PROCEEDINGS OF SCIENCE

$I = 2$ Two-Pion Wave Functions with Non-zero Total Momentum

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We calculate the two-pion wave function for the $I = 2$ S-wave two-pion system with a finite scattering momentum and estimate the interaction range between two pions. It allows us to examine the validity of the necessary condition for the finite-volume method for the scattering phase shift. A calculation is carried out with a plaquette gauge action for gluons and a clover-improved Wilson action for quarks at $1/a = 1.63$ GeV on $32^3 \times 120$ lattice in the quenched approximation. We conclude that the necessary condition is satisfied within statistical errors for the lattice size $L \geq 32$, when the quark mass is in the range $m_\pi^2 = 0.176 - 0.345$ GeV^2 and the scattering momentum in $k^2 < 0.026$ GeV^2. We also find that the energy dependence of the interaction range is small and it takes $1.2 - 1.7$ fm for our simulation parameters. We obtain the phase shift from the two-pion wave function with a smaller statistical error than that from the conventional analysis with the two-pion time correlator.

The XXV International Symposium on Lattice Field Theory
July 30 - August 4 2007
Regensburg, Germany

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PoS(LATTICE 2007)130

1. Introduction

The scattering phase shift is an important quantity for understanding a dynamical aspect of hadrons. For the $I = 2$ S-wave two-pion system, which is the simplest case, the phase shift has been calculated in Refs. [1, 2, 3, 4]. The calculations employed the finite-volume method, in which the phase shift is related to the energy on a finite volume. It has been proposed by Lüscher [5] and extended for the non-zero momentum system by Rummukainen and Gottlieb [6].

The derivation of Lüscher's and Rummukainen-Gottlieb's formula assumes the condition $R < L/2$ for the two-pion interaction range R and the lattice size L, so that the boundary condition does not distort the shape of the two-pion interaction. The CP-PACS collaboration calculated the two-pion wave function for the ground state of the $I = 2$ S-wave two-pion system and estimated the interaction range R from the wave function [7]. In their case the scattering momentum is highly small, $k \sim 0$, thus their work is an examination of the necessary condition for the calculation of the scattering length on the lattice.

In present work we extend their work for the scattering length (the scattering momentum $k \sim 0$) to that for the scattering phase shift ($k \neq 0$). For this aim we consider the ground state of the system having a non-zero total momentum $\mathbf{P} = (2\pi/L)\mathbf{e}_x$ in a L^3 box satisfying the periodic boundary condition. All calculations of this work has been done on VPP5000/80 at the Academic Computing and Communications Center of University of Tsukuba.

2. Finite size formula

In this section we briefly review the finite-volume method presented by Rummukainen and Gottlieb in Ref. [6], with emphasis on the role of the condition for the interaction range. The formula has also been derived from another approaches in Ref. [8] and [9]. We follow, however, from the original derivation in Ref. [6].

The two-pion wave function on a finite periodic box of volume L^3 is defined by

$$\Phi(\mathbf{x},t) = \langle 0|\ \pi^+(\mathbf{X}+\mathbf{x}/2, T+t/2)\ \pi^+(\mathbf{X}-\mathbf{x}/2, T-t/2)\ |\pi\pi; E, \mathbf{P}\rangle \cdot \mathrm{e}^{-i\mathbf{P}\cdot\mathbf{X}} \cdot \mathrm{e}^{E\cdot T}\ , \qquad (2.1)$$

where $|\pi\pi; E, \mathbf{P}\rangle$ is an energy eigenstate of the two-pion system with the energy E and the total momentum $\mathbf{P}$. $\pi^+(\mathbf{x},t)$ is an interpolating operator for π^+ at $(\mathbf{x},t)$. The two exponential factors are introduced to remove the trivial exponential factors for the center of mass coordinate $\mathbf{X}$ and T.

