

## Finite $ma$ Errors of the Overlap Fermion

---

**S.J. Dong\*** and **K.F. Liu**

*Dept. of Physics and Astronomy, University of Kentucky, Lexington, KY 40506, USA*

*E-mail: super124@uky.edu*

We shall discuss the finite  $ma$  errors of the overlap fermion in this talk. We present results on the speed of light from the dispersion relation and the hyperfine splitting between the vector and pseudoscalar mesons as a function to  $ma$  to reveal the  $m\Lambda_{QCD}a^2$  and  $m^2a^2$  errors. We conclude from this study that one should be limited to using  $ma$  less than 0.5 in order to keep the systematic  $ma$  errors below a few percent level.

*The XXV International Symposium on Lattice Field Theory  
July 30-4 August 2007  
Regensburg, Germany*

---

\*Speaker.

In the last few years, there have been a number of studies to check how well current algebra relations are satisfied with various numerical approximations for the overlap fermion and how feasible it is to carry out large scale calculations with realistically small quark masses. Although it takes two orders of magnitude more time to compute the propagators than those of the Wilson-type fermions, the overlap fermion has a host of desirable features, such as the fact that there are no  $O(a)$  errors, no additive renormalization for the quark masses, no mixing between different chiral sectors, and that the chiral Ward identity and other current algebra relations make the non-perturbative renormalization easier, etc. In addition, there are a number of pleasant surprises in that the  $O(a^2)$  errors, as judged from hadron masses, is small [1] and is about the smallest among the fermion actions studied in the quenched simulations [2]. The overlap fermion is local for lattice spacing as coarse as 0.2 fm with the range being roughly one lattice spacing for the Euclidean distance [2]. The renormalization factors from the chiral Ward identity and the regularization independent scheme [3] have very little dependence on  $ma$  [4] for  $ma$  as large as 0.7. The speed of light as calculated from the dispersion relation deviates from unity appreciably only for  $ma$  larger than  $\sim 0.55$  [5].

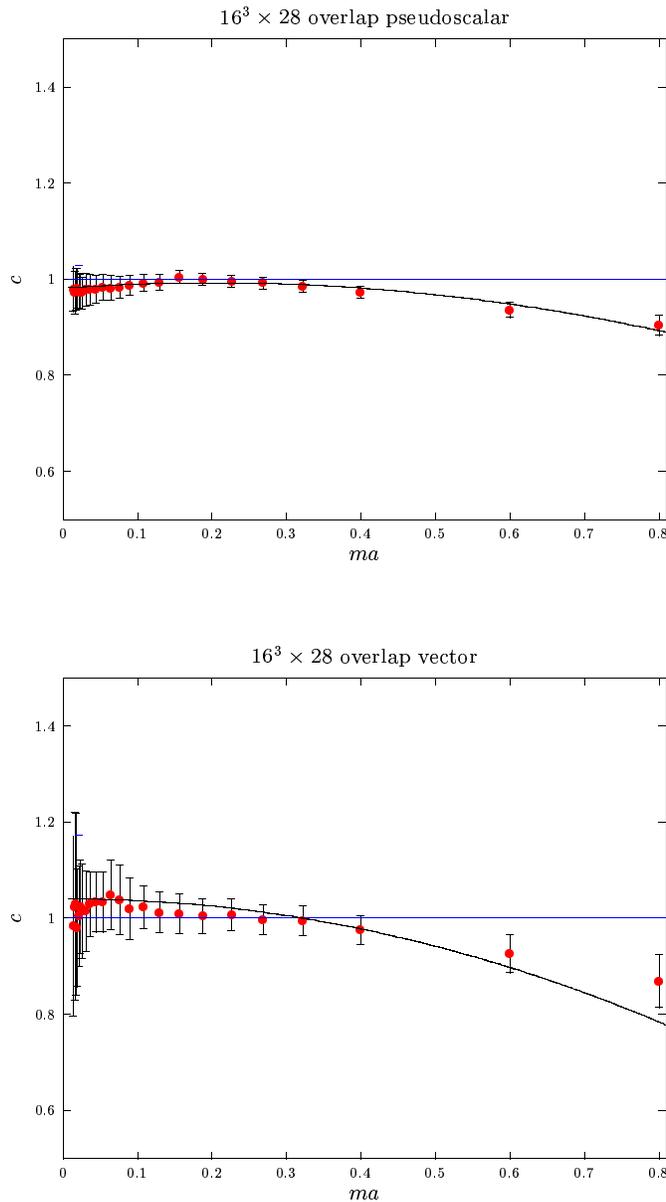
It was emphasized that the effective quark propagator for the overlap fermion has the same form as that in the continuum, i.e. the inverse effective propagator is just an anti-hermitian Dirac operator plus the bare quark mass term [5]. As such, the overlap fermion is equally applicable to the heavy as well as the light quarks. The only practical concern is how large the  $ma$  errors are for the heavy quarks. Thus, it is essential to assess the  $ma$  errors before one can confidently apply the overlap formalism to heavy quarks for a specific  $ma$ . For this purpose, we present results on the dispersion relation and the hyperfine splitting between the vector and pseudoscalar mesons as a function of  $ma$  to reveal the  $m\Lambda_{QCD}a^2$  and  $m^2a^2$  errors.

