μ in the SF scheme to the RGI one is shown in figure 8 for the operators $O^{\rm stat}_{\rm VA+AV}$ and $O^{\rm stat}_{\rm VA+AV} + 4O^{\rm stat}_{\rm SP+SP}$, which renormalize multiplicatively [49]. The figure refers to the computation in the quenched theory. Perturbation theory seems to work for $\mu \geq 1$ GeV for these quantities. In par-

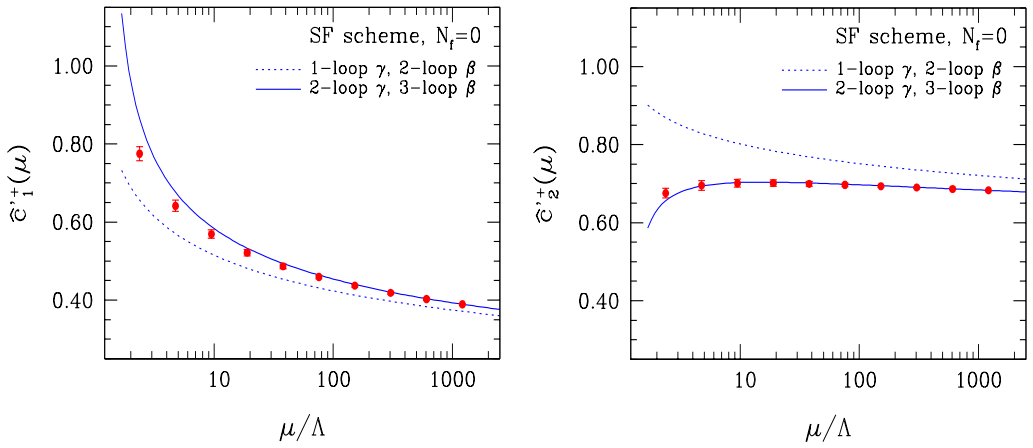

Figure 8: Non-perturbative running matrix elements for the operators $O^{\rm stat}_{\rm VA+AV}$ (left) and $O^{\rm stat}_{\rm VA+AV} + 4O^{\rm stat}_{\rm SP+SP}$ (right) in the SF scheme (cfr. figure 2). Figure from [49].

ticular for $\hat{c}_2^{\prime +}$ the series might seem badly convergent from the difference between the 1/2 and the 2/3-loop results, but quite surprisingly the non-perturbative value eventually agrees with the 2/3-loop one on all the energy range plotted. Similar findings apply to the $N_{\rm f} = 2$ theory as well [50]. In that case however the final errors on the running of the matrix elements are much larger (3% - 8%), which severely limits the eventual precision one can reach on the weak matrix elements. An improvement might be obtained by repeating the calculation at finer lattice spacings.

4. Form factors for heavy-light and heavy-heavy semi-leptonic decays

Semi-leptonic decays of B mesons are still the most precise channel for measuring e.g. $|V_{\rm ub}|$. On the theoretical side they are described in a well-understood way (compared to hadronic decays) and experimentally they are easier to study than the less abundant purely leptonic decays. Taking as prototype the $B \to \pi l \nu$ transition, the differential decay rate in the SM reads (ignoring the lepton mass)

$$\frac{d\Gamma}{dq^2} = \frac{G_{\rm F}^2}{24\pi^3} p_\pi^3 |V_{\rm ub}|^2 |f_+(q^2)|^2 \,, \tag{4.1}$$

where q is the lepton pair momentum. The form factor $f_+(q^2)$ can be extracted from the matrix element of the vector current

$$\langle \pi(p_\pi)|V^\mu|B(p_{\rm B})\rangle = f_+(q^2)(p_\pi + p_{\rm B} + q\Delta_{m^2})^\mu + f_0(q^2) q^\mu \Delta_{m^2} \,, \tag{4.2}$$

with $\Delta_{m^2} = (m_{\rm B}^2 - m_\pi^2)/q^2$. For vector to pseudoscalar transitions the decay rate is parameterized by four form factors which can be obtained from matrix elements of the axial and the vector current [51].

