PROCEEDINGS OF SCIENCE

Hamiltonian Lattice QCD near the Light Cone

H.J. Pirner[*ac] **, D. Grünewald**[a] **,E.-M. Ilgenfritz**[b] **and E.V. Prokhvatilov**[d]

[a] *Institut für Theoretische Physik, Universität Heidelberg, Germany*
[b] *Institut für Physik, Humboldt-Universität zu Berlin, Germany*
[c] *Max-Planck-Institut für Kernphysik Heidelberg, Germany*
[d] *Department of Theoretical Physics, St. Petersburg University, Russia*
E-mail: pirner@tphys.uni-heidelberg.de,
d.gruenewald@tphys.uni-heidelberg.de,
ilgenfri@physik.hu-berlin.de,
Evgeni.Prokhvat@pobox.spbu.ru

We give a status report of our work on light cone Hamiltonian lattice QCD. We have derived an effective Hamiltonian H_{eff} which is only quadratic in the momenta and therefore can be simulated by standard methods. For this Hamiltonian we determine variationally an approximate ground state wave functional in the light cone limit.

The XXV International Symposium on Lattice Field Theory
July 30-August 4,2007
Regensburg, Germany

[*] Speaker.

PoS(LATTICE 2007)276

1. Introduction

Although the Hamiltonian is not Lorentz invariant, the light cone Hamiltonian [1, 2] offers the advantage of being boost invariant and has - naively interpreted - a trivial vacuum. On the other hand, one would be surprised if QCD looses its non-perturbative vacuum structure in the light cone limit. Probably much of the complicated vacuum structure of QCD is hidden in the constraint equations appearing in light cone QCD. Remarkable progress has been made in light cone QCD with a color dielectric lattice theory as a starting point [3, 4]. This approach is based on "fat" links which arise from averaging gluon configurations [5]. With this method the spectrum of glue balls and the pion light cone wave function have been calculated [4]. On the light cone one is prevented from approaching the continuum limit, since the effective Hamiltonian for the link matrices $M \in GL(N)$ approaching $U \in SU(N)$ is not known. This is the reason why we propose to formulate QCD near the light cone using $SU(N)$ link variables. Near light cone time plays a similar role as ordinary Minkowski time, therefore we can follow the conventional method of the transfer matrix. The transversal fields are increased in magnitude due to the boost into the vicinity of the light cone whereas the longitudinal fields remain unchanged. Constraint equations arise in the light cone Hamiltonian framework which enforce the "equality" of the transverse chromo-electric and chromo-magnetic fields $E^a_k = F^a_{-k}$. The lattice Hamiltonian density depends on an effective constant which represents the product $\tilde{\eta} = \xi\eta$ of the asymmetry parameter $\xi = a_-/a_\perp$ and the near light cone parameter η. If one chooses $\eta = 1$ and lets $\xi \to 0$ one obtains a deformed system which is squeezed in the spatial a_- direction, if one uses $\xi = 1$ and lets $\eta \to 0$ one obtains the light cone limit. This equivalence has been found by Verlinde and Verlinde [6] and Arefeva [7]. These authors have proposed to implement such asymmetric lattices in order to study high energy scattering. This has motivated us to proceed further in this way. In the work of Balitsky [8] one approaches the light cone from time-like distances which is close to scattering experiments. However, we approach the light cone from space-like distances. The asymmetric lattice Hamiltonian itself is not usable for Monte Carlo methods since the electric field strengths i.e. the momenta appear linearly. Because of the translational invariance of the vacuum we can add a term $1/\eta^2 P_-$ to cancel the unwanted terms. Naively this amounts to returning to an effective lattice Hamiltonian which is proportional to the energy in ordinary Minkowski coordinates. This is a reasonable procedure to search the ground state in the vacuum channel. Applications of light cone coordinates in finite temperature field theory have followed the same route [9].

