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On Majorana fermions on the lattice

Jan M. Pawlowski[*]
Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, 69120 Heidelberg, Germany
E-mail: `j.pawlowski@thphys.uni-heidelberg.de`

Yuji Igarashi
Faculty of Education, Niigata University, Ikarashi, 950-2184, Niigata, Japan
E-mail: `igarashi@ed.niigata-u.ac.jp`

The construction of massless Majorana fermions with chiral Yukawa couplings on the lattice is considered. We find topological obstructions tightly linked to those underlying the Nielsen-Ninomiya no-go theorem. In contradistinction to chiral fermions the obstructions originate only from the combination of the Dirac action and the Yukawa term. These findings are used to construct a chirally invariant lattice action. We also show that the path intgral of this theory is given by the Pfaffian of the corresponding Dirac operator.

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[*]Speaker.

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1. Introduction

Massive neutrinos can be incorporated in the Standard model with Majorana fermions that become massive via spontaneous symmetry breaking. This mass generation relates to a chirally symmetric Yukawa term. Majorana fermions with chiral symmetry also play a rôle for physics beyond the standard model, e.g. in supersymmetric theories. A lattice approach to the related physics problems has to be based on an appropriate lattice formulation of Majorana fermions in the presence of chiral symmetry [1, 2, 3, 4], for Majorana fermions on the lattice see e.g. [5, 6, 7, 8, 9].

In the present note we want to address the related obstructions and provide a construction of chirally coupled Majorana fermions. We believe that this construction may also prove useful for the construction of supersymmetric theories on the lattice. Within a lattice formulation, chiral symmetry becomes non-trivial due to the Nielsen-Ninomiya no-go theorem [10, 11, 12, 13] , and we expect related obstructions for chirally coupled Majorana fermions. Indeed there appears a certain conflict between the definition of the Majorana fermions and lattice chiral symmetry in the presence of Yukawa couplings. The conflict is closely related to the requirements of locality and of avoiding species doubling, which are the basic issues of lattice chiral symmetry. It causes an obstruction in constructing the simplest supersymmetric model, the Wess-Zumino model on a lattice, and also in showing CP invariance of chiral gauge theory, see e.g.[14, 15].

Before we discuss the lattice obstruction we want to recapitulate the continuum formulation of chirally coupled Majorana fermions with an emphasis on 4-dim Euclidean space-time. Majorana fermions are neutral fermions and obey a reality constraint. In 4-dim Eulicdean space-time the charge conjugation operator C has the properties

$$C\gamma_\mu C^{-1} = -\gamma_\mu^T\,, \qquad C\gamma_5 C^{-1} = \gamma_5^T\,, \qquad C^\dagger C = 1\!\!1\,, \qquad C^T = -C\,. \tag{1.1}$$

Majorana fermions are defined via the reality constraint

$$\psi^* = B\psi\,, \tag{1.2}$$

where $C = B\gamma_5$. However, (1.1) implies $B^*B = -1\!\!1$ and hence we cannot implement the reality constraint (1.2) as it fails to satisfy the consistency condition $\psi^{**} = \psi$. Doubling the degrees of freedom suffices to implement the reality constraint with

$$\psi^* = \mathscr{B}\psi\,, \qquad \text{with} \qquad \psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\,, \qquad \text{and} \qquad \mathscr{B} = \begin{pmatrix}0 & B\\ -B & 0\end{pmatrix}\,. \tag{1.3}$$

The symplectic structure of $\mathscr{B}$ leads to $\mathscr{B}^*\mathscr{B} = 1\!\!1$ following from (1.1) with $B^*B = -1\!\!1$. Thus the reality constraint, $\psi^{**} = \psi$, is satisfied. The corresponding charge conjugation operator is provided by

$$\mathscr{C} = \begin{pmatrix}0 & C\\ C & 0\end{pmatrix} = \mathscr{B}\Gamma_5\,, \qquad \text{with} \qquad \Gamma_5 = \begin{pmatrix}-\gamma_5 & 0\\ 0 & \gamma_5\end{pmatrix}\,. \tag{1.4}$$