The differential decay rate in eq. (4.1) grows with the pion momentum and therefore experimental measurements are more precise for large values of p_π. On the lattice, on the other hand, only the low p_π (or large q^2) region is safe from large cutoff effects. Incidentally that is also the region where HQET is applicable. Notice however that the sensitivity of the matrix element in eq. (4.2) to $f_+(q^2)$ vanishes for $q \to q_{\rm max} = (m_{\rm B} - m_\pi, \vec{0})$ as the kinematical factor in front of $f_+(q^2)$ vanishes in that limit. The form factors are therefore directly computed on the lattice only for some large value of q^2 and then parameterized over the whole q^2 region using functional forms, which include kinematical constraints, HQET scaling and dispersion relations as originally proposed in [52].

The most recent lattice determination of f_+ and f_0 is due to the HPQCD Collaboration [53]. The three-point correlation functions needed to extract the form factors have been computed on the same set of $N_{\rm f} = 2+1$ configurations used for measuring $B_{\rm B_s}$ plus additional sets at lighter sea quark masses, down to $m_{\rm l}/m_{\rm s} = 0.125$. The b-quark has been simulated in the NRQCD formalism with one-loop matching of the currents to $O(1/m_{\rm b})$, i.e. including one-loop subtracted dimension-four operators. The subtraction in this case contributes about 5% of the final result on the matrix element. Four lattice momenta have been used for the pion, $\vec{p}_\pi = \{(0,0,0),\ (0,0,1),\ (0,1,1)$ and $(1,1,1)\} \times \frac{2\pi}{L}$. For each light quark mass the data are interpolated to fixed common values of E_π and then extrapolated to the physical point using SχPT and continuum χPT to assess the uncertainties in the extrapolation. In the chiral fits the coupling $g_{\rm B^*B\pi}$ entering the chiral logs is left free to vary in order to use the functional form suggested by SχPT also for $E_\pi > 2m_\pi$. The results are plotted in figure 9 together with the curve obtained from the 4-parameter Ball-Zwicky fit [54]. The errors shown are statistical and chiral extrapolation errors only. As expected

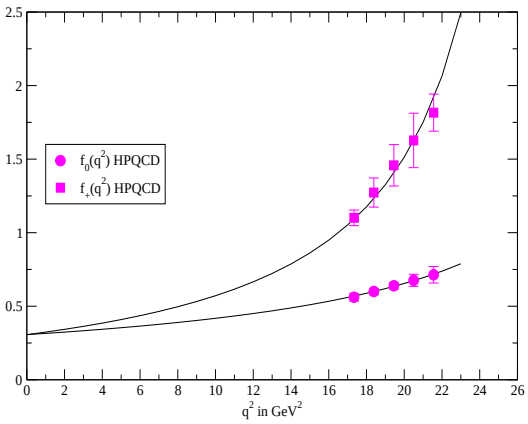

Figure 9: Form factors $f_+(q^2)$ and $f_0(q^2)$ from [53]. Errors are combined statistical and chiral extrapolation errors. The curve is the Ball-Zwicky parameterization fit. Courtesy of Junko Shigemitsu.

from the discussion at the beginning of this section they grow for large q^2 and statistic is being accumulated to reduce them. The total error in the final budget is 14%, mainly due to statistic, chiral extrapolation and matching of the currents. The parameterization of $f_+(q^2)$ is used to obtain

$$\frac{1}{|V_{\rm ub}|^2}\int_{16{\rm GeV}^2}^{q^2_{\rm max}} \frac{d\Gamma}{dq^2} dq^2 = 2.07(41)(39)\ {\rm ps}^{-1}\ , \tag{4.3}$$

which combined with the experimental result from the HFAG [3] for the integrated decay rate in the equation above gives $|V_{ub}| = 3.55(25)(50) \times 10^{-3}$. The tension with the inclusive determination $|V_{ub}| = 4.49(33) \times 10^{-3}$, which in the SM is dis-favored by the global Unitarity Triangle fits [55] and poses problems also for Minimal Flavor Violating extensions of the SM [56], is still there.

An alternative and complementary approach, which can provide precise results for large q^2, consists in using Heavy Flavor χPT. At leading order in chiral perturbation theory and through the order $1/m_b$ in the heavy quark expansion [57]

$$f_+(q^2) = -\frac{F_{B^*}}{2F_\pi}\left[g_{B^*B\pi}\left(\frac{1}{v\cdot k_\pi - m_{B^*} + m_B} - \frac{1}{m_B}\right) + \frac{F_B}{F_{B^*}}\right], \tag{4.4}$$

