Chiral violations from one-loop domain wall fermions

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We present results from lattice perturbation theory for the residual mass and other matrix elements measuring the breaking of chiral symmetry in domain-wall fermions.

We have used the exact propagators corresponding to a finite number of points in the fifth dimensions, and results were obtained for several choices of the domain-wall parameters.

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1. Introduction

Domain-wall simulations use lattices with a finite number of points N_s in the 5th dimension [1], and so a breaking of chiral symmetry occurs. Only in the theoretical limit in which $N_s = \infty$ the chiral modes can fully decouple from each other, yielding an exact chiral symmetry. Here we study these chiral violations using perturbative calculations and computing three quantities: the residual mass m_{res}, the difference $\Delta = Z_V - Z_A$, and c_{mix}, a chirally-forbidden mixing (which is then nonzero at finite N_s) of an operator which measures the lowest moment of the g_2 structure function.

We have studied the dependence of these three quantities on N_s and the domain-wall height M, and calculated the deviations from the $N_s = \infty$ results when N_s is limited to small values, of $O(10)$. We have hence repeated the computations for several choices of N_s and M. A thorough exploration of large regions in the two-dimensional space spanned by N_s and M would be instead quite expensive for Monte Carlo simulations, and perturbation theory seems the more practical and cheaper way to gather hints of what is happening when the parameters are moved in this space.

In order to carry out these calculations one must use the Feynman rules which correspond to the theory truncated at finite N_s, and thus we also had to compute the required propagator functions.

We have calculated the same quantities with the plaquette action [2] as well as with improved gauge actions [3], since in numerical simulations it was observed that these improved gauge actions (especially DBW2) reduce the chiral violations. We refer to [2, 3] for the actions, notations and conventions used, and in particular for the expressions of the domain-wall fermion propagators at finite N_s. Here we only remind that $0 < M < 2$ and that the chiral projectors are $P_\pm = (1 \pm \gamma_5)/2$.

This domain-wall formulation [4] corresponds to having several flavors of lattice Dirac fermions which are mixed via a mass matrix in a very special way, so that a large mass hierarchy is generated. To determine the chiral modes one must diagonalize (in the fifth dimension) this mass matrix, which however is not hermitian. Its square must then be considered, which means the second-order operators $DD^\dagger$ and $D^\dagger D$. They are hermitian and nonnegative and give a well-behaved spectrum.

A rotation of the 5-dimensional quark fields $\psi_s(x)$ to the basis which diagonalizes the mass matrix gives finally the expression of the chiral mode:

$$\chi_0(x) = \sqrt{1 - w_0^2} \sum_s (P_+ w_0^{s-1} \psi_s(x) + P_- w_0^{N_s - s} \psi_s(x)),$$

where from now on we put $w_0 = 1 - M$. We can see from the damping factors w_0^{s-1} and $w_0^{N_s-s}$ that the chiral mode is exponentially localized near the two walls at $s = 1$ and $s = N_s$. However, the domain-wall height M, which is not protected by chiral symmetry, undergoes an additive renormalization, so that w_0 is also additively renormalized.

The standard chiral mode used in Monte Carlo simulations is then not $\chi_0(x)$. It contains instead only the quark fields exactly located at the boundaries:

$$q(x) = P_+ \psi_1(x) + P_- \psi_{N_s}(x), \qquad \overline{q}(x) = \overline{\psi}_{N_s}(x) P_+ + \overline{\psi}_1(x) P_-.$$

These physical quark fields $q(x)$ are more convenient to use than $\chi_0(x)$: they do not contain w_0 and avoid the problem of its renormalization.

At finite N_s an additional issue arises: $\chi_0(x)$ itself is no longer the exact expression of the chiral mode. In fact, $\chi_0(x)$ at finite N_s is an eigenvector of the mass matrix only up to terms of order $N_s e^{-N_s \alpha(0)}$, where $\alpha(0)$ is a constant determined by $2 \cosh(\alpha(0)) = (1 + w_0^2)/|w_0|$.

