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Supersymmetric lattice models in one and two dimensions

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We study and simulate $\mathcal{N}=2$ supersymmetric Wess-Zumino models in one and two dimensions. For any choice of the lattice derivative, the theories can be made manifestly supersymmetric by adding appropriate improvement terms corresponding to discretizations of surface integrals. In particular, we check that fermionic and bosonic masses coincide and the unbroken Ward identities are fulfilled to high accuracy. Equally good results for the effective masses can be obtained in a model with the SLAC derivative (even without improvement terms). In two dimensions we introduce a non-standard Wilson term in such a way that the discretization errors of the kinetic terms are only of order $\mathcal{O}(a^2)$. Masses extracted from the corresponding manifestly supersymmetric model prove to approach their continuum values much quicker than those from a model containing the standard Wilson term. Again, a comparable enhancement can be achieved in a theory using the SLAC derivative.

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1. Introduction

Supersymmetry is nowadays an important ingredient in most theoretical developments of quantum field theory beyond the standard model. It allows for the unification of the three fundamental forces described by the standard model and is also incorporated in supergravity and string theory. In the low energy regime this symmetry is obviously not manifest and the question remains by which mechanism supersymmetry if realized in nature is broken. From non-renormalization theorems it is at least known that this has to be answered non-perturbatively. In this view the lattice might serve as an equally good approach as it has been before for gauge theories. However since supersymmetry is an extension of the Poincaré symmetry of spacetime it is inherently broken on a spacetime lattice.

Here we study and simulate $\mathscr{N} = 2$ Wess-Zumino models in one and two dimensions. Lattice theories with different lattice derivatives and discretization prescriptions which preserve parts of the supersymmetry are simulated. It is checked that fermionic and bosonic masses coincide and that unbroken Ward identities are fulfilled to high accuracy. By introducing a nonstandard Wilson term in the two-dimensional theory we can suppress common $\mathscr{O}(a)$ artifacts. To include dynamical fermions several algorithms are used and compared with each other. For a more thoroughfull presentation of our results we like to refer the reader to [1].

2. Quantum Mechanics

In the continuum, the action of our first model is given by the action

$$S_{\text{cont}} = \int d\tau \Big(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}W'^2 + \bar{\psi}\dot{\psi} + \bar{\psi}W''\psi \Big) \quad \text{with} \quad W'(\phi) \equiv \frac{dW(\phi)}{d\phi}; \tag{2.1}$$

it is invariant under the following supersymmetric variations:

$$\begin{array}{lll} \delta^{(1)}\phi = \bar{\varepsilon}\psi, & \delta^{(1)}\bar{\psi} = -\bar{\varepsilon}(\dot{\phi} + W'), & \delta^{(1)}\psi = 0, \\ \delta^{(2)}\phi = \bar{\psi}\varepsilon, & \delta^{(2)}\psi = (\dot{\phi} - W')\varepsilon, & \delta^{(2)}\bar{\psi} = 0. \end{array} \tag{2.2}$$

In order to perform numerical simulations and compare with previously results [2] we have fix the potential to

$$W(\phi) = \frac{m}{2}\phi^2 + \frac{g}{4}\phi^4. \tag{2.3}$$

A lattice version of this supersymmetric continuum theory raises a couple of questions. First we can ask whether the lattice model admits part of the continuum supersymmetry. Integrating out the fermions ψ and changing variables from the bosons ϕ to the so-called Nicolai variables

$$\xi = \dot{\phi} + W' \tag{2.4}$$

renders the bosonic continuum path integral purely Gaussian. Discretizing this sum of squares

$$\begin{aligned} S_{\text{bos}} = \frac{1}{2}\sum_x \xi_x \xi_x &= \frac{1}{2}\sum_x \big((\partial\phi) + W'(\phi) \big)_x^2 \\ &= S_{\text{naive}} + \sum_x (\partial\phi)_x W'(\phi)_x, \end{aligned} \tag{2.5}$$