where v is the velocity of the heavy meson (notice $[F_{B^*}] = 2$ in the formula). The method requires a computation of the coupling $g_{B^*B\pi}$, which in the static approximation is obtained from the matrix element of the light-light axial current between a B and a B^* state at zero momentum (and is called $\hat{g}$). A very precise determination of $\hat{g}$ in the quenched approximation has appeared this year in [58], while a preliminary $N_f = 2$ result has been presented at this conference [59]. In both cases the HYP1 static action [23] has been used and the required two- and three-point correlation functions have been evaluated adopting the all-to-all techniques introduced in [60]. In the quenched approximation 100 eigenvectors have been computed for the low-mode part of the correlators whereas in the dynamical case 200 eigenvectors were needed, the number of configurations used, on the other hand, was only 32 and 100, respectively. The lattice spacings in both cases were quite coarse, 0.1 fm in the quenched computation and 0.2 fm for $N_f = 2$. The results are collected in figure 10 (from [59]) as a function of the pseudoscalar meson mass. In the $N_f = 0$ case the final

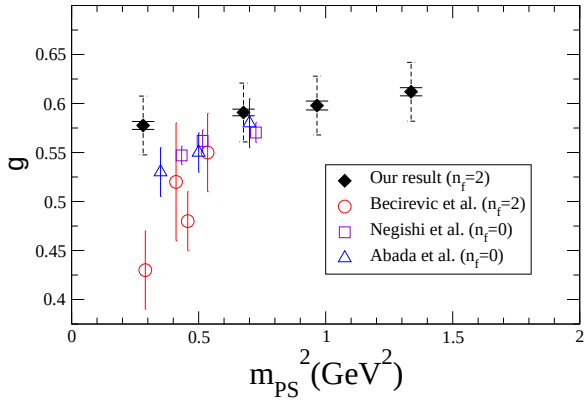

Figure 10: Compilation of results for $\hat{g} = g$ from [58, 59, 61, 62]. The two error bars on the diamonds correspond to statistical and perturbative errors. The empty squares and triangles are non-perturbatively renormalized. Figure from [59].

value $\hat{g} = 0.517(16)$ is obtained by extrapolating linearly the data in $a^2 m_\pi^2$ while for the preliminary dynamical result $\hat{g} = 0.55(1)(3)(3)(6)$ different extrapolations have been compared. For the latter value the first error is statistical, the second from the chiral extrapolation, the third from the renormalization factor (computed at one-loop only) and the fourth is an estimate of discretization effects.

Considering now heavy to heavy transitions, the Rome II group in [63] presented a quenched computation of the form factor $G(w)$, where w is the scalar product of the velocities of the meson

in the initial and in the final state, for the $B \to Dl\nu$ decay. The computation makes use of the step scaling method developed by the group in order to avoid resorting to effective theories. In a small volume of linear size $L_0 = 0.4$ fm and a very fine lattice resolution the form factor $G(w, L_0)$ is computed at the physical values of the bottom and charm quark masses. This number is of course affected by very large finite size effects, which are removed by multiplying (twice) by the step scaling function $\sigma(L, s, w, m_{\rm h})$ defined (for a heavy, would be bottom, quark of mass $m_{\rm h}$) as

$$\sigma(L, s, w, m_{\rm h}) = \frac{G(sL, w, m_{\rm h})}{G(L, w, m_{\rm h})}\,, \quad s > 1\,. \tag{4.5}$$

For $L > L_0$ the step scaling function can not be computed directly at the physical value of the b-quark mass, and the key idea of the approach is exactly that it is enough to compute it for $m_{\rm h} \simeq \frac{L_0}{L} m_{\rm b}$ (which typically means that in the last step $m_{\rm h} \simeq m_{\rm c}$) and then extrapolate it in $1/m_{\rm h}$ to $m_{\rm b}$. The extrapolation is expected to be smooth as finite size effects (which is what the step scaling function describes) shouldn't depend strongly on the heavy-mass scale. This is found to be true in all applications of the method (see [64] for recent ones where the step scaling functions have been computed also in HQET to turn the extrapolations into interpolations). For the case at hand the form factor is finally obtained in a $(1.2\ \text{fm})^3$ volume as

$$G(w) = \sigma(2L_0, 1.5, w, m_{\rm b})\sigma(L_0, 2, w, m_{\rm b})G(w, L_0)\,. \tag{4.6}$$

Each factor is computed in the continuum limit (although extrapolating in a^2 from two lattice resolutions only for the σ's) and the product is then linearly extrapolated in the light quark mass from masses above $m_{\rm s}/4$. Different values of w have been considered by adopting flavor twisted boundary conditions. The result is shown in figure 11.