We first examine the  $O(m^2a^2)$  and  $O(ma^2)$  errors in the dispersion relation. It is suggested that dispersion relation is one of the places where one can discern the  $ma$  error [6, 7]. We computed the pseudoscalar meson mass and energies at several lattice momentum, i.e.  $p_La = \sqrt{n}2\pi/La$  with  $n = 0, 1, 2, 3$ . The overlap quark propagators are calculated on the  $16^3 \times 28$  quenched lattice with 80 configurations generated from Iwasaki gauge action with  $a = 2.00$  fm as determined from  $f_\pi$  [8]. Following Refs. [6, 7], we fit the energies to the dispersion relation

$$(E(p)a)^2 = c^2(pa)^2 + (E(0)a)^2 \quad (1)$$

where  $p = 2\sin(p_La/2)$ . The dispersion relation is so defined such that the  $ma$  error is reflected in the deviation of  $c$  (the effective speed of light) from unity.

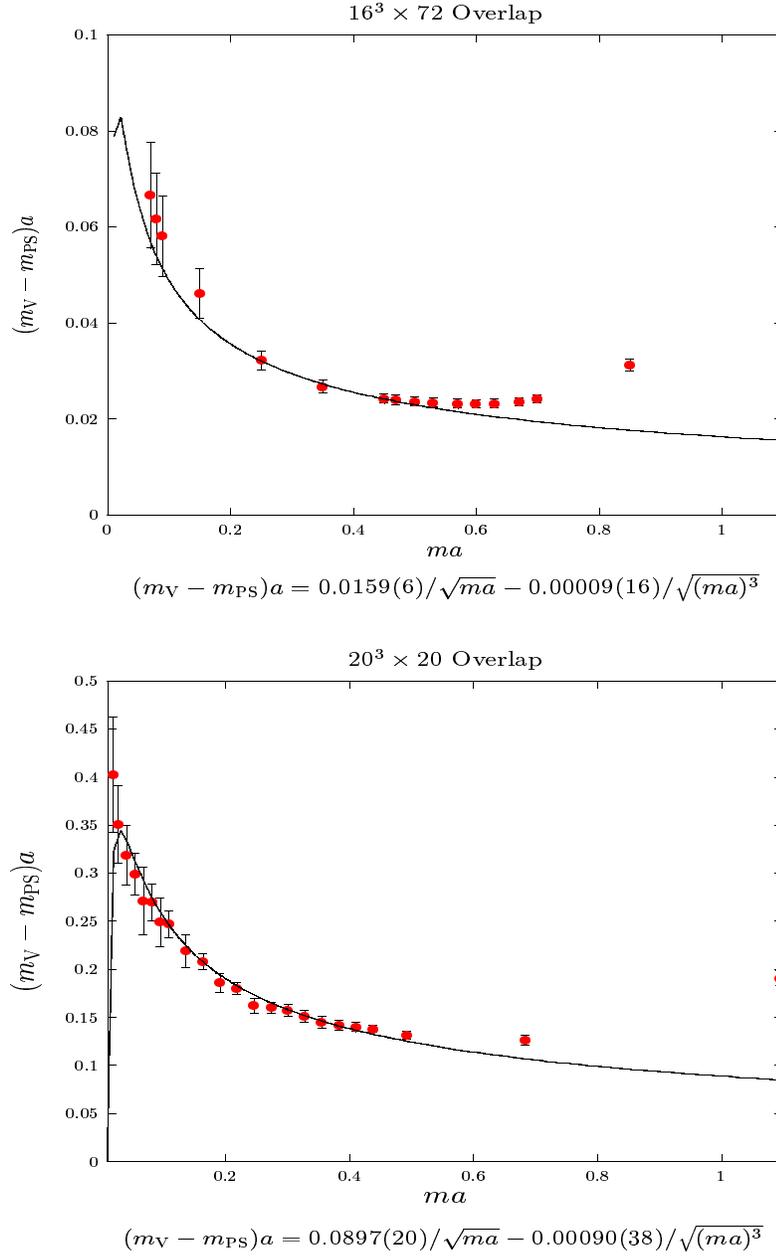
We see in Fig. 1 that the effective speed of light  $c$  is consistent with unity all the way to  $ma \sim 0.4$ . Since there is no  $O(ma)$  error, we fit it with the form quadratic in  $a$ , i.e.  $c = c_0 + b(\Lambda_{QCD}a)ma + dm^2a^2$  ( $\Lambda_{QCD}a = 0.188$  for  $a = 0.2$  fm), and find that  $c_0 = 0.982(10)$ ,  $b = 0.580(346)$ , and  $d = -0.279(87)$  with  $\chi^2/N_{dof} = 0.1$  for the pseudoscalar meson case and  $c_0 = 1.044(43)$ ,  $b = 0.016(38)$ , and  $d = -0.41(36)$  with  $\chi^2/N_{dof} = 0.1$  for the vector meson case. Using these to gauge how large the  $ma$  errors are, we see that the systematic error is less than  $\sim 4\%$  for both the pseudoscalar and vector mesons up to  $ma \sim 0.56$ . This  $ma$  is  $\sim 2.4$  times larger than that is admitted in the study of improved Wilson action [7] where it is found that the  $O(m^2a^2)$  errors from



**Figure 1:** The effective speed of light  $c$  from the pseudoscalar-meson (upper panel) and vector-meson (lower panel) dispersion relations as a function of  $ma$ .

the anisotropy of the dispersion relation for the pseudoscalar and vector mesons are less than  $\sim 5\%$  when  $m_Q a_t < 0.23$ .

The other physical quantity we calculate is the hyperfine splitting between the vector and pseudoscalar mesons as a function of  $ma$ . We should first point out that this hyperfine splitting is expected to go down with the square root of the quark mass for heavy quarks [9]. This is so because the spin-spin part of the one-gluon-exchange interaction which is expected to dominate the short



