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Critical behaviour of 3D U(1) LGT at finite temperature

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The three-dimensional compact U(1) LGT is studied at finite temperature. In particular, correlation functions of the Polyakov loops and the 't Hooft operator are computed perturbatively at high temperatures. Performing dimensional reduction the effective two-dimensional model is obtained which describes vortex–anti-vortex dynamics in the high-temperature regime. We explore this effective model to study in details the critical behaviour which is expected to be of the BKT type. Under the standard assumptions we compute critical indices and compare them with those of the two-dimensional XY model.

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1. Introduction

The main motivation to investigate the pure gauge compact three-dimensional (3d) $U(1)$ LGT is two-fold. At zero temperature the theory has a nonvanishing mass gap and a string tension at arbitrarily small coupling constant. This is a feature expected from 4d QCD. At finite temperatures the theory undergoes a deconfinement phase transition. The corresponding phenomenon takes place in 4d QCD as well. It thus appears that the 3d $U(1)$ gauge theory constitutes one of the simplest models with continuous gauge symmetry which possesses the same fundamental properties as QCD. Therefore, it is very important to understand in great details a mechanism which underlies the permanent confinement and deconfinement phase transition on the simpler example of three-dimensional abelian model.

First investigations of $U(1)$ LGT at finite temperature have been performed by Polyakov [1] and Susskind [2]. Their analysis, done for strong coupled Hamiltonian version of 4d model, showed the possibility of the deconfinement phase transition at high temperatures. 3d theory was studied by Parga using Lagrangian formulation of the theory [3]. The picture emerging from this study can be described as follows. At high temperatures the system becomes effectively two-dimensional, in particular the monopoles of the original $U(1)$ gauge theory become vortices of the 2d system. The partition function turns out to coincide (in the leading order of the high-temperature expansion) with the 2d XY model in the Villain representation. The XY model is known to have the Berezinskii-Kosterlitz-Thouless (BKT) phase transition of the infinite order [4, 5] (a rigorous proof of the BKT phase transition existence was done in [6]). According to the Svetitsky-Yaffe conjecture the finite-temperature phase transition in the 3d $U(1)$ LGT should belong to the universality class of the 2d XY model [7]. This means, firstly that the global $U(1)$ symmetry cannot be broken spontaneously because of the Mermin-Wagner theorem [8] and, consequently the absence of the local order parameter. Secondly, the correlation function of the Polyakov loops (which become spins of the XY model) decreases with the power law at $\beta \geq \beta_c$ implying a logarithmic potential between heavy electrons

$$P(R) \asymp \frac{1}{R^{\eta(T)}} \ , \tag{1.1}$$

where the $R \gg 1$ is the distance between test charges. The critical index $\eta(T)$ is known from the renormalization-group analysis of Ref.[5] and equals $\eta(T_c) = 1/4$ at the critical point of the BKT transition. For $\beta < \beta_c$, $t = \beta_c/\beta - 1$ one has

$$P(R) \asymp \exp\left[-R/\xi(t)\right] \ , \tag{1.2}$$

where the correlation length $\xi \sim e^{bt^{-\nu}}$ and the critical index $\nu = 1/2$. Therefore, the critical indices η and ν should be the same in the finite-temperature $U(1)$ model if the Svetitsky-Yaffe conjecture holds in this case. The numerical check of these predictions was performed on the lattices $N_s^2 \times N_t$ with $N_s = 16, 32$ and $N_t = 4, 6, 8$ in [9]. Though authors of [9] confirm the expected BKT nature of the phase transition, the reported critical index is almost three times larger of that predicted for the XY model, $\eta \approx 0.78$.

Thus, so far there is no numerical indications that critical indices of 3d $U(1)$ LGT coincide with those of the 2d XY model. Analytical calculations have been performed in the leading order of the high-temperature expansion only. In finite-temperature simulations the scaling was not reached.

The problem can be in the finite-size effects. In the XY model, due to logarithmic corrections, in order to reliably determine critical indices one should use the FSS technics and/or simulate the model on large thermodynamic lattices, i.e. $L \gg \xi$.

In what follows we concentrate on the studying of the universality problem. In the next section we introduce our conventions and give definition of the compact version of $U(1)$ LGT together with some expectation values. Investigation of the model at limiting values of anisotropic couplings is presented in the Section 3. In this limit the BKT critical behaviour is clearly seen. The perturbative calculations of the Polyakov loop correlations and the 't Hooft operator at high temperatures are the subject of the Section 4. In the Section 5 we derive the effective vortex model and give an analytical predictions for the critical indices of the theory. A summary of our results is presented in the Section 6.