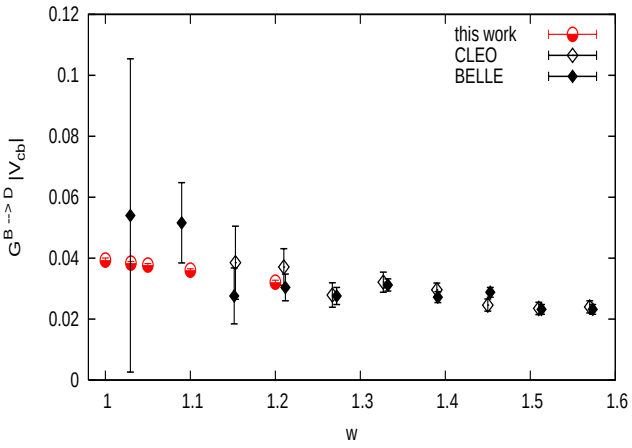

Figure 11: Comparison of $|V_{\rm cb}|G(w)$ with experimental data [65, 66]. The form factor has been computed in [63] and the figure has been obtained by extracting $|V_{\rm cb}|$ at $w = 1.2$, which yields $|V_{\rm cb}| = 3.84(9)_{\rm theo}(42)_{\rm exp} \times 10^{-2}$. Figure from [63].

The ETM Collaboration also computed the form factors for heavy pseudoscalar to pseudoscalar transitions [67] in the same $N_f = 2$ setup used for the computation of $F_{\rm D}$ and $F_{\rm D_s}$ [33] (i.e. with heavy quarks around the charm). The preliminary result at $a \simeq 0.09$ fm is consistent with the one in [63] within the still rather large statistical errors.

We close this section with the computation of the form factor for $B \to D^* l\nu$ at zero recoil from the Fermilab and MILC Collaborations [68]. This channel is less helicity suppressed than the $B \to Dl\nu$ one and it is therefore preferred for the extraction of $|V_{cb}|$. The computation simplifies in the zero recoil kinematics as in this limit only one, usually called h_{A_1}, of the four form factors contributes. It is obtained from the matrix element of the heavy-heavy axial current between B and D^* states. As proposed in [68] the form factor can actually be computed directly from the double ratio

$$\mathscr{R}_{A_1} = \frac{\langle D^*|\bar{c}\gamma_j\gamma_5 b|\overline{B}\rangle\langle\overline{B}|\bar{b}\gamma_j\gamma_5 c|D^*\rangle}{\langle D^*|\bar{c}\gamma_4 c|D^*\rangle\langle\overline{B}|b\gamma_4 b|\overline{B}\rangle} = |h_{A_1}|^2 \,. \tag{4.7}$$

Three similar double ratios had been introduced in [69] in order to compute $h_{A_1}(1)$ to $O(1/m_b^2)$ in the heavy quark expansion. The expression in eq. (4.7) gives the correct answer to all orders and preserves the feature that most of the lattice current renormalizations cancel in the ratio. Notice however that contrary to the double ratios in [69], $\mathscr{R}_{A_1}$ has a non-trivial value different from one already for $m_b = m_c$ and therefore the uncertainty on it doesn't strictly scale as $\mathscr{R}_{A_1} - 1$.

In [68] the method has been applied on the $N_f = 2+1$ MILC rooted staggered configurations together with the Fermilab formalism for heavy quarks. The results are collected in figure 12. The physical, continuum value obtained by using SχPT formulae for the chiral/continuum extrapolations is $h_{A_1}(1) = 0.924(12)(19)$ where the second error is the sum in quadrature of all the systematic ones. By combining it with the experimental measurement (see [3]), the estimate $|V_{cb}| = 3.87(9)_{theo}(7)_{exp} \times 10^{-2}$ is obtained.

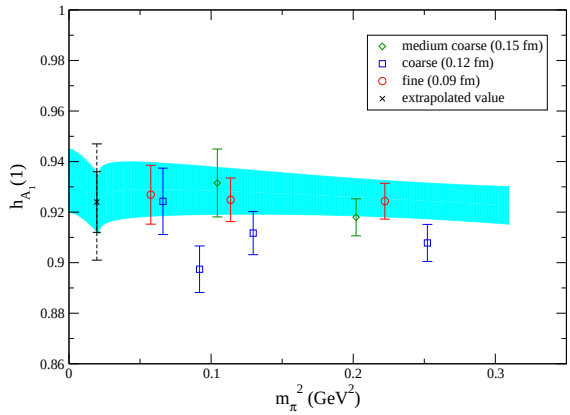

Figure 12: Form factor $h_{A_1}(1)$ from [68] as a function of the lightest pseudoscalar meson mass for unitary points. The band is the continuum extrapolated curve and the dashed line on the physical point is the total error after the inclusion in quadrature of the systematic ones. Figure taken from [68].