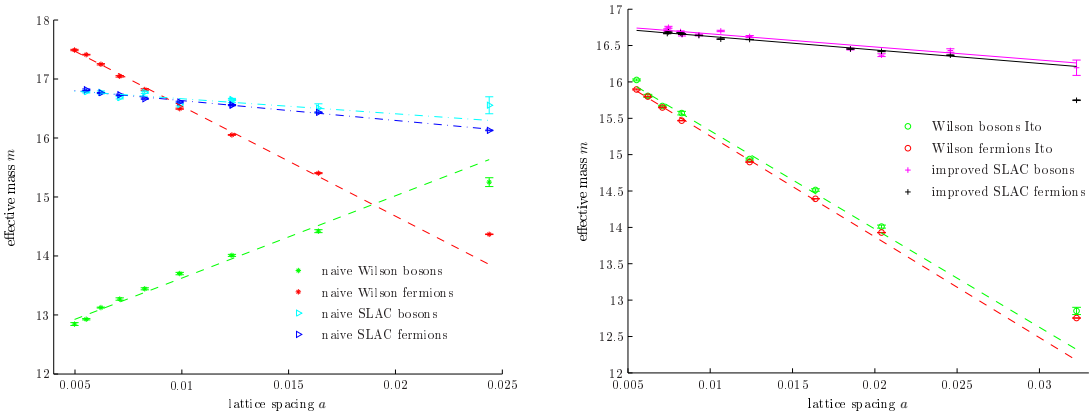

Figure 1: Left: The naive discretization of the continuum action fails to recover the correct supersymmetric continuum limit when Wilson fermions are used (green and red graphs). For every finite lattice spacing the extracted masses differ vastly from each other. The situation is different when SLAC fermions are used. Here the masses coincide within statistical bounds and tend towards the correct $a \to 0$ limit. **Right:** The same graph as on the left, now using the improved actions as discussed in the main text. Here also the model with Wilson fermions exhibits degenerate masses at finite lattice spacing and tends towards the correct continuum limit.

one easily verifies that one of the symmetries is preserved. Since the presence of the additional "surface" term improves the behavior of the action with respect to supersymmetry this action will be called *Nicolai improved*.

Second we have investigated whether there is an optimal lattice prescription for the Dirac operator. In particular it is a well-known fact that (ultra-)local hermitean Dirac operators will introduce fermionic doublers thus spoiling the balance between bosonic and fermionic degrees of freedom. Two strategies might be pursued, to double the bosonic spectrum as well or to use the non-local SLAC derivative. The former requires then to amend the superpotential with a corresponding Wilson term while the latter is free of any such modifications.

2.1 Degeneracy of mass spectra

The most obvious physical consequence of supersymmetric theories is the degeneracy of masses between the bosonic and fermionic channels which is simply due to the fact that supersymmetry transforms corresponding states into each other. In Monte-Carlo simulations the masses of the lowest lying state can be read off from the exponential decay of the connected twopoint function which can be readily measured. For various lattice spacings a we have measured the masses for all models in both channels, see Fig. 1. For all improved actions the presence of one unbroken supersymmetry suffices to find the degeneracy even at finite lattice spacing. However while naive Wilson fermions fail to recover the correct continuum limit as expected [3] and are still plagued by strong $\mathcal{O}(a)$ artifacts for the improved action, SLAC fermions show considerably smaller deviations for finite a and are much less sensitive to improvement terms.

2.2 Ward identities

Another important check for the presence of supersymmetry in the lattice theory is given by the