In order to relate the wave function to the scattering phase shift, we transform the wave function $\Phi(\mathbf{x},t)$ in (2.1) to that in the center of mass frame $\Phi_{\mathrm{CM}}(\mathbf{x},t)$ by the Lorentz transformation. Here we assume that the two-pion interaction range R is smaller than one half the lattice extent, *i.e.* there exists the region $R < |\mathbf{x}| < L/2$, where the two pions behave as free particles. In this region the wave function satisfies the following two equations [6].

$$(\nabla^2 + k^2)\ \Phi_{\mathrm{CM}}(\mathbf{x},t) = 0 \quad , \quad \frac{\partial}{\partial t}\ \Phi_{\mathrm{CM}}(\mathbf{x},t) = 0\ , \qquad (2.2)$$

where k is the scattering momentum related to the invariant mass by $\sqrt{s} = \sqrt{E^2 - P^2} = 2\sqrt{m_\pi^2 + k^2}$. $\Phi_{\mathrm{CM}}(\mathbf{x},t)$ also satisfies the boundary condition,

$$\Phi_{\mathrm{CM}}(\mathbf{x},t) = (-1)^{(L/2\pi)\mathbf{P}\cdot\mathbf{m}} \cdot \Phi_{\mathrm{CM}}(\mathbf{x} + L\hat{\gamma}[\mathbf{m}], t) \qquad \text{for } \mathbf{m} \in \mathbb{Z}^3\ , \qquad (2.3)$$

where $\hat{\gamma}$ is the vector operation $\hat{\gamma}[\mathbf{x}] = \gamma \mathbf{x}_{\parallel} + \mathbf{x}_{\perp}$ with the Lorentz boost factor $\gamma = E/\sqrt{s}$, $\mathbf{x}_{\parallel} = \mathbf{P}(\mathbf{P}\cdot\mathbf{x})/P^2$ and $\mathbf{x}_{\perp} = \mathbf{x} - \mathbf{x}_{\parallel}$.

The solution of the (2.2) under the condition (2.3) can be given by

$$\Phi_{\rm CM}(\mathbf{x},t) = \frac{1}{\gamma L^3} \sum_{\mathbf{p}\in\Gamma} \frac{1}{p^2-k^2} \cdot e^{i\mathbf{p}\cdot\mathbf{x}} , \quad \left(\Gamma = \left\{ \mathbf{p} \mid \mathbf{p} = (2\pi/L)\cdot \hat{\gamma}^{-1}[\mathbf{n}] + \mathbf{P}/2,\ \mathbf{n}\in\mathbb{Z}^3 \right\} \right) \quad (2.4)$$

$$= \frac{1}{\gamma L^3} \sum_{\mathbf{p}\in\Gamma} \frac{1}{p^2-k^2} \cdot j_0(kx) + \frac{k}{4\pi} \cdot n_0(kx) + \sum_{lm} C_{lm}(k) \cdot Y_l^m(\Omega) j_l(kx) , \quad (2.5)$$

up to overall constant, where $j_l(kx)$ is the spherical Bessel and $n_l(kx)$ is the spherical Neumann function. $C_{lm}(k)$ is some constant depending on the scattering momentum k. The first and second terms of (2.5) consist of the S-wave component and those coefficients give the S-wave phase shift $\delta(k)$,

$$\frac{1}{\tan\delta_0(k)} = \frac{4\pi}{k} \cdot \frac{1}{\gamma L^3} \sum_{\mathbf{p}\in\Gamma} \frac{1}{p^2-k^2} . \quad (2.6)$$

This is the Rummukainen-Gottlieb formula [6].

3. Details of simulation

In the present work we consider the ground state of the system with the total momentum $\mathbf{P} = \mathbf{0}$ and $\mathbf{P} = (2\pi/L)\mathbf{e}_x$. We can obtain the scattering length from the energy of the system with $\mathbf{P} = \mathbf{0}$ and the phase shift from that with $\mathbf{P} = (2\pi/L)\mathbf{e}_x$ through the Rummukainen-Gottlieb formula (2.6).