The above properties of the symplectic Majorana fermion ψ fix its behaviour under chiral rotations,

$$\psi \to (1 + i\varepsilon\Gamma_5)\psi\,. \tag{1.5}$$

Now we are in the position to construct a chirally invariant Majorana action. We summarise the necessary properties,

$$\mathscr{C}=\mathscr{B}\Gamma_5\,,\qquad \mathscr{C}\Gamma_5\mathscr{C}^{-1}=-\Gamma_5^T\,,\qquad \mathscr{C}^\dagger\mathscr{C}=1\!\mathrm{l}\,,\qquad \mathscr{C}^T=-\mathscr{C}\,, \tag{1.6}$$

and construct the corresponding chirally invariant Majorana action

$$S[\psi]=\int d^4x\,\psi^T\mathscr{C}\mathscr{D}\psi=\int d^4x\,\left(\psi_1^T CD\psi_1+\psi_2^T CD\psi_2\right) \tag{1.7}$$

with

$$\mathscr{D}=\begin{pmatrix}0 & D\\ D & 0\end{pmatrix},\qquad \text{and}\qquad (\mathscr{C}\mathscr{D})^T=-\mathscr{C}\mathscr{D}\,. \tag{1.8}$$

The action (1.7) could also be obtained by a Majorana reduction, see e.g. [16, 17]. We remark that skew symmetry of $\mathscr{C}\mathscr{D}$ is not required but only the skew-symmetric part of $\mathscr{C}\mathscr{D}$ contributes to the action S. The definitions (1.8) imply

$$(CD)^T=-CD\,,\qquad \text{and}\qquad D^*=BDB^{-1}\,, \tag{1.9}$$

for the Dirac operator D. The combined properties (1.9) hold for the standard chiral Dirac operator with

$$\gamma_5 D+D\gamma_5=0\,, \tag{1.10}$$

such as $D=\gamma_\mu\partial_\mu$, for which (1.9) can be deduced from (1.1). The action (1.8) is chirally invariant under a chiral transformation with (1.5) if

$$\Gamma_5\mathscr{D}-\mathscr{D}\Gamma_5=0\,, \tag{1.11}$$

which is valid for $\mathscr{D}$ with the standard Dirac operator (1.10). Finally we remark that the action (1.7) is real, as follows from (1.9). It is instructive to make this reality explicit by rewriting the action (1.7) with the help of the above relations,

$$S[\psi]=\int d^4x\,\left(\psi_2^\dagger\gamma_5 D\psi_1+\psi_1^\dagger\gamma_5 D^\dagger\psi_2\right)\,. \tag{1.12}$$

Note that (1.12) is even real for unconstrained Dirac fermions ψ_1,ψ_2 in contrast to (1.7). For the construction of a chirally invariant Yukawa term we introduce chiral projection operators related to Γ_5 in (1.5),

$$\mathscr{P}=\frac{1}{2}(1+\Gamma_5)=\begin{pmatrix}P & 0\\ 0 & 1-P\end{pmatrix},\qquad \text{with}\qquad P=\frac{1}{2}(1-\gamma_5)\,. \tag{1.13}$$

The chiral projection operators $\mathscr{P}$, $(1-\mathscr{P})$ allows us to project on left-handed and right-handed spinors. With these projections we can couple the Majorana fermions to a chirally invariant Yukawa term,

$$\begin{aligned}S_Y[\psi,\phi] &= g\int d^4x\left(\psi^T\mathscr{C}\mathscr{P}\phi^\dagger\,\psi+\psi^T\mathscr{C}(1-\mathscr{P})\phi\psi\right)\\ &= g\int d^4x\big(\psi_1{}^T CP\varphi\,\psi_1+\psi_1{}^T C(1-P)\varphi^\dagger\psi_1\\ &\qquad +\psi_2{}^T CP\varphi^\dagger\,\psi_2+\psi_2{}^T C(1-P)\varphi\psi_2\big)\,,\end{aligned} \tag{1.14}$$

where φ is a complex scalar,

$$\phi = \begin{pmatrix} 0 & \varphi \\ \varphi & 0 \end{pmatrix}, \qquad \text{with} \qquad \phi \to (1-2i\varepsilon)\phi \,. \tag{1.15}$$