2. Lattice conventions and definition of the model

We work on a $3d$ lattice $\Lambda = L^2 \times N_t$ with spatial extension L and temporal extension N_t. Periodic boundary conditions on gauge fields are imposed in all directions. In what follows we keep notations of the original lattice also for the dual lattice. Introduce anisotropic dimensionless couplings in a standard way as

$$\beta_t = \frac{1}{g^2 a_t} \ , \ \ \beta_s = \frac{\xi}{g^2 a_s} \ = \ \beta_t \ \xi^2 \ , \tag{2.1}$$

where a_t (a_s) is lattice spacing in the time (space) direction and $\xi = \frac{a_t}{a_s}$ is a ratio of lattice spacings. g^2 is a continuum coupling constant with dimension a^{-1}.

$3d$ $U(1)$ gauge theory on the anisotropic lattice is defined through its partition function as

$$Z \equiv Z(\Lambda;\beta_t,\beta_s) = \int_0^{2\pi} \prod_l \frac{d\omega_l}{2\pi} \ \exp[\beta_s \sum_{p_s} \cos\omega(p_s) + \beta_t \sum_{p_t} \cos\omega(p_t)] \ , \tag{2.2}$$

and the plaquette angles $\omega(p)$ are defined in a standard way. In the following we shall also need the plaquette and dual formulations of the model (2.2). The plaquette formulation on the dual lattice can be easily obtained from the corresponding formulation on the isotropic lattice [10, 11] and takes the form

$$Z = \int_0^{2\pi} \prod_l \frac{d\omega_l}{2\pi} \ \exp\left[\beta_s \sum_{l_t} \cos\omega(l_t) + \beta_t \sum_{l_s} \cos\omega(l_s)\right] \prod_x J(x), \tag{2.3}$$

where $J(x)$ is the periodic delta-function which expresses the lattice Bianchi identity

$$J(x) = \sum_{r=-\infty}^{\infty} e^{ir\omega_x} \ , \ \omega_x = \sum_n [\omega_n(x) - \omega_n(x-e_n)] \ . \tag{2.4}$$

Integration over plaquette (dual link) variables leads to the corresponding dual representation of the anisotropic model. The Villain formulation of $3D$ $U(1)$ gauge theory on the anisotropic lattice can be deduced from last formulae. In particular, the dual formulation reads

$$Z = \sum_{r(x)=-\infty}^{\infty} \exp\left[-\sum_x \sum_{n=0}^{2} \frac{1}{2\beta_n} (r(x) - r(x+e_n))^2\right] \ . \tag{2.5}$$

Clearly, the representations (2.3)-(2.5) can be viewed as the dimensional continuations of the link and dual representations of the $2d$ XY model [12, 13, 14].

The correlation function of Polyakov loops in representation j can be written in the dual formulation as a ratio of partition functions

$$P_j(R) \;=\; \left\langle \exp\left[ij\sum_{x_0=0}^{N_t-1}(\omega_0(x)-\omega_0(R))\right]\right\rangle \;=\; \frac{Z_j}{Z_0}\;, \tag{2.6}$$

where $Z_0 = Z$ and

$$Z_j = \sum_{r(x)=-\infty}^{\infty}\;\prod_x\prod_{n=0}^{2} I_{r(x)-r(x+e_n)+\eta_n(x)}(\beta_n)\;. \tag{2.7}$$

Here we have introduced sources $\eta_n(x) = \eta(l)$ as

$$\eta(l) \;=\; \begin{cases} j, & l\in S_d\;,\; l=(x,n) \\ -j, & l\in S_d\;,\; l=(x-e_n,n) \\ 0, & \text{otherwise} \end{cases} \tag{2.8}$$

where S_d is a surface enclosed between two Polyakov loops.

The standard 't Hooft operator which measures a free energy of the monopole-antimonopole pair is given in the dual formulation by the following expectation value

$$D(x,y) = \left\langle (-1)^{r(x)-r(y)} \right\rangle . \tag{2.9}$$

3. Limiting values of anisotropic couplings

We start by examining the limiting values of the anisotropic couplings.

1. The limit $\beta_t = 0$. This is the simplest limit because here the model reduces to a product of non-interacting two-dimensional gauge models. The solution of $2d$ gauge models is well known. For $U(1)$ LGT we thus get

$$Z(\beta_t=0,\beta_s) \;=\; \left[\sum_{r=-\infty}^{\infty}\; I_r^{L^2}(\beta_s)\right]^{N_t}\;. \tag{3.1}$$

The model is in the confined phase at all values of β_s. The temporal Wilson loop, the Polyakov loop and all the correlations of the Polyakov loops are vanishing in the limit $\beta_t = 0$. The spatial Wilson loop in the thermodynamic limit behaves as

$$W_j(C) \;=\; \exp[-\alpha S]\;,\; \alpha \;=\; \ln\frac{I_0(\beta_s)}{I_j(\beta_s)}\;, \tag{3.2}$$

where S is the area of the loop C.