2. Residual mass at tree level

The calculation of the propagator of the (approximate) chiral fields $q(x)$ gives (for $m = 0$)

$$\langle q(-p)\overline{q}(p)\rangle = \frac{i\gamma_\mu \sin p_\mu \left(1 - e^{-2N_s\alpha(p)}\right) + e^{-N_s\alpha(p)} \cdot 2W(p)\sinh(\alpha(p))}{1 - W(p)e^{\alpha(p)} - e^{-2N_s\alpha(p)}\left(1 - W(p)e^{-\alpha(p)}\right)},$$

where $W(p) = 1 - M + 2\sum_\lambda \sin^2 \frac{p_\lambda}{2}$ and $2\cosh(\alpha(p)) = (1 + W^2(p) + \sum_\lambda \sin^2 p_\lambda)/|W(p)|$. In the limit of small momentum this 4-dimensional propagator becomes

$$\langle q(-p)\overline{q}(p)\rangle\Big|_{p\ll 1} = -(1 - w_0^2)\frac{i\slashed{p} + w_0^{N_s}(1 - w_0^2)}{p^2 + w_0^{2N_s}(1 - w_0^2)^2}.$$

We can thus see that, although in the bare Lagrangian all quark fields are massless, the truncation of domain-wall fermions at finite N_s generates already at the tree level a nonvanishing residual mass of the physical fields: $am_{res}^{(0)} = -w_0^{N_s}(1 - w_0^2) = -(1 - M)^{N_s}M(2 - M)$. As expected, this tree-level residual mass vanishes when N_s becomes infinite[1]. Its sign can be inferred from the general expression of a fermion propagator of mass μ for small momentum in Euclidean space: $(-i\slashed{p} + \mu)/(p^2 + \mu^2) = 1/(i\slashed{p} + \mu)$. Since we work with even N_s (where the fermion determinant can be proven to be positive), $m_{res}^{(0)}$ is always a negative quantity. With our calculations we have thus reproduced, up to a sign, the result for $m_{res}^{(0)}$ found in [4, 5, 6, 7, 8], where it was derived by considering the quadratic operator $D^\dagger D$, which could perhaps explain the sign discrepancy.

3. Physical propagator at one loop

At one loop we can write[2]

$$\langle q(-p)\overline{q}(p)\rangle_{1\ loop} = \frac{1 - w_0^2}{i\slashed{p} - w_0^{N_s}(1 - w_0^2)} + \frac{1 - w_0^2}{i\slashed{p} - w_0^{N_s}(1 - w_0^2)}\Sigma_q(p)\frac{1 - w_0^2}{i\slashed{p} - w_0^{N_s}(1 - w_0^2)}$$

$$= \frac{1 - w_0^2}{i\slashed{p} - w_0^{N_s}(1 - w_0^2) - (1 - w_0^2)\Sigma_q(p)}.$$

The general form of $\Sigma_q(p)$ for $m = 0$ is (where we call for brevity $\bar{g}^2 = (g_0^2/16\pi^2)C_F$)

$$\Sigma_q(p) = \frac{\bar{g}^2}{1 - w_0^2}\left[\frac{\Sigma_0}{a} + i\slashed{p}\left(c_{\Sigma_1}^{(N_s,M)}\log a^2 p^2 + \Sigma_1\right) - \left(i\slashed{p} - w_0^{N_s}(1 - w_0^2)\right)\frac{2w_0}{1 - w_0^2}\Sigma_3\right].$$

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[1]Indeed, since $w_0 = e^{-\alpha(0)}$, it is easy to see that the terms which are proportional to $w_0^{N_s} = e^{-N_s\alpha(0)}$ rapidly approach zero when N_s becomes large.

[2]Evidencing the damping factors of the external legs and the loop integral $\Sigma_{st}(p)$, the structure of $\Sigma_q(p)$ is

$$\Sigma_q(p) = \sum_{s=1}^{N_s}\sum_{t=1}^{N_s} \frac{1}{1 - w_0^{2N_s}}\left[\left(w_0^{N_s-s} - w_0^{2N_s}w_0^{-(N_s-s)}\right)P_+ + \left(w_0^{s-1} - w_0^{2N_s}w_0^{-(s-1)}\right)P_-\right.$$

$$\left. - w_0\frac{i\slashed{p} - w_0^{N_s}(1 - w_0^2)}{1 - w_0^2}\left(\left(w_0^{s-1} - w_0^{2(N_s-1)}w_0^{-(s-1)}\right)P_+ + \left(w_0^{N_s-s} - w_0^{2(N_s-1)}w_0^{-(N_s-s)}\right)P_-\right)\right]$$

$$\cdot\Sigma_{st}(p)\cdot\frac{1}{1 - w_0^{2N_s}}\left[\left(w_0^{N_s-t} - w_0^{2N_s}w_0^{-(N_s-t)}\right)P_- + \left(w_0^{t-1} - w_0^{2N_s}w_0^{-(t-1)}\right)P_+\right. \tag{3.1}$$

$$\left. - w_0\left(\left(w_0^{t-1} - w_0^{2(N_s-1)}w_0^{-(t-1)}\right)P_- + \left(w_0^{N_s-t} - w_0^{2(N_s-1)}w_0^{-(N_s-t)}\right)P_+\right)\frac{i\slashed{p} - w_0^{N_s}(1 - w_0^2)}{1 - w_0^2}\right].$$