2. The QCD Hamiltonian near the light cone

We introduce near light cone (nlc) coordinates, first proposed by [10]

$$\begin{aligned} x^+ &= \frac{1}{\sqrt{2}}\left[\left(1+\frac{\eta^2}{2}\right)x^0 + \left(1-\frac{\eta^2}{2}\right)x^3\right] \\ x^- &= \frac{1}{\sqrt{2}}\left[x^0 - x^3\right] . \end{aligned} \tag{2.1}$$

The transversal coordinates x^1 and x^2 remain unchanged. The near light cone parameter η may be interpreted as parameterizing a Lorentz boost into a frame which is moving with velocity $\beta =$

$(1-\eta^2/2)/(1+\eta^2/2)$ along the longitudinal direction relative to the laboratory frame. Then, the nlc energy p_+ and longitudinal momentum p_- expressed in terms of the laboratory energy E_{lab} and longitudinal momentum p^3_{lab} are given by

$$\begin{aligned} p_+ &= \frac{1}{\eta}\left(E_{lab} - p^3_{lab}\right) \\ p_- &= \eta\, p^3_{lab}\,. \end{aligned} \tag{2.2}$$

The second relation in eqn. (2.2) shows that large longitudinal momenta p^3_{lab} become accessible by a nlc lattice with a cut-off $p_- \propto 1/a_-$.

The definition of nlc coordinates eqn. (2.1) induces the following metric:

$$g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -\eta^2 \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} \eta^2 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{2.3}$$

with $\mu,\nu = +,1,2,-, \det g = 1$. This defines the scalar product

$$\begin{aligned} x_\mu y^\mu &= x^- y^+ + x^+ y^- - \eta^2 x^- y^- - \vec{x}_\perp \vec{y}_\perp \\ &= x_- y_+ + x_+ y_- + \eta^2 x_+ y_+ - \vec{x}_\perp \vec{y}_\perp\,. \end{aligned} \tag{2.4}$$

Note, that the metric has off-diagonal terms which implies that there are terms mixing temporal and longitudinal spatial coordinates in the scalar product. This yields a form of the pure gluonic Lagrange density eqn. (2.5) which has severe consequences for a numerical treatment

$$\mathscr{L} = \sum_a \left[\frac{1}{2} F^a_{+-} F^a_{+-} + \sum_{k=1}^{2} \left(F^a_{+k} F^a_{-k} + \frac{\eta^2}{2} F^a_{+k} F^a_{+k} \right) - \frac{1}{2} F^a_{12} F^a_{12} \right]\,. \tag{2.5}$$

The Lagrange density is linear in one of the temporal field strengths, namely $F^a_{+k} F^a_{-k}$. Therefore, a standard Monte Carlo sampling of the Euclidean path integral does not work for nlc coordinates and we rather use a Hamiltonian formulation.

The energy momentum tensor in its most general form is given by

$$T^{\mu\nu} = \sum_r \frac{\delta \mathscr{L}}{\delta\left(\partial_\mu \Phi_r\right)} \partial^\nu \Phi_r - g^{\mu\nu} \mathscr{L}\,. \tag{2.6}$$

It defines the Hamiltonian density $\mathscr{H} = T^+_{\ +}$ and the longitudinal momentum density $\mathscr{P}_- = T^+_{\ -}$ as

$$\begin{aligned} \mathscr{H} &= \frac{1}{2} \sum_a \left[\Pi^a_- \Pi^a_- + F^a_{12} F^a_{12} + \sum_{k=1}^{2} \frac{1}{\eta^2} \left(\Pi^a_k - F^a_{-k} \right)^2 \right] \\ \mathscr{P}_- &= \Pi^a_- \partial_- A^a_- + \sum_{k=1}^{2} \Pi^a_k \partial_- A^a_k\,. \end{aligned} \tag{2.7}$$

This form of the local integrand for the generator $\mathscr{P}_-$ of longitudinal translations is not manifestly gauge invariant. However, if one uses Gauss' law and the definition of the field strength tensor one can rewrite $\mathscr{P}_-$ in a symmetrized form

$$\mathscr{P}_- = \frac{1}{2}\left(\Pi^a_k F^a_{-k} + F^a_{-k} \Pi^a_k \right)\,. \tag{2.8}$$