In order to calculate the wave function we consider the correlator,

$$F(\mathbf{x},\tau) = \langle 0 |\ \Omega(\mathbf{x},\tau)\ \overline{\Omega}(\mathbf{P},\tau_s)\ | 0 \rangle . \quad (3.1)$$

The operator $\Omega(\mathbf{x},\tau)$ is defined by

$$\Omega(\mathbf{x},\tau) = \sum_{\hat{R}} \sum_{\mathbf{X}} e^{i\mathbf{P}\cdot\mathbf{X}}\ \pi^+(\mathbf{X} + \hat{R}[\mathbf{x}],\tau)\ \pi^+(\mathbf{X},\tau) . \quad (3.2)$$

The vector operation $\hat{R}$ represents an element of the cubic group (O_h) for $\mathbf{P} = \mathbf{0}$ and the tetragonal group (D_{4h}) for $\mathbf{P} = (2\pi/L)\mathbf{e}_x$. The summation over $\hat{R}$ projects out $\mathbf{A}_1^+$ representation of these groups, which equals to the S-wave state ignoring the effects from higher angular momentum $l \geq 2$.

The operator $\overline{\Omega}(\mathbf{P},\tau_s)$ in (3.1) is defined by

$$\overline{\Omega}(\mathbf{P},\tau_s) = \frac{1}{N_R} \sum_{j=1}^{N_R} \left[\pi^+(\mathbf{P},\tau_s;\xi_j)\pi^+(\mathbf{0},\tau_s;\eta_j) \right]^{\dagger} , \quad (3.3)$$

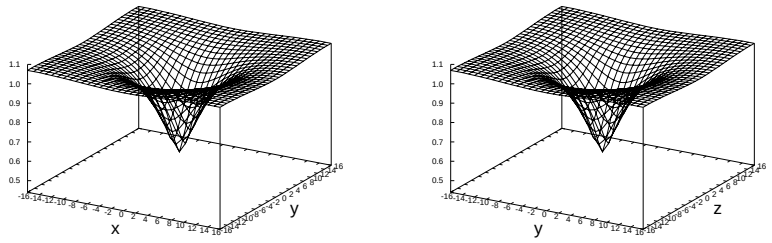
where

$$\pi^+(\mathbf{P},\tau_s;\xi_j) = \left[\sum_{\mathbf{x}} e^{i\mathbf{P}\cdot\mathbf{x}} \overline{u}(\mathbf{x},\tau_s) \xi_j^{\dagger}(\mathbf{x}) \right] \gamma_5 \left[\sum_{\mathbf{y}} d(\mathbf{y},\tau_s)\xi_j(\mathbf{y}) \right] . \quad (3.4)$$

The operator $\pi^+(\mathbf{P},\tau_s;\eta_j)$ is defined as $\pi^+(\mathbf{P},\tau_s;\xi_j)$ by changing $\xi_j(\mathbf{x})$ to $\eta_j(\mathbf{x})$. The functions $\xi_j(\mathbf{x})$ and $\eta_j(\mathbf{x})$ are $U(1)$ noise whose property is

$$\lim_{N_R\to\infty} \frac{1}{N_R} \sum_{j=1}^{N_R} \xi_j^*(\mathbf{x})\xi_j(\mathbf{y}) = \delta^3(\mathbf{x}-\mathbf{y}) . \quad (3.5)$$

(a) $\mathbf{P} = \mathbf{0}$



(b) $\mathbf{P} = (2\pi/L)\mathbf{e}_x$

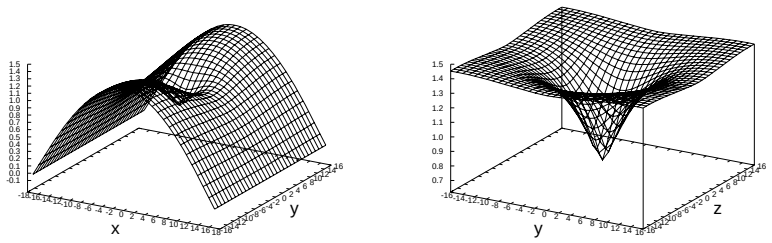


Figure 1: The two-pion wave functions $\Phi_{\rm CM}(\mathbf{x})$ at $m_\pi^2 = 0.176$ GeV^2 for $\mathbf{P} = \mathbf{0}$ and $\mathbf{P} = (2\pi/L)\mathbf{e}_x$. The left and right panels for each momentum show the wave function on *xy*- and *yz*-plane.

In the present work we take $N_R = 2$ in (3.3).