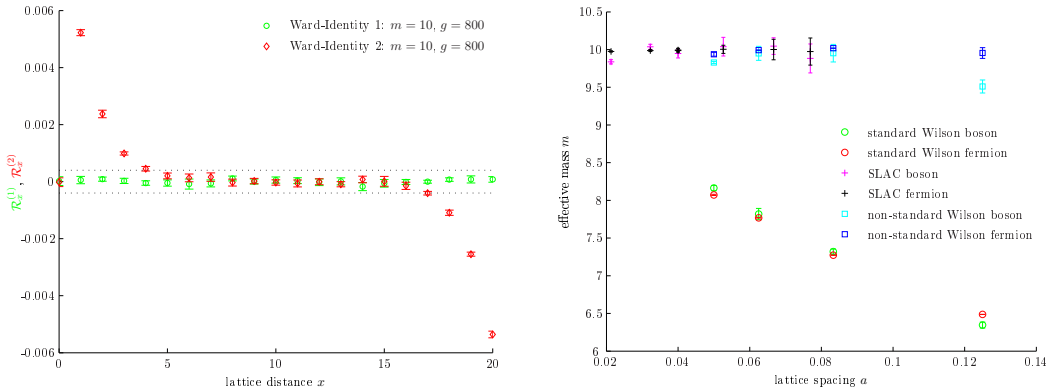

Figure 2: Left: The Ward identities mentioned in the text as a function of $|x-y|$. The green graph shows the "intact" identity while in red the deviation from the continuum result is clearly visible. The data were taken from a lattice with 20 lattice points using Wilson fermions. The analysis of SLAC fermions is somewhat more involved and all signals are worse by an order of magnitude than what is shown here. **Right:** The masses of the various two-dimensional models as a function of the lattice spacing. All spectra are degenerate in the scaling region, however lattice artifacts are much more pronounced for Wilson fermions than for SLAC or modified Wilson fermions.

computation of several Ward identities. For any observable O and supersymmetry variation δ one should find that

$$\delta\langle O\rangle = \langle \delta O\rangle = 0 \tag{2.6}$$

holds. With the particular choice for $O = \phi_x\bar{\psi}_y$ and $\delta = \delta^{(1)}$ we have checked explicitly the relation

$$\langle \psi_x\bar{\psi}_y\rangle - \langle \phi_x\xi_y\rangle = \langle \psi_x\bar{\psi}_y\rangle - \langle \phi_x(\dot{\phi}_y + W'_y)\rangle = 0, \tag{2.7}$$

the results are shown in green on the left of Fig. 2. On the other hand, since $\delta^{(2)}$ is not respected by the lattice action one might expect

$$\langle \psi_x\bar{\psi}_y\rangle - \langle \phi_x(\dot{\phi}_y - W'_y)\rangle = \langle \delta^{(2)}S\rangle \neq 0 \tag{2.8}$$

to hold, cf. the left of Fig. 2, too.

By considering Ward identities we have seen that indeed one supersymmetry is preserved while the other is clearly broken at finite lattice spacing. Moreover the breaking of Ward Identities vanishes rapidly with decreasing lattice spacing and weak coupling g.

3. Wess-Zumino model in two dimensions

The action we start from now reads

$$S_{\text{cont}} = \int d^2x\Big(2\bar{\partial}\bar{\phi}\partial\phi + \frac{1}{2}|W'|^2 + \bar{\psi}M\psi\Big), \tag{3.1}$$

where W' denotes the first derivative of the holomorphic superpotential and M is given by

$$M = \not{\partial} + W''P_+ + \bar{W}''P_-, \quad P_\pm = \frac{1}{2}(\mathbb{1}\pm\gamma_3), \quad \psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}. \tag{3.2}$$

Again this model is invariant under the following set of supersymmetric variations:

$$\begin{aligned} \delta\phi &= \bar{\psi}^1\varepsilon_1 + \bar{\varepsilon}_1\psi^1, & \delta\bar{\phi} &= \bar{\psi}^2\varepsilon_2 + \bar{\varepsilon}_2\psi^2 \\ \delta\psi^1 &= -\tfrac{1}{2}\bar{W}'\varepsilon_1 + \bar{\partial}\phi\varepsilon_2, & \delta\bar{\psi}^1 &= -\tfrac{1}{2}\bar{W}'\bar{\varepsilon}_1 - \partial\phi\bar{\varepsilon}_2, \\ \delta\psi^2 &= \partial\phi\varepsilon_1 - \tfrac{1}{2}W'\varepsilon_2, & \delta\bar{\psi}^2 &= -\partial\bar{\phi}\bar{\varepsilon}_1 - \tfrac{1}{2}W'\bar{\varepsilon}_2. \end{aligned} \tag{3.3}$$