5. b-quark mass and B meson decay constant in HQET at O($1/m_b$)

HQET on the lattice was introduced in [70, 71] twenty years ago. It offers a theoretically very sound approach to non-perturbative B-physics as it provides the correct asymptotic description of QCD correlation functions in the limit $m_b \to \infty$. Subleading effects are described by higher dimensional operators whose coupling constants are formally $O(1/m_b)$ to the appropriate power. The

theory can be treated in a completely non-perturbative way including renormalization and matching, in principle to an arbitrary order in $1/m_{\rm b}$, as it was shown in [72]. This implies the existence of the continuum limit at any fixed order in the expansion. However precise computations have been hampered for a long time by the poor signal to noise ratio in heavy-light correlation functions at large time separations, which affects the Eichten-Hill action. The signal can be exponentially improved by considering minimal modifications of the action, where the link in the time covariant derivative is replaced by a smeared link [73, 23]. The inclusion of dynamical quarks is straightforward and the approach can be used together with other methods, as for example the one proposed by the Rome II group (see [64] for such recent applications).

To fix the notation we write the HQET action at $\mathrm{O}(1/m_{\rm b})$ as

$$S_{\rm HQET} = a^4 \sum_x \left\{ \bar{\psi}_{\rm h}(D_0 + \delta_{\rm m})\psi_{\rm h} + \omega_{\rm spin}\bar{\psi}_{\rm h}(-\sigma\mathbf{B})\psi_{\rm h} + \omega_{\rm kin}\bar{\psi}_{\rm h}\left(-\mathbf{D}^2\right)\psi_{\rm h}\right\} \tag{5.1}$$

with $\psi_{\rm h}$ satisfying $P_+\psi_{\rm h} = \psi_{\rm h}$ and $P_+ = \frac{1+\gamma_0}{2}$. The parameters $\omega_{\rm spin}$ and $\omega_{\rm kin}$ are formally $\mathrm{O}(1/m_{\rm b})$. For the computation of the b-quark mass the task is to fix $\delta_{\rm m}$, $\omega_{\rm kin}$ and $\omega_{\rm spin}$ non-perturbatively by performing a matching to QCD. Actually, by considering spin averaged quantities we can immediately get rid of the contributions proportional to $\omega_{\rm spin}$. I will give here a short overview of the computation and present the final results, precise definitions can be found in the corresponding publications [12, 25]. Let us start by remarking that in order not to spoil the asymptotic convergence of the series the matching must be done non-perturbatively (at least for the leading, static piece) as soon as the $1/m_{\rm b}$ corrections are included. Following [74], one can imagine having computed a matching coefficient $C_{\rm match}$ for the static theory at order $l-1$ in perturbation theory. The truncation error $\Delta C_{\rm match}$ is

$$\Delta C_{\rm match} \propto [\bar{g}^2(m_{\rm b})]^l \simeq \left\{ \frac{1}{2b_0 \ln(m_{\rm b}/\Lambda_{\rm QCD})} \right\}^l \gg \frac{\Lambda_{\rm QCD}}{m_{\rm b}} \quad \text{as } m_{\rm b} \to \infty\,, \tag{5.2}$$

where $\bar{g}^2$ is a renormalized coupling at the scale $m_{\rm b}$ and b_0 is the first coefficient of the β function. In other words the perturbative error due to the matching coefficient of the static term is much larger than the power corrections in the large $m_{\rm b}$ limit. In our framework matching and renormalization are performed simultaneously and non-perturbatively.