In large τ region, we can obtain the wave function for the ground state in (2.1) by $\Phi(\mathbf{x},0) = F(\mathbf{x},\tau)/F(\mathbf{x}_0,\tau)$ introducing the reference position $\mathbf{x}_0$. In the present work we set $\tau_s = 20$, $\tau - \tau_s = 40$ and $\mathbf{x}_0 = (7,5,2)$. In the estimation of the interaction range, we analyze the wave function in the center of mass frame $\Phi_{\rm CM}(\mathbf{x},t)$ at $t = 0$. It is related to $\Phi(\mathbf{x},t)$ by $\Phi_{\rm CM}(\hat{\gamma}[\mathbf{x}],0) = \Phi(\mathbf{x},0)$ outside of the interacting region ($|\mathbf{x}| > R$) from the second equation in (2.2).

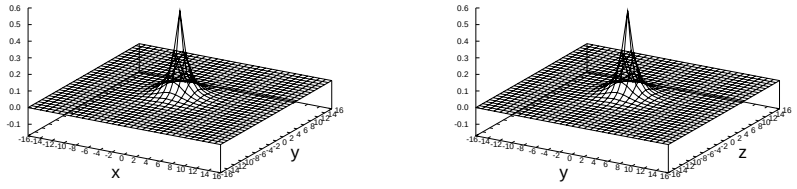
Gauge configurations are generated in a quenched approximation with a plaquette gauge action at $\beta = 5.9$ on a $32^3 \times 120$ lattice. The physical quantities are measured every 200 sweeps independently for each quark mass parameter. A clover fermion action with $C_{SW} = 1.364$ is used. The quark propagators are imposed to the Dirichlet boundary condition in the time direction and to the periodic boundary condition in the spatial one. The lattice cutoff is estimated as $1/a = 1.63(5)$ GeV ($a = 0.121(3)$ fm) from the ρ meson mass. Three quark masses are chosen to give $m_\pi^2 = 0.176$, 0.238 and 0.345 GeV^2. The numbers of configurations are 400, 212 and 212 for each quark masses.

4. Results of interaction range

In Fig. 1, we show the two-pion wave functions $\Phi_{\rm CM}(\mathbf{x})$ at $m_\pi^2 = 0.176$ GeV^2 for $\mathbf{P} = \mathbf{0}$ and for $\mathbf{P} = (2\pi/L)\mathbf{e}_x$. The left and right panels for each momentum show the wave function on *xy*- and *yz*-plane. We find a very clear signal.

We now consider the two-pion interaction from the ratio, $V(\mathbf{x}) = \nabla^2\Phi_{\rm CM}(\mathbf{x})/\Phi_{\rm CM}(\mathbf{x})$. Away from the two-pion interaction range, *i.e.* $|\mathbf{x}| > R$, we expect that $V(\mathbf{x})$ is independent of $\mathbf{x}$ and equals to $-k^2$ from (2.2). In Fig. 2, $V(\mathbf{x})$ for the same parameters as for Fig. 1 are plotted. We find a very

(a) $\mathbf{P} = \mathbf{0}$



(b) $\mathbf{P} = (2\pi/L)\mathbf{e}_x$

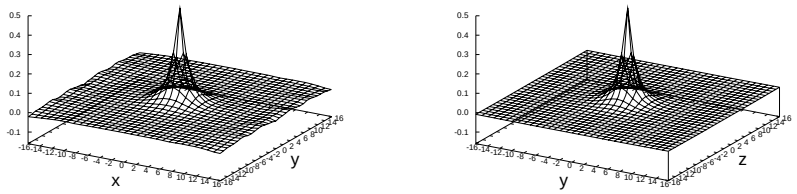


Figure 2: $V(\mathbf{x})$ at $m_\pi^2 = 0.176$ GeV^2 for $\mathbf{P} = \mathbf{0}$ and $\mathbf{P} = (2\pi/L)\mathbf{e}_x$. The left and right panels for each momentum show the $V(\mathbf{x})$ on *xy*- and *yz*-plane.

clear signal and $V(\mathbf{x})$ seems to be constant for $|\mathbf{x}| > 10$. We observe a strong repulsive interaction at the origin consistent with the negative phase shift of the $I = 2$ two-pion system.