Note that the scalar field ϕ is off-diagonal and hence does not commute with the projection operators, we have e.g. $\mathscr{P}\phi^\dagger = \phi^\dagger(1-\mathscr{P})$. The action $S[\psi]+S_Y[\psi,\phi]$ is invariant under the transformation (1.5) of the fermions and that in (1.15) for the scalar field ϕ related to $\varphi \to (1-2i\varepsilon)\varphi$. This concludes our brief summary of chirally coupled Majorana fermions in the continuum. Due to chiral symmetry, and in particular the use of chiral projections in (1.14) we expect obstructions for putting the above theory on the lattice.

2. Lattice formulation and topological obstructions for Majorana fermions

Chiral symmetry on the lattice differs from that in the continuum as consistent chiral transformations necessarily depend on the Dirac operator. Hence we first dicuss the properties of the lattice version of the Dirac action (1.7)

$$S[\psi] = \sum_{x,y\in\Lambda} \psi^T(x)\mathscr{C}\mathscr{D}(x-y)\psi(y)\,, \tag{2.1}$$

with the lattice Dirac operator $D(x-y)$ used in the definition of $\mathscr{D}$ as defined in (1.8). Assume for the moment that $D(x-y)$ is of Ginsparg-Wilson type [1] with

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D\,. \tag{2.2}$$

Then the chiral transformation

$$\psi \to (1+i\varepsilon\Gamma_5(1-\frac{1}{2}aD))\psi\,, \tag{2.3}$$

is an invariance of (2.1). However, smooth chiral projections P and $\mathscr{P}$ cannot be constructed, which is reflected in the fact that the transformation (2.3) vanishes at the doublers. This is a consequence of the well-known Nielsen Ninomiya (NN) no-go theorem [10, 11, 12, 13] , which provides obstructions for putting chiral fermions on the lattice. Ginsparg-Wilson fermions [1] circumvent the no-go theorem with a modified chiral symmetry (2.2), which can be reformulated as

$$\gamma_5 D + D\hat\gamma_5 = 0, \qquad \text{with} \qquad \hat\gamma_5 = \gamma_5(1-aD)\,, \tag{2.4}$$

and chiral projections

$$P = \tfrac{1}{2}(1-\gamma_5), \qquad \hat P = \tfrac{1}{2}(1-\hat\gamma_5)\,. \tag{2.5}$$

The general case going beyond Ginsparg-Wilson fermions, including e.g.[18, 19], only resorts to general chiral projections $P, \hat P$, which are compatible:

$$(1-P)D = D\hat P\,. \tag{2.6}$$

It has been shown in [20] that projection operators $P, \hat{P}$ carry a winding number that is related to the total chirality χ of the system at hand,

$$\chi = n[\hat{P}] - n[1-P]\,, \qquad \text{with} \qquad n[P] \equiv \frac{1}{2!}\left(\frac{i}{2\pi}\right)^2 \int_{T^4} \mathrm{tr}\, P(\mathrm{d}P)^4\,. \in \mathbf{Z}\,, \tag{2.7}$$

if $\hat{P}\psi = \psi$ in the action. Eq. (2.7) also entails that for odd chirality χ, $\hat{P}\psi$ and $P\psi$ live in topologically different spaces, and hence have to be different. In the present case the total chirality χ is even due to the symplectic construction. The continuum Yukawa action, however, contains projection operators $\mathscr{P}, 1-\mathscr{P}$ with $P, 1-P$ on chiral sub-spaces with $\mathscr{P}\psi \neq \psi$, that is on fermionic sub-systems with odd chirality. Thus we have to worry about the use of projection operators in the Yukawa action S_Y.