As the action in eq. (5.1) would produce a non-renormalizable theory, we treat the $1/m_{\rm b}$ corrections to the static, renormalizable theory as space-time insertions in correlations functions. For correlation functions of some multilocal fields O this means

$$\langle O\rangle = \langle O\rangle_{\rm stat} + \omega_{\rm kin}a^4\sum_x\langle OO_{\rm kin}(x)\rangle_{\rm stat} + \omega_{\rm spin}a^4\sum_x\langle OO_{\rm spin}(x)\rangle_{\rm stat}\,, \tag{5.3}$$

where $\langle O\rangle_{\rm stat}$ denotes the expectation value in the static approximation and $O_{\rm kin}(x)$ and $O_{\rm spin}(x)$ are given by $\bar{\psi}_{\rm h}(x)\sigma\mathbf{B}\psi_{\rm h}(x)$ and $\bar{\psi}_{\rm h}(x)\mathbf{D}^2\psi_{\rm h}(x)$, respectively. We work with Schrödinger functional boundary conditions, i.e. we consider QCD with Dirichlet boundary conditions in time and periodic boundary conditions in space (up to a phase θ for the fermions). For the computation in [12] we remain in the quenched approximation. In a small volume of extent $L_1 \simeq 0.4$ fm, one can afford lattice spacings a sufficiently smaller than $1/m_{\rm b}$, in such a way that the b-quark propagates correctly up to discretization errors of $\mathrm{O}(a^2)$. QCD observables defined in this volume are described in HQET

up to effects of $\mathrm{O}\left(\frac{\Lambda_{\rm QCD}}{m_{\rm b}}\right)^2$ and $\mathrm{O}\left(\frac{1}{L_1 m_{\rm b}}\right)^2$. The size L_1 is chosen in order to have the two effects of the same size. We consider two quantities, $\Phi_1^{\rm QCD}(L,m_{\rm h})$ defined exploiting the sensitivity of SF-correlation functions to the angle θ and $\Phi_2^{\rm QCD}(L,m_{\rm h})$, which is given by $L\Gamma_1$ where Γ_1 is a finite volume effective energy. When expanded in HQET[2], $\Phi_1^{\rm HQET}(L)$ is given by $\omega_{\rm kin}$ times a quantity defined in the effective theory (which we call $R_1^{\rm kin}(\theta,\theta')$) while $\Phi_2^{\rm HQET}(L)$ is a function of $\omega_{\rm kin}$ and $m_{\rm bare}=\delta_{\rm m}+m_{\rm h}$ involving two other HQET quantities ($\Gamma_1^{\rm stat}$ and $\Gamma_1^{\rm kin}$) . Obviously, by equating $\Phi_{\rm i}^{\rm QCD}(L_1,m_{\rm h})$ to $\Phi_{\rm i}^{\rm HQET}(L_1)$ one can determine the bare parameters $m_{\rm bare}$ and $\omega_{\rm kin}$ as functions of $m_{\rm h}$ at the lattice spacings used for the volume L_1^3. To eventually compute $m_{\rm b}$ we need the phenomenological, large volume, input of the spin-averaged vector-pseudoscalar B-meson mass, $m_{\rm B}^{\rm av}$. Here we introduce the step scaling functions $\sigma_{\rm ij}(L)$ to evolve the $\Phi_{\rm i}$'s to larger volumes and write

$$\Phi_{\rm i}^{\rm HQET}(2L_1,m_{\rm h})=\sum_{\rm j}\sigma_{\rm ij}(L_1)\Phi_{\rm j}^{\rm QCD}(L_1,m_{\rm h})+\delta_{\rm i2}\sigma_{\rm m}(L_1)\ . \tag{5.4}$$

Notice that $\Phi_{\rm i}^{\rm HQET}(2L_1,m_{\rm h})$ constructed in this way still has a dependence on $m_{\rm h}$, which is inherited from the matching to QCD in L_1. The step scaling functions on the other hand are defined in HQET and have a continuum limit there. After two evolution steps volumes of extent roughly 1.5 fm are reached and the bare parameters $m_{\rm bare}$ and $\omega_{\rm kin}$ can be computed again *as functions of* $m_{\rm h}$ for the corresponding lattice spacings. They are expressed in terms of step scaling functions, $\Phi_{\rm i}^{\rm QCD}(L_1,m_{\rm h})$ and quantities computed in HQET (the large volume version of $R_1^{\rm kin}(\theta,\theta')$, $\Gamma_1^{\rm stat}$ and $\Gamma_1^{\rm kin}$). At this point the b-quark mass can finally be determined by solving for $m_{\rm h}$ the equation

$$m_{\rm B}^{\rm av}=E^{\rm stat}+\omega_{\rm kin}(m_{\rm h})E^{\rm kin}+m_{\rm bare}(m_{\rm h})\ , \tag{5.5}$$