It is still possible to construct a local Nicolai map given by

$$\xi = 2(\bar{\partial}\bar{\phi}) + W', \; \bar{\xi} = 2(\partial\phi) + \bar{W}' \tag{3.4}$$

and the *Nicolai improved* bosonic action is thus

$$S_{\text{bos}} = \sum_x \left(2(\bar{\partial}\bar{\phi})_x(\partial\phi)_x + W'_x(\partial\phi)_x + \bar{W}'_x(\bar{\partial}\bar{\phi})_x + \frac{1}{2}|W'_x|^2 \right) \tag{3.5}$$

while the fermionic part reads

$$S_{\text{ferm}} = \sum_{x,y} \bar{\psi}_x M_{xy}\psi_y, \quad M = M_0 + W''(\phi_x)\delta_{xy}P_+ + \bar{W}''(\phi_x)\delta_{xy}P_-. \tag{3.6}$$

The chosen superpotential differs from the quantum mechanical one and now reads

$$W(\phi) = \frac{m}{2}\phi^2 + \frac{g}{3}\phi^3. \tag{3.7}$$

This particular lattice actions leaves one of the four continuum supersymmetries intact. Unlike in the quantum mechanical case we consider only improved actions but choose different realizations of the Dirac operator. Standard Wilson fermions are certainly a natural choice since they are free of doublers and ultralocal and hence easy and fast to simulate. However, they suffer from large $\mathcal{O}(a)$ discretization errors and in our case necessitate a modification of the bosonic kinetic operator as well. SLAC fermions are again another choice. In order to obtain reasonable results, we have checked that the theory remains one-loop renormalizable via an explicit perturbative calculation. A third option emerges from a modification of the standard Wilson term reading

$$M_0 = \gamma^\mu\partial_\mu + \frac{ar}{2}i\gamma_3\Delta. \tag{3.8}$$

By this twist it can be shown for the free Dirac operator that all $\mathcal{O}(a)$ artifacts vanish and the corrections become $\mathcal{O}(a^2)$. Moreover on correllators of spatially averaged operators the corrections become even $\mathcal{O}(a^4)$ for the free theory. The right panel of Fig. 2 shows the masses of the lightest boson and fermion state respectively as a function of the lattice spacing. Both SLAC and twisted Wilson fermions are much less disturbed by lattice artifacts than standard Wilson fermions are, although the improvement of the action ensures the supersymmetric mass degeneracy in all three cases.

4. Algorithms

Since low-dimensional theories are less demanding than four-dimensional LQCD, several strategies to handle the fermion determinant on top of the standard hybrid Monte Carlo might be put to use.

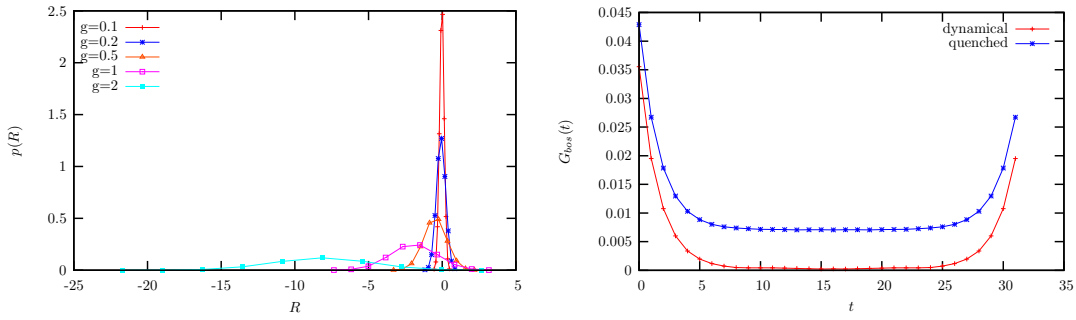

Figure 3: Left: Distribution of the reweighting factor plotted as logarithm of the fermion determinant normalized to the free field determinant for different coupling strengths g using SLAC fermions on a 31×31 lattice. The more pronounced the peak, the better statistical errors are under control and the more reliably estimates can be measured. It is obvious that the reweighting technique will fail for $g \geq 1$. For each distribution 20,000 configurations were evaluated. **Right:** Comparison of the bosonic two point function between the quenched and reweighted ensemble at $g = 0.5$ on a 32×32 lattice with Wilson fermions. The inclusion of fermionic fluctuations in the path integral are clearly vital for the correct computation of correlation functions and physical observables.