The first question that arises in this context is whether the lattice Yukawa action can be constructed such that it is left invariant under the chiral transformations (2.3), and tends toward the continuum action. This would require the existence of a smooth operator $\tilde{P}$ which reduces $\tilde{P} \to P$ in the continuum limit, and ensures invariance of the Yukawa term under the combined transformation (2.3) and (1.15). However, as $\gamma_5(1-\frac{a}{2}D)$ is not normalised and even vanishes at the doublers such an operator $\tilde{P}$ cannot exist, even if one relaxes the projection property $\tilde{P}^2 = \tilde{P}$, see also [21]. This important results will be detailed elsewhere.

In turn it is required that the chiral transformation must be compatible with the projection operators used in the Yukawa term. This already excludes (2.3). Without loss of generality we can restrict ourselves to the chiral transformation

$$\psi \to (1 + i\varepsilon\hat{\Gamma}_5)\psi\,, \qquad \longrightarrow \qquad \psi^T\hat{\mathscr{C}} \to \psi^T\hat{\mathscr{C}}\,[\hat{\mathscr{C}}^{-1}(1+i\varepsilon\hat{\Gamma}_5^T)\hat{\mathscr{C}}]\,, \tag{2.8}$$

where $\hat{\mathscr{C}}$ is a lattice generalisation of $\mathscr{C}$. Then, chiral invariance of the action S in (2.1) leads to the constraint

$$\hat{\mathscr{C}}^{-1}\hat{\Gamma}_5^T\hat{\mathscr{C}} = -\Gamma_5\,, \qquad \text{with} \qquad \Gamma_5\mathscr{D} = \mathscr{D}\hat{\Gamma}_5\,. \tag{2.9}$$

We conclude that invariance of the lattice action (2.1) under the chiral transformations (2.8) would require

$$\hat{\gamma}_5^T = \hat{C}\gamma_5\hat{C}^{-1}\,, \tag{2.10}$$

which maps $\hat{\gamma}_5$ carrying the winding number $n[\hat{P}]$ to γ_5 carrying the winding number $n[P]$. Note that using different γ_5's in the definition of Γ_5 still leads to the same conclusion (2.10). In order to elucidate this obstruction we use Ginsparg-Wilson fermions as an example. There the relation (2.10) reads

$$(1 - aD^T)\gamma_5^T = \hat{C}\gamma_5\hat{C}^{-1}\,, \tag{2.11}$$

with a possible solution

$$\hat{C} = C(1 - \tfrac{1}{2}aD)\,. \tag{2.12}$$

The $\hat{C}$ in (2.12) has zeros at the doublers, and the relative winding number is carried by these zeros. Inserting a lattice $\hat{C}$ in (2.12) into the action (2.1) we encounter zeros or singularities of the operator $\hat{C}^{-1}D$ at the positions of the doublers. This brings back the doubling problem. Consequently we have to use *independent* Majorana fields ψ, ψ' with different chiral transformation properties for the construction of Majorana actions.

3. Construction of Majorana actions on the lattice

Now we are in the position to construct chirally coupled Majorana fermions on the lattice. In line with the arguments of the last section we introduce a copy of the original symplectic Majorana fermion, ψ'. Then chiral invariance is easily arranged for with appropriate, different, chiral transformations for ψ and ψ' respectively. Furthermore we have to ensure that our path integral results in Pfaffians of the Dirac operator which signals Majorana fermions. The corresponding lattice action reads

$$S[\psi,\psi'] = \sum_{x,y\in\Lambda} \psi'^T(x)\mathscr{C}\mathscr{D}(x-y)\psi(y)\,, \tag{3.1}$$

