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Power-counting theorem for staggered fermions

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One of the assumptions that is used in Reisz's power-counting theorem does not hold for staggered fermions, as was pointed out long ago by Lüscher. Here, we generalize the power-counting theorem, and the methods of Reisz's proof, such that the difficulties posed by staggered fermions are overcome.

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1. Motivation and summary

Staggered fermions (SFs) do not entirely overcome the fermion doubling problem. The doubler multiplicity is *taste.* Fermion doubling creates difficulties for lattice power-counting, as was pointed out some years ago by Lüscher [1].

To better understand perturbative renormalization of SFs it is of course useful to have a lattice power-counting theorem. Reisz's lattice power-counting theorem [2] was a significant achievement because on the lattice Feynman integrands are trigonometric rather than rational functions of momenta; this can lead to results that differ from those of the continuum in important ways. It is often stated that no power-counting theorem exists for SFs [3–6].

However, it is also widely believed that the theory of SFs coupled to Yang-Mills (denoted here SF-QCD) yields the right quantum continuum limit in perturbation theory. That is to say, the lattice perturbation series can be renormalized and matched to a continuum renormalization scheme at every order in the gauge coupling g. This conclusion is supported by an analysis of the types of non-irrelevant operators that are allowed by the symmetries of SF-QCD. One finds that all such operators are already present at tree-level. (See for example [7] and refs. therein.) That is, from a Wilsonian point of view one concludes that SF-QCD is in the same universality class as continuum QCD. It is reasonable to believe that by an adjustment of the bare parameters of the lattice action, one can arbitrarily adjust the coefficients of all non-irrelevant operators in the infrared, in order to obtain the desired theory.

The belief that SF-QCD is renormalizable also follows from a consideration of powers of the lattice spacing a that arise in vertices and propagators of the theory, and how they appear in loop diagrams, an early example being [8]. In fact, for 1-loop diagrams, it is easy to power-count by partitioning the loop integration domain in a sensible way and estimating the integrand and measure for each of those domains. But this is nothing other than a limited version lattice power-counting. So, in fact, a version of power-counting already exists, though it is not as general as we would like.

In actuality, this sort of partitioning is exactly what is done in Reisz's proof of his lattice power-counting theorem. However, the complexities that occur at high orders—where the number of domains increases factorially—are best addressed by a more sophisticated mathematical approach, just as in the continuum proofs of Weinberg [9] or Hahn and Zimmermann [10]. It is this sort of general method of power-counting that is aimed at in the present study. The subtraction of divergences, and renormalization, are left for future work.

2. Brief summary

(1) Domain decomposition to isolate poles. Insert a resolution of identity for each line momentum[1] ℓ_i that partitions the integration domain into ε-near and ε-far regions. The former are

[1] Line momenta are the linear combinations of loop momenta k and external momenta q that flow through propagators (lines) in the diagram.

balls of radius $\varepsilon\pi/a$ concentric about the lattice poles, including SF doublers. $\ell_i \in J$ correspond to ε-near and $z_i \in \mathbf{Z}^4$ locates the pole: $\ell_i = (\pi/a)z_i$. The result is that the lattice Feynman integral $\hat{I}$ is written as a sum:

$$\hat{I} = \sum_{Jz} \hat{I}_{Jz}. \tag{2.1}$$

The number of terms that are nonvanishing is finite. Our job is to bound each one.

(2) Shift loop momenta such that $\ell_i \approx 0$ for ε-near line momenta. Here, $k_i \to k_i + \Delta_i$ where $\Delta_i \in (\pi/a)\mathbf{Z}^4$, so that all $\ell_i \in J$ are near to the trivial poles $\ell_i = 0$. Thereby, denominators can be bounded by continuum expressions.

(3) Eliminate explicit π/a using trig. identities. Due to lack of π/a-periodicity, the k-shifted numerator $V(k+\Delta, q; m, a)$ of the Feynman integrand may contain explicit π/a. Application of trig. identities leads to a new function: $V_A(k,q;m,a) \equiv V(k+\Delta, q; m, a)$.

We are now in a position to follow Reisz in every remaining step.

(4) Apply Reisz's rational bounding methods. Dropping the subscript A, we write the numerator as $V(k,q;m,a) = P(k,q;m) + R(k,q;m,a)$. The integral thus decomposes as $\hat{I}_{Jz} = \hat{I}^0_{Jz} + \hat{I}^R_{Jz}$. The denominator has been bounded by rational expressions; the a-dependent numerator R can also be bounded by a set of polynomials $Q_{ib}(k,q;m)$ and powers of a. Thus one obtains bounds:

$$|\hat{I}^0_{Jz}| \leq \bar{I}^0_{Jz}, \quad |\hat{I}^R_{Jz}| \leq \sum_b \bar{I}^b_{Jz}, \tag{2.2}$$

where "barred" quantities indicate rational expressions that are subject to Reisz's *auxiliary power-counting theorem.*

We thereby prove that $\hat{I}_{Jz}$ with negative UV degree are finite, establishing the power-counting theorem.