where $E^{\rm stat}=\lim_{L\to\infty}\Gamma_1^{\rm stat}$ and $E^{\rm kin}=-\langle B|a^3\sum_{\mathbf{z}}O_{\rm kin}(0,\mathbf{z})|B\rangle_{\rm stat}$ with $\langle B|B\rangle=1$. In eq. (5.5) I have emphasized the dependence of the bare parameters $m_{\rm bare}$ and $\omega_{\rm kin}$ on $m_{\rm h}$. However when those are re-expressed in terms of $\Phi_{\rm i}^{\rm QCD}(L_1)$, $\sigma_{\rm ij}$, $R_1^{\rm kin}(\theta,\theta',L_2)$, $\Gamma_1^{\rm stat}(L_2)$ and $\Gamma_1^{\rm kin}(L_2)$ the equation involves only quantities which have a continuum limit either in QCD or HQET. This in particular implies that in the procedure all power divergences have been non-perturbatively subtracted.

To see how the different pieces combine together in the final result it is instructive to consider more explicitly the relatively simple case of the computation in the static approximation. In this situation only the parameter $m_{\rm bare}$ needs to be determined. In the small volume the matching condition reads

$$\Gamma_1(L_1,m_{\rm h})=\Gamma_1^{\rm stat}(L_1)+m_{\rm bare}\ , \tag{5.6}$$

and its large volume version is

$$m_{\rm B}^{\rm av}=E^{\rm stat}+m_{\rm bare}\ , \tag{5.7}$$

to this order we could have just as well used $m_{\rm B}$ or $m_{\rm B^*}$ in the previous equation. If we were able to simulate the small and the large volumes at the same lattice spacings we could insert $m_{\rm bare}$ from eq. (5.6) into eq. (5.7) and obtain the master equation

$$m_{\rm B}^{\rm av}=(E^{\rm stat}-\Gamma_1^{\rm stat}(L_1))+\Gamma_1(L_1,m_{\rm h})\ , \tag{5.8}$$

[2]We set the mass counterterm $\delta_{\rm m}$ in the action to zero here. Its contribution is taken into account in the overall energy shift $m_{\rm bare}$ between the effective theory and QCD.

whose solution is the b-quark mass in the static limit. To circumvent the problem we bridge the gap in volume in two steps by inserting a step scaling function $\sigma_{\rm m}(L_1) = 2L_1(\Gamma_1^{\rm stat}(2L_1) - \Gamma_1^{\rm stat}(L_1))$ into the master equation (i.e. we add and subtract $\Gamma_1^{\rm stat}(2L_1)$), which then becomes

$$2L_1 m_{\rm B}^{\rm av} - 2L_1[E^{\rm stat} - \Gamma_1^{\rm stat}] - \sigma_{\rm m}(L_1) = 2L_1\Gamma_1(L_1, m_{\rm h}) \; . \tag{5.9}$$

Now the quantities in the square brackets can be computed at the same values of a and their difference has a well-defined continuum limit in HQET because in the combination we are non-perturbatively removing all the divergences, particularly the linear one. As announced any reference to bare parameters has disappeared in the final equation.

The graphical solution of eq. (5.9) is shown in figure 13. On the horizontal axis we plot $z = L_1 M_{\rm h}$ where $M_{\rm h}$ is the heavy quark mass in the RGI scheme. The result is $M_{\rm b}^{\rm stat} = 6.806(79)$

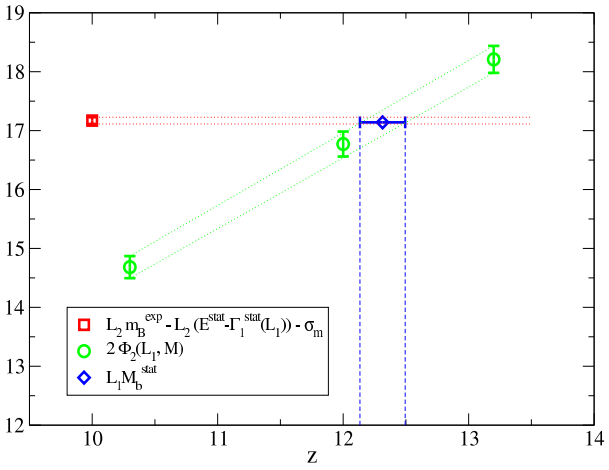

Figure 13: Graphical solution of eq. (5.9) in the quenched approximation. In the caption $\Phi_2(L_1, M) = L_1\Gamma_1(L_1, M_{\rm h})$. Data from [12].

GeV using $r_0 = 0.5$ fm to set the scale [75, 76].