In order to estimate the interaction range R, we consider,

$$U(\mathbf{x}) = \nabla^2 \Phi_{\rm CM}(\mathbf{x}) / \Phi_{\rm CM}(\mathbf{x}) + k^2 \ , \tag{4.1}$$

where k^2 is obtained from the two-pion time correlator. According to Ref. [7], we employ the operational definition of the interaction range R as the scale where $U(\mathbf{x})$ is sufficiently small compared to the statistical error. Strictly speaking, even if $|\mathbf{x}|$ takes a large value, $U(\mathbf{x})$ does not vanish and has a finite tail. However, the systematic error for the final results of the phase shift due to the existence of the tail is buried into the statistical error in this definition.

In Fig. 3, we show $U(\mathbf{x})$ as a function of $|\mathbf{x}|$ for $\mathbf{P} = \mathbf{0}$ and $\mathbf{P} = (2\pi/L)\mathbf{e}_x$ at the three quark masses. We find that the interaction range R takes

m_π^2 (GeV^2)	0.176	0.238	0.345
R for $\mathbf{P} = \mathbf{0}$	13.0	14.0	12.0
R for $\mathbf{P} = (2\pi/L)\mathbf{e}_x$	10.0	11.0	12.0 .

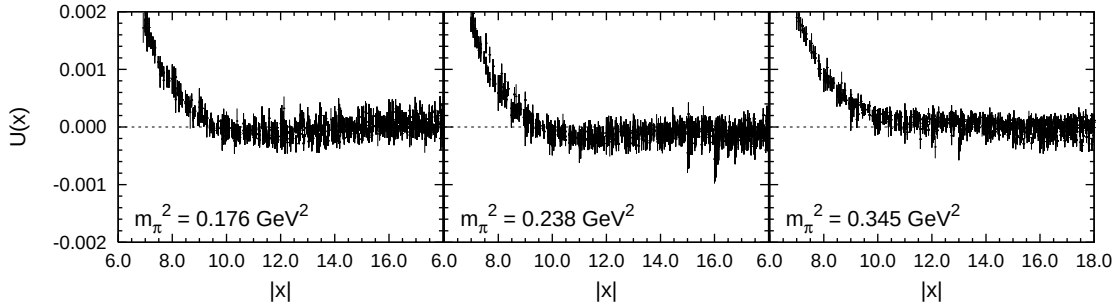
(4.2)

It is at most 14.0 (1.69 fm) and smaller than $L/2 = 16$. Thus the necessary condition for the Rummukainen-Gottlieb formula (2.6) is satisfied within statistical errors at our simulation points.

5. Results of scattering phase shift

We estimate the phase shift with the following three methods :

(a) $\mathbf{P} = \mathbf{0}$



(b) $\mathbf{P} = (2\pi/L)\mathbf{e}_x$

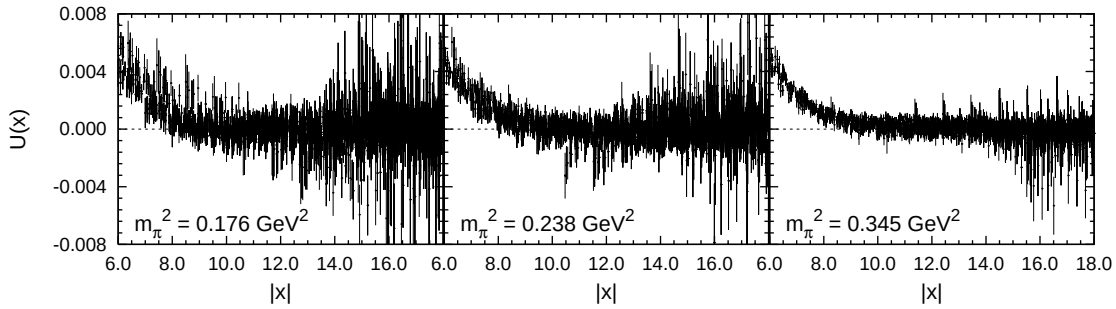


Figure 3: $U(\mathbf{x})$ as a function of $|\mathbf{x}|$ for $\mathbf{P} = \mathbf{0}$ and $\mathbf{P} = (2\pi/L)\mathbf{e}_x$ for several quark masses.