The most important difference in Σ_q from its expression at infinite N_s is the appearance of a totally new contribution, Σ_0, proportional to $1/a$ and associated with the breaking of chiral symmetry. Σ_0 comes from the terms of $\Sigma_{st}(p)$ which are of order zero in p, and acts as a mass correction term. Since

$$\langle q(-p)\overline{q}(p)\rangle_{1\ loop} = \frac{1-w_0^2}{i\not{p} - w_0^{N_s}(1-w_0^2) - (1-w_0^2)\Sigma_q(p)} = \frac{1-w_0^2}{i\not{p}\, Z_2^{-1} + m_{res}^{(1)}}Z_w,$$

the one-loop radiatively induced mass is given by

$$a m_{res}^{(1)} = -w_0^{N_s}(1-w_0^2) - \bar{g}^2\Sigma_0.$$

Thus, Σ_0 generates a finite additive renormalization to the residual mass when N_s is not infinite[3]. The factor $Z_w = 1 - 2\bar{g}^2\Sigma_3 w_0/(1-w_0^2) = 1 + \bar{g}^2 z_w$ generates the additive renormalization to w_0 at this order [9], as we can see from $(1-w_0^2)Z_w = 1 - (w_0 + \bar{g}^2\Sigma_3)^2 + O(\bar{g}^4)$.

The renormalization of a composite operator $\overline{q}(x)\,O\,q(x)$ which is multiplicatively renormalizable can also be expressed in a simple way:

$$\langle (\overline{q}Oq)\, q\overline{q}\rangle_{1\ loop} = \frac{1-w_0^2}{i\not{p} - w_0^{N_s}(1-w_0^2)} \cdot A_O(p) \cdot \frac{1-w_0^2}{i\not{p} - w_0^{N_s}(1-w_0^2)},$$

where $A_O(p)$ contains also the contribution of the damping factors, and takes the form

$$A_O(p) = \bar{g}^2\left(-\gamma_O^{(N_s,M)}\log a^2 p^2 + B_O\right)$$

for a logarithmically divergent operator. The coefficients of the divergences turn out to be different from their continuum values, and they depend on N_s and M. It is only when $N_s = \infty$ that the anomalous dimensions become the ones calculated in the continuum theory. In particular, the vector and axial-vector currents acquire a nonzero anomalous dimension at any finite N_s:

$$\gamma_V^{(N_s,M)} = 2w_0^{2N_s}\left(1 - N_s w_0^{2N_s}\frac{1-w_0^2}{1-w_0^{2N_s}}\right)\left(N_s\frac{1-w_0^2}{1-w_0^{2N_s}}\left(2 + \frac{1}{1-w_0^{2N_s}}\right) - 2 - \frac{w_0^2}{1-w_0^{2N_s}}\right).$$

Furthermore, the residual mass as well as renormalization factors and mixing coefficients turn out to lose gauge invariance when N_s is not infinite. Although numerically the deviations from gauge invariance remain in most practical cases rather small, this is another of the pathological features of the domain-wall theory truncated at finite N_s. It could be that this is a limitation of perturbation theory, but it could be that a small gauge dependence is also present in numerical simulations.

These pathologies could actually be related to the mismatch between the (simplified) chiral modes which are actually used, and the true chiral modes (the ones which contain w_0). Notice that at finite N_s there is an additional mismatch, because terms of order $N_s e^{-N_s\alpha(0)}$ and higher, which are present in the true chiral modes for $N_s < \infty$, are here missing as well. Thus, if calculations with the true chiral modes would be gauge invariant and reproduce the continuum anomalous dimensions, the missing pieces from these mismatches could then account for the above pathologies.

At one loop two diagrams contribute to Σ_0 and so enter in the calculation of the residual mass: the half-circle (or sunset) and the tadpole diagrams. We have automated the calculations of the half-circle diagram (as well as the vertex diagrams for Δ and c_{mix}) by developing suitable FORM codes [10], integrating afterwards the corresponding expressions by means of Fortran codes. With these programs we are able to compute matrix elements for general values of N_s and M.

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[3]Of course higher loops and nonperturbative effects give further contributions to the shift of the residual mass.

4. The tadpoles

The behavior of the tadpole diagrams as N_s and M change is particularly important.