2. The limit $\beta_s = 0$. This is a non-trivial limit which cannot be solved exactly but in which the $U(1)$ model reduces to the XY-like model. Integrating out spatial gauge fields one can prove that

$$Z(\beta_t,\beta_s=0) \;=\; \int_0^{2\pi}\prod_x\frac{d\omega_x}{2\pi}\prod_{x,n}\left[\sum_{r=-\infty}^{\infty} I_r^{N_t}(\beta_t)\;\exp\left[ir(\omega_x-\omega_{x+e_n})\right]\right]\;. \tag{3.3}$$

Here, $e^{ir\omega_x}$ is the Polyakov loop in the representation r.

For $N_t = 1$ using the formula $\sum_r I_r(x) e^{ir\omega} = e^{x\cos\omega}$ one finds

$$Z(\beta_t, \beta_s = 0, N_t = 1) \ = \ \int_0^{2\pi} \prod_x \frac{d\omega_x}{2\pi} \ \exp\left[\beta_t \sum_{x,n} \cos(\omega_x - \omega_{x+e_n})\right] \tag{3.4}$$

which is the partition function of the XY model. Thus, in this case the dynamics of the system is governed by the XY model with the inverse temperature β_t. For $N_t \geq 2$ the model (3.3) is of the XY-type, i.e. it describes the interaction between the nearest neighbour spins (Polyakov loops) and possesses the global $U(1)$ symmetry. There is a little doubt that the critical behaviour for all N_t is the same as that of the XY model.

4. Perturbative calculation of the Polyakov loop correlator at high temperatures

Perturbative calculations for abelian models are especially simple in the plaquette formulation (2.3). The perturbation theory on isotropic lattice in the plaquette formulation has been developed in [11]. In that paper, a calculation of the first two perturbative coefficients of the Wilson loop can be found. An extension of those calculations to the anisotropic lattice is straightforward. For the Polyakov loop we write the result in the form

$$P_j(C) \ = \ 1 - g^2 \, C_1 + g^4 \, C_2 \ + \ \mathscr{O}(g^6) \ , \tag{4.1}$$

$$C_1 \ = \ \frac{1}{2} \, j^2 \, \beta \, D(R) \ , \ \ C_2 \ = \ \frac{1}{8} \, j^4 \, \beta^2 \, D^2(R) \ - \ \frac{1}{4} \, j^2 \, a_t \beta \, D(R) \ \left(1 - \beta_t^{-1} D_{n_1}\right) \ . \tag{4.2}$$

Here, $D(R) = G(0) - G(R) \asymp \mathscr{O}(\ln R)$ is the two-dimensional Green function appearing in (4.4) and $D_{n_1} = G_0 - G_{n_1}$. At high temperatures $\beta_t^{-1} D_{n_1} \approx (2N_t)^{-1}$. Substituting last formulae into (4.1) one can extract the potential between test charges

$$V_j(R) \ = \ -\frac{1}{\beta} \, \ln P_j(R) \ = \ \frac{1}{2} \, g^2 j^2 \, \left[1 + \frac{1}{2\beta_t}(1 - \beta_t^{-1} D_{n_1})\right] \, D(R) \ . \tag{4.3}$$

These formulae are to be compared with the perturbative expansion of the two-point correlation function of the $2d$ XY model. If g^2 is a dimensionless coupling constant of the XY model then the perturbative expansion takes the form [14, 15]

$$\Gamma_{XY}(R) \ = \ 1 \ - \ \frac{g^2}{2} \, D(R) \ + \ \frac{g^4}{8} \, D(R) \, [D(R) - 1] \ + \ \mathscr{O}(g^6) \ . \tag{4.4}$$

Comparing (4.4) with (4.2) one sees the perturbative coefficients of the Polyakov loop behave qualitatively and quantitatively similar to those of the 2-point function of the $2d$ XY model.

Now we present perturbative results for the 't Hooft loop. From (2.9) one can easily calculate the leading perturbative contribution

$$D(R) \ = \ \exp\left[-\frac{1}{2} B(R)\right] \ . \tag{4.5}$$

At high temperature and for $R = (x_1^2 + x_2^2)^{1/2} \gg 1$, $\tau = 0$

$$B(R,0) \ = \ \frac{1}{\pi g^2 \beta} \ \ln R \ . \tag{4.6}$$

At high temperature and for $R = 0$

$$B(0,\tau) \;=\; \frac{\beta}{g^2 a_s^2}\left(\frac{\tau}{\beta}\right) \;,\; \tau = a_t x_0 \;. \tag{4.7}$$

These results agree qualitatively with numerical simulations of [16]. Also, this behaviour is qualitatively similar to the behaviour of the disorder operator in the XY model.