In any case the models deviate at least in two points from the more familiar scenario of LQCD. First our theories involve a only single flavor in order to keep the fermionic an bosonic degrees of freedom balanced and second, γ_5-hermiticity is broken by the Yukawa coupling terms. In view of this and to start the investigation on safe grounds, various treatments of the fermion determinant are used in parallel and compared to each other. In the simplest case, the quantum-mechanical model with Wilson fermions, the explicit formula for the fermion determinant reads

$$\det M_W[\phi] = \prod_x \left(1 + m + 3g\phi_x^2\right) - 1, \tag{4.1}$$

and can be applied directly to include fermionic contributions in a HMC integration scheme. Since the computational effort is rather small very high statistics are attainable. In the second related method one computes the determinant and the inverse of the fermion matrix by direct methods such as LU-factorization. Again additional noise originating from the use of pseudo-fermions is absent. While easily applicable in one dimension the method soon becomes infeasible in two dimensions as the lattices grow in size. A third possibility is given by reweighting the fermionic contribution from quenched ensembles. This method can generate configurations very quickly and is still exact in its treatment of fermionic fluctuations. Nonetheless it fails rapidly with increasing coupling constants since the fluctuations might then overstretch more than twenty orders of magnitude, see the left of Fig. 3, thereby reducing the effective number of configurations to order one. Finally pseudo-fermions are a well-known approach to estimate the fermion determinant stochastically. Recent algorithms such as PHMC and RHMC allow for the treatment of fractional powers of $M^\dagger M$. Thus these algorithms can also be used to simulate supersymmetric single-flavor field theories. However, the annoying problems with small eigenvalues of the fermion matrix will remain and may hamper the numerical treatment of these models.

5. Conclusions and Outlook

We have tested several lattice constructions of supersymmetric $\mathscr{N} = 2$ Wess-Zumino models in one and two dimensions. The extended supersymmetry algebra admits the construction of a lattice action which preserves one supersymmetry. Using Wilson fermions this single remnant of the continuum symmetry suffices to observe important features of the theory such as a degenerate mass spectrum and the validity of associated Ward identities independent of the choosen lattice derivative. For the SLAC derivative in one dimension we have found that the results do not differ vastly between the naive and improved action respectively. With the help of the derived Ward identities it is possible to check explicitly that one supersymmetry is respected while the other is broken.

The two-dimensional models are numerically more demanding since upon integrating out the fermion fields one ends up with an (in general not strictly positive) determinant. This situation worsens when the coupling is made stronger leaving this regime inaccessible for reweighting techniques. However the correct treatment of fermionic fluctuations is again crucial for the expected "supersymmetric" physics to show up as can be seen from the right of Fig. 3. With the introduction of a modified Wilson term the typical $\mathscr{O}(a)$ scaling is circumvented yielding results of about the same quality as the non-local SLAC fermions.

In order to investigate the whole parameter space and/or models in more than two dimensions some technical obstacles related to the treatment of the fermion determinant must be readdressed. In particular we know from first experiments that preconditioning the linear systems before applying iterative solver schemes would lead to a significant gain. Furthermore other acceleration techniques such as Fourier accelaration, multiple time-scales or higher order integrators are under investigation. With the help of the PHMC algorithm we hope to extend the stability of the algorithm into regions of parameter space which are inaccessible at the moment. Apart from this, it is already possible to study further supersymmetric models such as the $\mathscr{N} = 1$ Wess-Zumino model in two dimensions, nonlinear supersymmetric σ-models, Wess-Zumino models in higher spacetime dimensions and Super-Yang-Mills theories with the help of our existing codes.

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