The tadpoles do not contain pure 5-dimensional quark propagators, and so for them the $\Sigma_{st}(p)$ of Eq. (3.1) is diagonal in the fifth-dimensional index, and also proportional to $(i\rlap{/}{k} - 4r/a)\, G_{\mu\nu}(k)$, where $G_{\mu\nu}$ is the gluon propagator. This is the same integrand of the tadpoles for Wilson fermions, where in the case of the tadpole diagram contributing to Σ_1 it gives the result (in a general covariant gauge) $T_l = 8\pi^2 Z_0 (1 - 1/4\,(1 - \lambda))$, with $Z_0 = 0.154933390231\ldots$ a well-known integral [11].

It is then clear that for domain-wall fermions the behavior of the tadpole diagrams as a function of N_s (and M) is completely determined by the damping factors in the fifth dimension, (see Eq. (3.1)). Their general effect can be already seen by looking at their leading contributions for large N_s. In this approximation the damping factors enter the game in either of these combinations:

$$\sum_{s=1}^{N_s} w_0^{s-1} w_0^{N_s-s} = N_s w_0^{N_s-1}, \qquad \sum_{s=1}^{N_s} (w_0^2)^{s-1} = \sum_{s=1}^{N_s} (w_0^2)^{N_s-s} = \frac{1 - w_0^{2N_s}}{1 - w_0^2}.$$

These are indeed the leading expressions, in units of $T_d = (1 - w_0^2) T_l / (1 - w_0^{2N_s})^2$, for the tadpole contributions to (respectively) Σ_0 and Σ_1, in the limit of large N_s. Already from these asymptotic expressions (before computing the exact results) we can immediately see that the tadpole of Σ_0 vanishes when $N_s = \infty$, while the tadpole of Σ_1 gives in this limit the known Wilson number, T_l.

Thus, the damping factors play a primary rôle in determining the values of the domain-wall tadpoles. After calculating their exact expressions, which include all subleading terms in N_s, the tadpole contribution to Σ_0 turns out to be equal to

$$4 T_d \left[N_s (1 + w_0^{2(N_s+1)}) w_0^{N_s-1} - 2 w_0^{N_s+1} \frac{1 - w_0^{2N_s}}{1 - w_0^2} \right],$$

while the tadpole contribution to Σ_1 turns out to be equal to

$$T_d \left[(1 + w_0^{2(N_s+1)}) \frac{1 - w_0^{2N_s}}{1 - w_0^2} - 2 N_s w_0^{2N_s} \right].$$

The values of these tadpoles present wide variations with N_s and M, so that sometimes they turn out to be small while in other situations they become large. This suggests that some care should be used when talking about tadpole dominance in relation to domain-wall fermions. It also happens that both tadpoles (of Σ_0 and Σ_1) even decrease toward zero for $M \to 0$ or $M \to 2$.

A central point is that there are two kinds of tadpoles in the game here:

- the tadpole of order zero in p, which tends to zero for $N_s \to \infty$, and which contributes to Σ_0 and the residual mass;

- the tadpole of order ap, which tends to its Wilson value for $N_s \to \infty$, and which contributes to Σ_1 and the renormalization factors.

They behave quite differently, and this is because the $i\rlap{/}{p}$ of the first order of the self-energy flips the chirality of some damping factors, which then combine in a different way. We stress that this is quite unlike the Wilson case, where the tadpole of Σ_0 is just proportional to the tadpole of Σ_1:

$$T_{(\Sigma_0)}^{(Wilson)} = -4 \, T_{(\Sigma_1)}^{(Wilson)}.$$

Tadpole improvement seems then not to be appropriate for the residual mass: the tadpole which contributes to m_{res} goes to zero for large N_s or for $M \to 1$, and for small N_s it assumes a wide spectrum of values. Our interest is small N_s, where it is unclear what tadpole improvement (or resummations) could mean. Moreover, no tadpole enters at all in the calculations of Δ and c_{mix}.

For large N_s the tadpole of Σ_1 is rather close to its Wilson value, and that is why tadpole improvement could be used in the calculations of the renormalization factors (in the large N_s limit).