1. We extract the energy E from the two-pion time correlator and calculate the scattering momentum by $k^2 = (E^2 - P^2)/4 - m_\pi^2$. The phase shift is calculated by substituting k^2 into (2.6). The results of the phase shift are shown in the left panel of Fig. 4 (labeled "from T"), where the scattering amplitudes,

$$A(m_\pi, k) = \tan\delta_0(k)/k \cdot \sqrt{m_\pi^2 + k^2} \tag{5.1}$$

   are plotted.

2. We extract k^2 by fitting the wave function $\Phi_{\rm CM}(\mathbf{x})$ with the fitting function given in (2.4) taking k^2 and an overall constant as the fitting parameters. We choose the fitting range $|\mathbf{x}| > R$ with R given in (4.2). The results are plotted in the left panel of Fig. 4 (labeled "from W").

3. k^2 is extracted by fitting $V(\mathbf{x})$ to a constant in the region $|\mathbf{x}| > R$ with R given in (4.2). We show the results in the left panel of Fig. 4 (labeled "from V").

As shown in the left panel of Fig. 4, the results given by the three method are consistent within the statistical errors. The data given by the method 3 ("from V") provides the smallest statistical error, so the following analysis is performed with this data.

In order to obtain the phase shift at the physical quark mass, we extrapolate our results with the fit form,

$$A(m_\pi, k) = A_{10} m_\pi^2 + A_{20} m_\pi^4 + A_{01} k^2 + A_{11} m_\pi^2 k^2 \,. \tag{5.2}$$

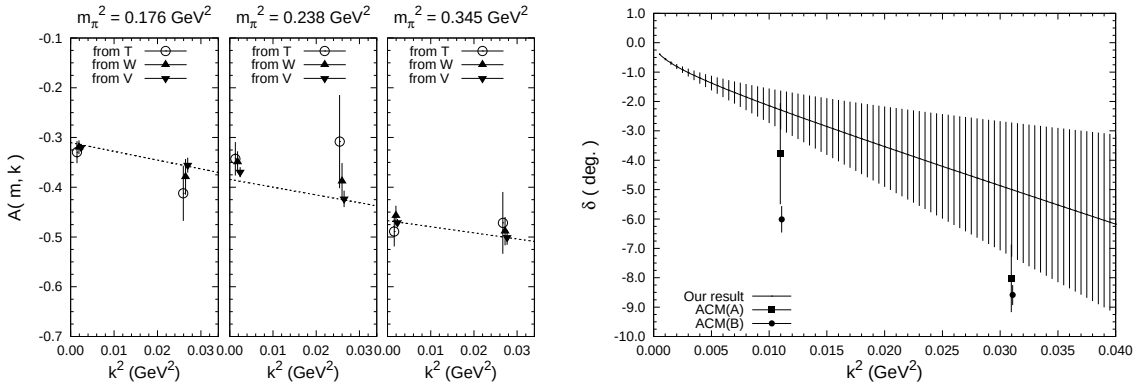


Figure 4: In the left panel $A(m_\pi, k)$ given by the three methods are shown as a function of k^2. The dotted line is the fit curve for the data given by the method ("from V"). In the right panel our final results of the scattering phase shift at the physical quark mass are plotted and compared with the experiment [10].

The fit curves for this fitting are also plotted in the left panel of Fig. 4. As shown in the figure the fitting is carried out well. Our final results of the phase shift at the physical quark mass are shown in the right panel of Fig.4 and compared with the experiment [10]. Our results are slightly larger than the experiment. A possible origin of the discrepancy is finite lattice spacing effects. We must leave the confirmation of this to studies in the future.

6. Conclusion

In the present work, we have studied the $I = 2$ two-pion wave functions with the scattering momentum $k^2 \sim 0$ and $k^2 \neq 0$. We have estimated the two-pion interaction ranges R from those. It has been confirmed that the necessary condition for the Rummukainen-Gottlieb formula (2.6) satisfies within statistical errors in our parameters. Moreover, we have estimated the scattering phase shift with the two-pion wave function. These methods provide a smaller statistical error than that from the conventional analysis with the two-pion time correlator.

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