3. Resolutions of identity

3.1 Resolution on $\mathscr{B}_a$

Here $\mathscr{B}_a = (-\frac{\pi}{a}, \frac{\pi}{a}]^4$ is the usual Brillouin zone. Balls of size $\pi\varepsilon/a$ around each point in the reciprocal lattice $(2\pi/a)\mathbf{Z}^4$ define ε-near regions; the remainder of momentum space consists of ε-far regions:

$$\begin{array}{rl} \varepsilon\text{-near:} & ||\ell - \frac{2\pi}{a}z|| < \frac{\pi}{a}\varepsilon \text{ for some } z \in \mathbf{Z}^4, \\ \varepsilon\text{-far:} & \text{otherwise.} \end{array} \tag{3.1}$$

Reisz introduces the following step function:

$$\Theta^B_\varepsilon(\ell) = \begin{cases} 0 & \varepsilon\text{-near} \\ 1 & \varepsilon\text{-far} \end{cases} \tag{3.2}$$

Next note (Θ is Heaviside's unit step function):

$$\sum_{z\in\mathbf{Z}^4}\Theta\left(\frac{\pi}{a}\varepsilon-\left|\left|\ell-\frac{2\pi}{a}z\right|\right|\right)=\begin{cases}1 & \varepsilon\text{-near}\\ 0 & \varepsilon\text{-far}\end{cases} \tag{3.3}$$

Thus: for any ℓ one can resolve identity as

$$1=1_B(\ell)\;\equiv\;\Theta^B_\varepsilon(\ell)+\sum_{z\in\mathbf{Z}^4}\Theta\left(\frac{\pi}{a}\varepsilon-\left|\left|\ell-\frac{2\pi}{a}z\right|\right|\right). \tag{3.4}$$

3.2 Resolution on reduced Brillouin zone $\mathscr{B}_{2a}$

Here $\mathscr{B}_{2a}=(-\frac{\pi}{2a},\frac{\pi}{2a}]^4$ is the Brillouin zone for reciprocal lattice corresponding to the staggered fermion poles. We use it to define a step-function that isolates all SF poles:

$$\Theta^F_\varepsilon(\ell)=\begin{cases}0 & \text{if } ||\ell-\frac{\pi}{a}z||<\frac{\pi}{a}\varepsilon \text{ for some } z\in\mathbf{Z}^4 \text{ (= modified } \varepsilon\text{-near)}\\ 1 & \text{otherwise (= modified } \varepsilon\text{-far)}\end{cases} \tag{3.5}$$

Now ε-near regions of the *reduced reciprocal lattice* $(\pi/a)\mathbf{Z}^4$ have been isolated. For any ℓ one can resolve identity as:

$$1=1_F(\ell)\;\equiv\;\Theta^F_\varepsilon(\ell)+\sum_{z\in\mathbf{Z}^4}\Theta\left(\frac{\pi}{a}\varepsilon-\left|\left|\ell-\frac{\pi}{a}z\right|\right|\right). \tag{3.6}$$

4. The Jz sum

One breaks up the line momenta into those corresponding to bosons (gluons) and fermions (quarks): $\ell_1^B,\ldots,\ell_{I_B}^B$ and $\ell_1^F,\ldots,\ell_{I_F}^F$ resp. Then one inserts into the Feynman integral, for each ℓ_i, the resolutions of identity that are described above: $1_B(\ell_i^B)$ defined in (3.4) and $1_F(\ell_j^F)$ defined in (3.6).

One obtains an expression analogous to Reisz's—a sum of integrals that comprises a domain decomposition:

$$\begin{aligned}&\hat{I}=\sum_{J_B,J_F}\sum_{z_B,z_F}\hat{I}(J_B,J_F,z_B,z_F)\equiv\sum_{Jz}\hat{I}_{Jz},\\ &J_B\subseteq\{1,\ldots,I_B\},\quad J_F\subseteq\{1,\ldots,I_F\},\\ &z_B=(z_{Bi}|i\in J_B),\quad z_F=(z_{Fj}|j\in J_F),\end{aligned} \tag{4.1}$$

with individual terms of the form:

$$\begin{aligned}\hat{I}_{Jz}\;=\;&\int_{\mathscr{B}_a{}^L}\frac{d^{4L}k\;\;V(k,q;m,a)}{\prod_{i=1}^{I_B}C_B(\ell_i^B;\lambda,a)\prod_{j=1}^{I_F}C_F(\ell_j^F;m,a)}\\ &\times\prod_{i\in J_B}\Theta\left(\frac{\pi}{a}\varepsilon-||\ell_i^B-\frac{2\pi}{a}z_{Bi}||\right)\prod_{i\notin J_B}\Theta^B_\varepsilon(\ell_i^B)\\ &\times\prod_{j\in J_F}\Theta\left(\frac{\pi}{a}\varepsilon-||\ell_j^F-\frac{\pi}{a}z_{Fj}||\right)\prod_{j\notin J_F}\Theta^F_\varepsilon(\ell_j^F).\end{aligned} \tag{4.2}$$

Note that J collectively denotes J_B,J_F, and so on. The decomposition has the following intuitive meaning: $\ell_i\in J$ are "ε-near" to a lattice pole, whereas $\ell_i\notin J$ are "ε-far" from a lattice pole.

5. The shift

For ε, a sufficiently small, the arguments of Reisz extend in an obvious way to show that there exists $k^{(0)} \equiv (k_1^{(0)}, \ldots, k_L^{(0)}) \in \mathscr{B}_a{}^L$ s.t.:

$$K_i^B(k^{(0)}) = \frac{2\pi}{a} z_{Bi}, \quad K_j^F(k^{(0)}) = \frac{\pi}{a} z_{Fj}, \quad i \in J_B, \quad j \in J_F. \tag{5.1}$$

Note that $K_i(k)$ was defined by

$$\ell_i(k,q) = \sum_{j=1}^{L} C_{ij} k_j + \sum_{\ell=1}^{E} D_{i\ell} q_\ell \equiv K_i(k) + Q_i(q). \tag{5.2}$$

Using the fact that ℓ_i are *natural,*[2] it is a trivial extension of one of Reisz's lemmas to prove that there exist reduced reciprocal lattice vectors $\Delta_1, \ldots, \Delta_L \in \frac{\pi}{a}\mathbf{Z}^4$ such that for $i \in J_B,\ j \in J_F$

$$K_i^B(\Delta) = \frac{2\pi}{a} z_{Bi}, \quad K_j^F(\Delta) = \frac{\pi}{a} z_{Fj}. \tag{5.3}$$

The Δ_i are determined in terms of a basis chosen from $\{K_i^B, K_j^F\}$, as explained by Reisz. Thus we define new loop momenta k_i' through:

$$k_i = k_i' + \Delta_i \equiv k_i' + \frac{\pi}{a}\delta_i \quad \forall \quad i = 1, \ldots, L, \tag{5.4}$$

where in the last step integer-valued 4-vectors δ_i have been introduced for convenience, following Reisz. A new domain of integration results:

$$\sigma_J = \left\{ k' \in \mathbf{R}^{4L} \;\middle|\; -\frac{\pi}{a} - \Delta_{i\mu} < k'_{i\mu} \le \frac{\pi}{a} - \Delta_{i\mu} \right\}, \tag{5.5}$$

identical to the result of Reisz.

For the line momenta $\ell_i^B \in J_B, \ell_j^F \in J_F$, (5.4) has the effect

$$\ell_i^B(k) = \ell_i^B(k') + \frac{2\pi}{a} z_i^B, \quad \ell_j^F(k) = \ell_j^F(k') + \frac{\pi}{a} z_j^F, \tag{5.6}$$

a generalization of Reisz corresponding expression. When this is accounted for in (4.2), the Heaviside step functions in (4.2) just force $\ell_i^B(k') \in J_B,\ \ell_j^F(k') \in J_F$ into the ε-neighborhood of the (unique) pole in $\mathscr{B}_a$ and $\mathscr{B}_{2a}$ respectively. As a consequence the following bounds hold:

$$\begin{aligned} C_B^{-1}(\ell_i^B \in J_B) &\le \alpha_B(\ell_i^B(k')^2 + \lambda^2)^{-1}, \\ C_F^{-1}(\ell_j^F \in J_F) &\le \alpha_F(\ell_j^F(k')^2 + m^2)^{-1}, \end{aligned} \tag{5.7}$$

generalizations of Reisz's corresponding expression. Here, α_B, α_F are constants that always exist for ε, a sufficiently small. For $\ell_i \notin J$, the line momenta are outside of the balls of radius $\varepsilon\pi/a$

[2] This is a standard term, the definition of which can be found in [2].

that are centered on sites of the (reduced) reciprocal lattice for (quarks) gluons. Therefore they are bounded by:

$$C_B^{-1}(\ell_i^B \notin J_B) \le \gamma_B a^2, \quad C_F^{-1}(\ell_j^F \notin J_F) \le \gamma_F a^2, \tag{5.8}$$

generalizations of Reisz's corresponding expression. Here, γ_B, γ_F are constants that always exist for ε, a sufficiently small.