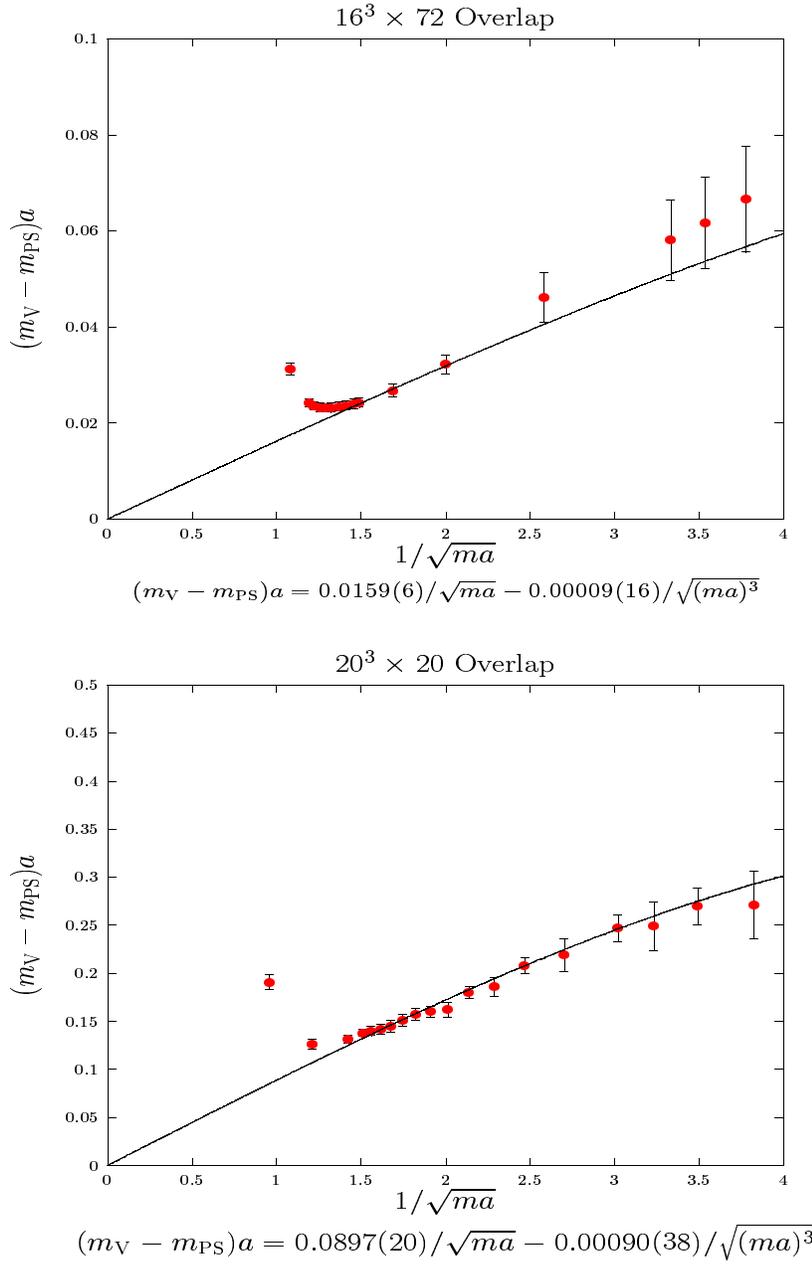
**Figure 2:** The hyperfine splitting on two lattices as a function of  $ma$ . The upper panel is for the  $16^3 \times 72$  lattice with  $a = 0.0561$  fm and the lower panel is for the  $20^4$  lattice with  $a = 0.133$  fm.

distance behavior between the heavy quarks in the quarkonium has the form

$$V_{SS} \propto \frac{\alpha_s \lambda_1 \cdot \lambda_2}{m_1 m_2} \sigma_1 \cdot \sigma_2 \delta(\vec{r}_1 - \vec{r}_2), \quad (2)$$

which leads to a hyperfine splitting between the equal-mass vector and pseudoscalar mesons in first order perturbation in  $\alpha_s$

$$h.f.s \propto \frac{|\Psi(0)|^2}{m^2}, \quad (3)$$



**Figure 3:** The same as in Fig. 2 except as a function of  $1/\sqrt{ma}$ .

where  $\Psi(0)$  is the wavefunction of the quarkonium at the origin. In view of the fact that the  $2S - 1S$  radial excitation and the splitting between the averaged  ${}^3P_2, {}^3P_1$  and  ${}^3P_0$  and the  ${}^3S_1$  state (i.e.  ${}^3P_{avg} - {}^3S_1$ ) of the vector mesons  $J/\Psi$  and  $\Upsilon$  are almost the same, one deduce from the non-relativistic potential model that the size of these mesons scale like

$$r \propto \frac{1}{\sqrt{m}} \quad (4)$$

in order to keep the excitation independent of the quark mass. Since  $|\Psi(0)|^2$  scales like  $r^{-3}$ , hence