5. Effective vortex model and critical behaviour

Here we calculate the effective vortex model at finite temperatures using the Villain version of the theory (2.5). Simple computations lead to familiar result $Z = Z_{\rm SW} Z_{\rm mon}$, where $Z_{\rm SW}$ is the standard spin-wave contribution. The monopole contribution, which describes the standard Coulomb interaction mGm between dynamical monopoles, reads

$$Z_{\rm mon} \;=\; \sum_{m(x)=-\infty}^{\infty} \exp\left[-\pi^2 \sum_{x,x'} m(x) G_{xx'} m(x')\right] \;. \tag{5.1}$$

Here and below the sums over all repeating indices are understood. The Green function G_x on the anisotropic lattice can be written as $(k_0 + k_1 + k_2 \neq 0)$

$$G_x \;=\; \frac{1}{L^2}\sum_{k_n=0}^{L-1} \frac{1}{N_t}\sum_{k_0=0}^{N_t-1} \frac{\exp\left[\frac{2\pi i}{L}\sum_{n=1}^{2} k_n x_n + \frac{2\pi i}{N_t} k_0 x_0\right]}{\frac{1}{\beta_s}\left(1-\cos\frac{2\pi k_0}{N_t}\right) + \frac{1}{\beta_t}\sum_{n=1}^{2}\left(1-\cos\frac{2\pi k_n}{L}\right)} \;. \tag{5.2}$$

Now we study the effective monopole theory (5.1). We follow the conventional strategy developed in the context of the XY model and described in many reviews and books. The first step is to substitute the full Green function by its asymptotics at high temperatures which can be derived from Eq.(5.2)

$$G_x \;=\; \frac{1}{g^2\beta} G_x^{2d} + \frac{\beta}{g^2 a_s^2} B_2(\tau/\beta)\delta_{x,0} + \frac{\beta^3}{6g^2 a_s^4} B_4(\tau/\beta)\Delta_x + \mathscr{O}(\beta^5) \;, \tag{5.3}$$

where $\tau \;=\; a_t x_0$, G_x^{2d} is the Green function of the $2d$ model, Δ_x is the Laplace operator and $B_n(z)$ are the Bernoulli polynomials. Using this expansion one finds in the leading order $\tau/\beta \ll 1$ the effective $2d$ vortex model, $x = (x_1, x_2)$

$$Z_{\rm vor} \;=\; \sum_{m(x)=-\infty}^{\infty} \delta\left[\sum_x m(x)\right] \exp\left[-\frac{\pi^2}{g^2\beta}\sum_{x,x'} m(x) G_{xx'} m(x') - \kappa_0 \sum_x m^2(x) - \kappa_1 m(x)\Delta_{xx'} m(x')\right] \;, \tag{5.4}$$

$$\kappa_0 \;=\; \kappa\,\frac{\pi^2\beta}{6g^2 a_s^2} \;,\; \kappa_1 \;=\; \kappa\,\frac{\pi^2\beta^3}{180 g^2 a_s^4} \;. \tag{5.5}$$

This vortex model can be exactly mapped onto the model of the sine-Gordon type

$$Z_{\rm SG} = \int \prod_x d\alpha_x \,\exp\left[-\sum_{x,x'} \alpha_x B_{xx'} \alpha_{x'} + y \sum_x \cos\alpha_x\right] , B_{xx'} = \frac{g^2\beta}{4\pi}\Delta_{xx'} + \kappa\frac{g^2\beta^5}{720 a_s^4}\Delta_{xy}\Delta_{yy'}\Delta_{y'x'} \tag{5.6}$$

where Δ_{xy} is the lattice Laplace operator and effective fugacity of the XY model is $y = 2\exp[-\frac{\gamma\pi^2}{g^2\beta} + \kappa\,\frac{\pi^2\beta}{6g^2 a_s^2}]$. The sine-Gordon model can be analyzed by the conventional RG methods [17]. Terms proportional to κ are treated perturbatively. We skip these well-known calculations which predict the XY critical indices $\eta(T_c) = 1/4$ and $\nu = 1/2$ also for our effective vortex theory.

6. Summary

At zero temperature $3d$ $U(1)$ compact gauge theory exhibits permanent confinement at all values of coupling constant. At finite temperature a deconfinement phase transition takes place to a phase where the potential between test charges grows logarithmically. This is seen, e.g. from the behaviour of the correlation function of the Polyakov loops which have been computed perturbatively at high temperature. In the limit $\beta_s = 0$ this is the BKT phase transition which belongs to the XY model universality class. At large values of β_s we have computed effective static model for monopoles and studied it at high temperature. Assuming validity of the conventional RG methods we have obtained analytical predictions for the critical indices of the model. Our result implies that these indices coincide with those of the XY model at all values of couplings. Nevertheless, since this result relies on certain approximations the numerical check is very desirable. Such MC simulations are now in progress.

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