with the Yukawa term

$$S_Y[\psi,\psi',\phi] \,=\, g\sum_{x,y\in\Lambda}\Big(\psi'^T\mathscr{C}\mathscr{P}\,\phi^\dagger(1-\hat{\mathscr{P}})\psi+\psi'^T\mathscr{C}(1-\mathscr{P})\phi\,\hat{\mathscr{P}}\psi\Big) \tag{3.2}$$

where $\mathscr{P}=(1+\Gamma_5)/2$, $\hat{\mathscr{P}}=(1+\hat{\Gamma}_5)/2$. We emphasise that in contradisctinction to the continuum theory in general we have $\mathscr{P}\phi^\dagger\neq\phi^\dagger(1-\mathscr{P})$ and $\hat{\mathscr{P}}\phi^\dagger\neq\phi^\dagger(1-\hat{\mathscr{P}})$ as the projection operators $\mathscr{P},\hat{\mathscr{P}}$ depend on the Dirac operator. The action $S+S_Y$ is invariant under the chiral transformations

$$\psi\to(1+i\varepsilon\hat{\Gamma}_5)\psi\,,\qquad \psi'\to(1+i\varepsilon\Gamma_5)\psi'\,,\qquad \phi\to(1-2i\varepsilon)\phi\,. \tag{3.3}$$

The action $S+S_Y$ reduces to the continuum action in the continuum limit, but with a doubling of the field content. This doubling can be removed by appropriate prefactors in the action, or by simply taking roots of the generating functional Z. However, it is left to prove the Pfaffian nature of the path integral. Since we have doubled the degrees of freedom we could have constructed a Dirac fermion out of two Majorana fermions. To that end we concentrate on the path integral of the pure Majorana action [22, 23] including a mass term for dealing with the zero modes. The generating functional is given by

$$Z=\int\prod_x d\psi_1 d\psi_1^* d\psi_1' d\psi_1'^*\, e^{-S[\psi,\psi']}\,, \tag{3.4}$$

with the action

$$\begin{aligned} S[\psi,\psi'] \,&=\, \sum_{x,y\in\Lambda}\psi'^T(x)\mathscr{C}\mathscr{D}(x-y)\psi(y)-im\sum_{x,y\in\Lambda}\psi'^T(x)\mathscr{C}\begin{pmatrix}0&1\\1&0\end{pmatrix}\Gamma_5\psi(y)\\ &=\,-\sum_n\left[(\lambda_n+im)(b_n'c_n+b_nc_n')+c.c.\right]\,, \end{aligned} \tag{3.5}$$

where we have used the following expansion in terms of eigenfunctions of $\gamma_5 D$:

$$\psi_1=\sum_n\left(c_n\varphi_n+b_n\phi_n\right)\,,\qquad\text{with}\qquad \gamma_5 D\varphi_n=\lambda_n\varphi_n\,, \tag{3.6}$$

and $\phi_n=\gamma_5 C^{-1}\varphi_n^*$. The above relations allow us to show that

$$Z=m^{2(n_++n_-)}\left(\frac{4}{a^2}+m^2\right)^{N_++N_-}\prod_{0<\lambda_n\neq 2/a}(\lambda_n^2+m^2)^4\,. \tag{3.7}$$

with the massless limit $Z = \left(\frac{4}{a^2}\right)^{N_+ + N_-} \prod_{0<\lambda_n \neq 2/a} \lambda_n^8$. In conclusion we find that

$$Z = \mathrm{PF}(CD)^2\, \mathrm{PF}(C^* D^*)^2\,. \tag{3.8}$$

We close with a short summary. We have shown that the construction of a theory with chirally coupled Majorana fermions on the lattice has to deal with the usual topological obstructions well-known from the construction of chiral fermions, even though the total chirality is even. The obstruction is related to the use of chiral projection operators in the Yukawa term. This problem is resolved by doubling the degrees of freedom, and the Pfaffian nature of the path integral is proven.

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