The inclusion of the subleading $1/m_{\rm b}$ effects is more involved and I report here the final results summarized in table 2 . The different numbers correspond to various matching conditions, identified by the choice of the angle(s) θ and by the strategy adopted, "main strategy" for the first line and "alternative strategy" for the second to fourth line. The details can be found in [12], what should be emphasized here is that while there are some differences among the static results depending on the matching condition chosen, those are completely gone once the $1/m_{\rm b}$ terms are included, signalling practically negligible higher order corrections. The value eventually quoted in [12] is $m_{\rm b}(m_{\rm b}) = 4.347(48)$ MeV in the $\overline{\rm MS}$ scheme.

As a further application the decay constant of the $B_{\rm s}$ meson has been computed in quenched QCD including $O(1/m_{\rm b})$ in HQET [25]. Four quantities are needed for the matching, which again has been performed in several different ways. The preliminary results in table 3 show the same pattern discussed for the b-quark mass. Notice however that the difference at $O(1/m_{\rm b})$ is more significant than the errors suggest as most of the uncertainties from the large volume part of the computation cancel in the difference. This indeed yields for instance

$$F_{\rm B_s}^{\rm stat+(1)}(\theta_0 = 0, \theta_1 = 1, \theta_2 = 0) - F_{\rm B_s}^{\rm stat+(1)}(\theta_0 = 1, \theta_1 = 0, \theta_2 = 0.5) = 4 \pm 2 \;\; \text{MeV} \; . \tag{5.10}$$

θ_0	$r_0 M_{\rm b}^{(0)}$	$r_0 M_{\rm b} = r_0 (M_{\rm b}^{(0)} + M_{\rm b}^{(1a)} + M_{\rm b}^{(1b)})$		
		$\theta_1 = 0$	$\theta_1 = 1/2$	$\theta_1 = 1$
		$\theta_2 = 1/2$	$\theta_2 = 1$	$\theta_2 = 0$
0	17.25(20)	17.12(22)	17.12(22)	17.12(22)
0	17.05(25)	17.25(28)	17.23(27)	17.24(27)
1/2	17.01(22)	17.23(28)	17.21(27)	17.22(28)
1	16.78(28)	17.17(32)	17.14(30)	17.15(30)

Table 2: Results for the RGI mass $M_{\rm b}$ from [12].

θ_0	$F_{\rm B_s}^{\rm stat}$ [MeV]	$F_{\rm B_s}^{\rm stat} + F_{\rm B_s}^{(1)}$[MeV]		
		$\theta_1 = 0$	$\theta_1 = 0.5$	$\theta_1 = 1$
		$\theta_2 = 0.5$	$\theta_2 = 1$	$\theta_2 = 0$
0	224 ± 3	185 ± 21	186 ± 22	189 ± 22
0.5	220 ± 3	185 ± 21	187 ± 22	189 ± 22
1	209 ± 3	184 ± 21	185 ± 21	188 ± 22

Table 3: Results for $F_{\rm B_s}$ from [25].

Finally, the results are in good agreement with the determinations in [24, 64], which also go beyond the static approximation.

6. Conclusions

A big effort has been devoted in the last years to removing the quenched approximation from lattice computations. This is absolutely necessary in order to provide precise theoretical estimates to test the Standard Model or to look for signals of New Physics. Several lessons have been learnt from these works. Quenching effects have been proven to be large and chiral extrapolations to be more delicate than in the $N_{\rm f} = 0$ approximation, as partly expected [80, 81].

However B-flavor physics is going to become high-precision physics and other systematics may significantly affect the results. I have shown that the uncertainties associated to renormalization, matching, chiral and continuum extrapolations can easily reach the 5 to 10 percent level. When choosing an approach for performing a first-principle computation the possibility to keep these systematics under control should be included among the requirements.

I have described how these problems can be solved non-perturbatively in Heavy Quark Effective Theory on the lattice. The computations I discussed in this framework on the other hand have been performed in the quenched approximation only and therefore the results in principle are not immediately applicable to phenomenology. The extension to dynamical light fermions is ongoing and first steps have been reported at this conference [77]. The method can be used for several quantities, the b-quark mass and the B-meson decay constant discussed here but also $B - \bar{B}$ mixing parameters and form factors for semi-leptonic decays.

More generally, the non-perturbative matching procedure between HQET and QCD in small volume can be adopted also for other effective theories, as it has been done in [78, 79] for a version of the Fermilab action.

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