For the line momenta $\ell_i^B \notin J_B, \ell_j^F \notin J_F$, the shift (5.4) is only guaranteed to have

$$\ell_i^B(k) - \ell_i^B(k') = C_{im}^B \Delta_m \in \frac{\pi}{a}\mathbf{Z}^4, \quad \ell_j^F(k) - \ell_j^F(k') = C_{jm}^F \Delta_m \in \frac{\pi}{a}\mathbf{Z}^4. \tag{5.9}$$

6. The generalized UV degree and theorem

These considerations lead to the following generalization of Reisz's theorem, which incorporates all possible changes of UV degree due to odd-π/a shifts. As a preliminary step, we define the following set of four-vectors:

$$\mathscr{K} \equiv \{(0^4), (\underline{1,0^3}), (\underline{1^2,0^2}), (\underline{1^3,0}), (1^4)\}. \tag{6.1}$$

The notation is as follows. In the definition of the set of 4-vectors $\mathscr{K}$, powers indicate how many times a 0 or 1 appears. Underlining indicates that all permutations of entries are to be included.

Now for the degree and theorem:

Definition. Let $F_A = V_A/C_A$, $A \in \mathscr{K}^L$ denote the transformed Feynman integrand. That is:

$$\begin{aligned} V_A(k,q;m,a) &= V(k + (\pi/a)A, q; m, a), \\ C_A(k,q;m,a) &= C(k + (\pi/a)A, q; m, a). \end{aligned} \tag{6.2}$$

Generalize the UV degree as follows:

$$\begin{aligned} &\overline{\overline{\mathrm{degr}}}_{\hat{u}} F = \max_{A \in \mathscr{K}^L} \overline{\mathrm{degr}}_{\hat{u}} F_A, \quad \overline{\mathrm{degr}}_{\hat{u}} F_A = \overline{\mathrm{degr}}_{\hat{u}} V_A - \overline{\mathrm{degr}}_{\hat{u}} C_A; \\ &\overline{\overline{\mathrm{degr}}}_{\hat{H}} \hat{I} = 4d + \overline{\overline{\mathrm{degr}}}_{\hat{u}} F. \end{aligned} \tag{6.3}$$

Recall that $u_1, \ldots, u_d$ parameterizes the Zimmermann subspace H.

Proposition. Suppose that

$$\overline{\overline{\mathrm{degr}}}_{\hat{H}} \hat{I} < 0 \quad \forall \quad H \in \mathscr{H}. \tag{6.4}$$

Then $\hat{I}$ converges, and

$$\lim_{a \to 0} \hat{I} = \sum_{A \in \mathscr{K}^L} \int_{-\infty}^{\infty} d^{4L}k \, \frac{P_A(k,q;m)}{E_A(k,q;m)}, \tag{6.5}$$

where

$$P_A(k,q;m) = \lim_{a \to 0} V_A(k,q;m,a), \quad E_A(k,q;m) = \lim_{a \to 0} C_A(k,q;m,a) \tag{6.6}$$

are just the continuum limits of the numerator and denominator resp. This indicates that various regions of loop momenta may contribute to the continuum limit, due to the presence of doublers in the fermion spectrum.

7. The proof

Starting with (4.2), one makes the redefinition (5.4). Then the numerator is replaced by $V_A(k', q; m, a)$, as in (6.2). Once this has been done, $\hat{I}_{Jz}$ is in the form considered by Reisz. Due to the assumption (6.4), the remainder R_A in the decomposition

$$V_A(k,q;m,a) = P_A(k,q;m) + R_A(k,q;m,a) \tag{7.1}$$

does not contribute in the continuum limit, as follows from Reisz's arguments in §7 of Reisz's work. Thus one can replace V_A by the rational function P_A in the numerator of $\hat{I}_{Jz}$. Furthermore, Reisz's arguments show that the $\hat{I}_{Jz}$ term that maps to the index $A \in \mathscr{K}^L$ just yields

$$I_A = \int_{-\infty}^{\infty} d^{4L}k \, \frac{P_A(k,q;m)}{E_A(k,q;m)} \tag{7.2}$$

in the continuum limit. The result (6.5) follows immediately.

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