5. Residual mass at one loop

Our one-loop perturbative calculations show that the numerical deviations from the case of infinite N_s depend, apart from N_s (and to a smaller extent from g_0), very strongly on the choice of M. We can observe that the deviations from the case of exact chiral symmetry are rather pronounced when $M \sim 0.1$ or $M \sim 1.9$. The values of $am_{res}^{(1)}$ turn out to be positive only for $M \geq 1.2$ (at least for even N_s and if the coupling is not very small), and our results suggest that the minimal amount of chiral violations is attained for $M \sim 1.2$. This is then the optimal choice of M from the point of view of one-loop calculations, and corresponds to the renormalization of M, which is not protected by chiral symmetry and is then moved by radiative corrections away from its free field value $M = 1$. One can conjecture that higher-loop corrections and nonperturbative effects would shift this optimal value further on, until the minimal point is eventually reached around $M \sim 1.8$ (which provides the smallest residual mass in Monte Carlo simulations).

We also can observe that for $M = 1.9$ the residual mass at $N_s = 12$ is larger than at $N_s = 8$, and at $N_s = 16$ is even larger. For a detailed discussion of these phenomena which occur near the borders of the allowed values for M we refer to [2].

With improved gauge actions we can still see that the residual mass $am_{res}^{(1)}$ is positive only for $M \geq 1.2$. This also shows that improved gauge actions do not behave too differently in terms of the additive renormalization undergone by w_0. Employing improved gauge actions produces, not surprisingly, a suppression of m_{res} when one carries out the comparisons at the same value of the coupling. The Iwasaki action gives a stronger suppression than the Lüscher-Weisz action, and there seems to be a monotonic decrease of the residual mass as c_1 grows. The DBW2 action is indeed the most effective in generating large suppressions.

If comparisons between the various actions are instead made at the same energy scale, the picture that comes out is different from naive expectations. For example, for quenched QCD at 2 GeV one has to take $\beta = 5.7$ for the Lüscher-Weisz action, $\beta = 2.6$ for the Iwasaki action, and $\beta = 1.04$ for the DBW2 action[4]. The m_{res} numbers for the Iwasaki action are then rather close to those of the DBW2 action, and surprisingly they lie in general slightly above the plaquette values.

However, for the quenched DBW2 action at 2 GeV one has $g_0^2 = 5.77$, which is rather large, and so the one loop results cannot perhaps be trusted so easily[5]. Moreover, these values of the couplings are determined from numerical simulations, and they then contain informations of a nonperturbative nature, so that a mismatch can arise when one only takes into account the results of the one-loop diagrams calculated for these values of the couplings.

Many numerical results, which we cannot include here for lack of space, can be found in [2, 3].

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[4]We use $\beta = 6/g_0^2$ also for improved actions, instead of $\beta' = 6(1 - 8c_1)/g_0^2$.

[5]In this case the Lüscher-Weisz action gives the largest m_{res} suppression, and indeed g_0^2 at 2 GeV is still close to 1.

6. Bilinear differences, and a power-divergent mixing

Since $Z_V \neq Z_A$ when chiral symmetry is broken, the difference between these renormalization constants, $\Delta = Z_V - Z_A = -(Z_S - Z_P)/2$, provides an estimate of chirality-breaking effects.

The amount of chirality breaking connected to Δ follows a pattern similar to the one of the residual mass: Δ is rather large for small N_s or $|1-M| \sim 1$, it decreases when N_s grows or when $|1-M|$ tends towards zero, and the violations of gauge invariance are very small.

The numbers for Δ come out much smaller, at a given M and N_s, than the ones for the residual mass. For quantities such as four-quark operators it was suggested in [12, 13] that their chiral violations are of $O(m_{res}^2)$. Given the smallness of the numbers that we have obtained for Δ, it is possible that something similar is also occurring here.

We have also calculated the mixing of the antisymmetric operator

$$O_{d_1} = \bar{q}(x)\,\gamma_{[4}\gamma_5 D_{1]}\,q(x)$$

with an operator of lower dimension,

$$c_{mix} \cdot \frac{i}{a}\,\bar{q}(x)\,\sigma_{41}\gamma_5\,q(x).$$

The operator O_{d_1} enters in the calculation of the first moment of the g_2 structure function, and has been simulated using quenched domain-wall fermions with the DBW2 gauge action [14].

The power-divergent mixing of O_{d_1} on the lattice is only caused by the breaking of chirality, and hence it provides a quantitative measure of chiral violations. In the theoretical limit $N_s = \infty$ one has instead $c_{mix} = 0$ and so O_{d_1} becomes multiplicatively renormalized.

The chiral violations associated with c_{mix} are rather small, and thus they also seem to be of higher order in m_{res}. The pattern of the deviations from the case of exact chirality is the usual one.

References

[1] For an overview of recent Monte Carlo results with domain-wall fermions we refer to the plenary talk of Peter Boyle in these Proceedings, and to the references therein.

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