one obtains

$$h.f.s \propto \frac{1}{\sqrt{m}}. \quad (5)$$

We show in Fig. 2 the hyperfine splitting between the vector meson and pseudoscalar meson for the  $16^3 \times 72$  lattice with  $a = 0.0561$  fm [10] and the  $20^4$  lattice with  $a = 0.133$  fm [11] as a function of  $ma$ . We notice first that, despite of the fact that the lattice spacings of these two lattices differ by a factor of 2.37, their behaviors in  $ma$  are very similar. Furthermore, the hyperfine splittings in both cases do not approach zero at large quark mass as they should and this is obviously due to the  $ma$  errors. To assess the errors, we plot in Fig. 3 the hyperfine splitting as a function of  $1/\sqrt{ma}$ . It is clear that there is a broad range of  $ma$  where the hyperfine splitting is largely proportional to  $\sqrt{ma}$  as in Eq. (5). But there are a few outliers at large  $ma$  which deviate substantially from the  $1/\sqrt{m}$  behavior. These are due to the systematic  $ma$  errors. We fit the region which is largely linear in  $1/\sqrt{ma}$  with a form which also takes into account the  $1/m$  correction, i. e.

$$h.f.s. = \frac{a}{\sqrt{ma}} \left(1 + \frac{b}{ma}\right). \quad (6)$$

This form fits well in the range of  $ma$  from 0.07 to 0.47 for the  $16^3 \times 72$  lattice and from 0.1094 to 0.438 for the  $20^4$  lattice. The fits are drawn as solid lines in Figs. 2 and 3. We see in both cases, the lattice results start to deviate from the fits around  $ma = 0.5$  and correspondingly  $1/\sqrt{ma} = 1.4$ . For  $ma = 0.6$ , the  $m^2 a^2$  error is about 7%. By the time  $ma$  reaches 0.85, the  $m^2 a^2$  error is about 50%.

By examining the  $ma$  errors of the deviation from the effective speed of light and the hyperfine splitting, we conclude that it is prudent to use  $ma$  smaller than 0.5 in the overlap fermion formalism in order to keep the systematic  $O(ma^2)$  and  $O(m^2 a^2)$  errors to less than 3 to 4 %. This study is done with the Iwasaki gauge action. We have not explored if and how this conclusion varies with different gauge actions.

This work is partially supported by DOE Grants DE-FG05-84ER40154 and DE-FG02-95ER40907.

## References

- [1] S.J. Dong, F.X. Lee, K.F. Liu, and J.B. Zhang, *Phys. Rev. Lett.* **85**, 5051 (2000), [hep-lat/0006004].
- [2] T. Draper, N. Mathur, J. Zhang, A. Alexandru, Y. Chen, S.J. Dong, I. Horvath, F.X. Lee, and S. Tamhankar, [hep-lat/0609034]; *PoS LAT2005*, 120 (2005), [hep-lat/0510075].
- [3] G. Martinelli, C. Pittori, C.T. Sachrajda, M. Testa, and A. Vladikas, *Nucl. Phys.* **B445**, 81 (1995), [hep-lat/
- [4] J.B. Zhang, N. Mathur, S.J. Dong, T. Draper, I. Horváth, F.X. Lee, D.B. Leinweber, K.F. Liu, and A.G. Williams, *Phys. Rev.* **D72**, 114509 (2005), [hep-lat/0507022].
- [5] K.F. Liu and S.J. Dong, *Int. J. Mod. Phys.* **A20**, 7241 (2005), [hep-lat/0206002].
- [6] T.R. Klassen, *Nucl. Phys. Proc. Suppl.* **B73**, 918 (1999), [hep-lat/9809174].
- [7] J. Harada, H. Matsufuru, T. Onogi, and A. Sugita, *Phys. Rev.* **D66**, 014509 (2002), [hep-lat/0203025].

- 
- [8] Y. Chen, S.J. Dong, T. Draper, I. Horváth, F.X. Lee, K.F. Liu, N. Mathur, and J.B. Zhang, *Phys. Rev.* **D70**, 034502 (2004), [hep-lat/0304005]
- [9] K.F. Liu and C.W. Wong, *Phys. Lett.* **B73**, 223 (1978).
- [10] S. Tamhankar, A. Alexandru, Y. Chen, S.J. Dong, T. Draper, I. Horvath, F.X. Lee, K.F. Liu, N. Mathur, and J.B. Zhang ( $\chi$ QCD collaboration), *Phys. Lett.* **B638**, 55 (2006).
- [11] S.J. Dong, T. Draper, I. Horvath, F.X. Lee, K.F. Liu, and J.B. Zhang. *Phys. Rev.* **D65**, 054507 (2002).