In order to solve the Hamiltonian we are interested in translation-invariant states which are eigenstates of the longitudinal momentum operator, i.e. with eigenvalue equal zero. In vacuum, with light cone momentum $P_- = 0$, we can add $(1/\eta^2)\, P_-$ to define an effective Hamiltonian density $\mathscr{H}_{eff}$ which is only quadratic in momenta:

$$\begin{aligned}\mathscr{H}_{eff} &= \mathscr{H} + \frac{1}{\eta^2}\mathscr{P}_- \\ &= \frac{1}{2}\sum_a \left[\Pi^a_- \Pi^a_- + F^a_{12}F^a_{12} + \sum_{k=1}^{2} \frac{1}{\eta^2}\left(\Pi^a_k \Pi^a_k + F^a_{-k}F^a_{-k}\right)\right] . \end{aligned} \tag{2.9}$$

In a forthcoming paper we show how to derive [11] the effective lattice Hamiltonian with the coupling constant $\lambda = 4/g^4$ using the transfer matrix:

$$\begin{aligned}\mathscr{H}_{\text{eff,lat}} = \frac{1}{N_- N_\perp^2}\frac{1}{a_\perp^4}\frac{2}{\sqrt{\lambda}}\sum_{\vec{x}} \Bigg\{ &\frac{1}{2}\sum_a \Pi^a_-(\vec{x})^2 + \frac{1}{2}\,\lambda\, \mathrm{Tr}\left[\mathbb{1} - \mathrm{Re}\Big(U_{12}(\vec{x})\Big)\right] \\ &+ \sum_{k,a}\frac{1}{2}\,\frac{1}{\tilde{\eta}^2}\left[\Pi^a_k(\vec{x})^2 + \lambda\left(\mathrm{Tr}\left[\frac{\sigma_a}{2}\,\mathrm{Im}\Big(U_{-k}(\vec{x})\Big)\right]\right)^2\right]\Bigg\} . \end{aligned} \tag{2.10}$$

For $\tilde{\eta} = 1$ this effective Hamiltonian is very similar to the traditional Hamiltonian used in equal time lattice theory. It differs only in the potential energy terms for the U_{-k} plaquettes. Instead of the usual $\mathrm{Tr}[\mathbb{1} - \mathrm{Re}(U_{-k})]$ term resembling the field strength squared in the naive continuum limit, the nlc Hamiltonian has the form $(\mathrm{Tr}[\sigma^a/2\ \mathrm{Im}(U_{-k})])^2$ which corresponds to the plaquette in the adjoint representation and which yields an additional $Z(2)$ symmetry in comparison with the full lattice Hamiltonian [11]. The light cone limit $\tilde{\eta} \to 0$ enhances the importance of transverse electric and magnetic fields without generating unwanted linear terms in the momenta. The resulting vacuum solution should be a plausible extrapolation of the vacuum solution of QCD.

3. Variational optimization of the ground state wave functional

We analytically solve the effective lattice Hamiltonian for the ground state wave functional both in the strong and weak coupling limit [11]. Both solutions can be described by a product of single plaquette wave functionals for $\tilde{\eta}$ sufficiently close to one. In order to cover the whole coupling range, we make a variational ansatz of the ground state wave functional which is given by a product of single plaquette wave functionals with two variational parameters ρ and δ and with the normalization constant N

$$\Psi_0(\rho,\delta) = N\prod_{\vec{x}} \exp\left\{\sum_{k=1}^{2}\rho\,\mathrm{Tr}\left[\mathrm{Re}\Big(U_{-k}(\vec{x})\Big)\right] + \delta\,\mathrm{Tr}\left[\mathrm{Re}\Big(U_{12}(\vec{x})\Big)\right]\right\} . \tag{3.1}$$

With this normalized wave function we optimize the energy expectation value $\varepsilon_0(\rho,\delta)$ of the effective Hamiltonian for fixed values of the coupling λ and the near light cone parameter $\tilde{\eta}$.

In fig. (1) and fig. (2), we present the variationally optimized wave functional parameters ρ_0 and δ_0 rescaled by a factor $1/\sqrt{\lambda}$ such that they approach a constant in the asymptotic weak

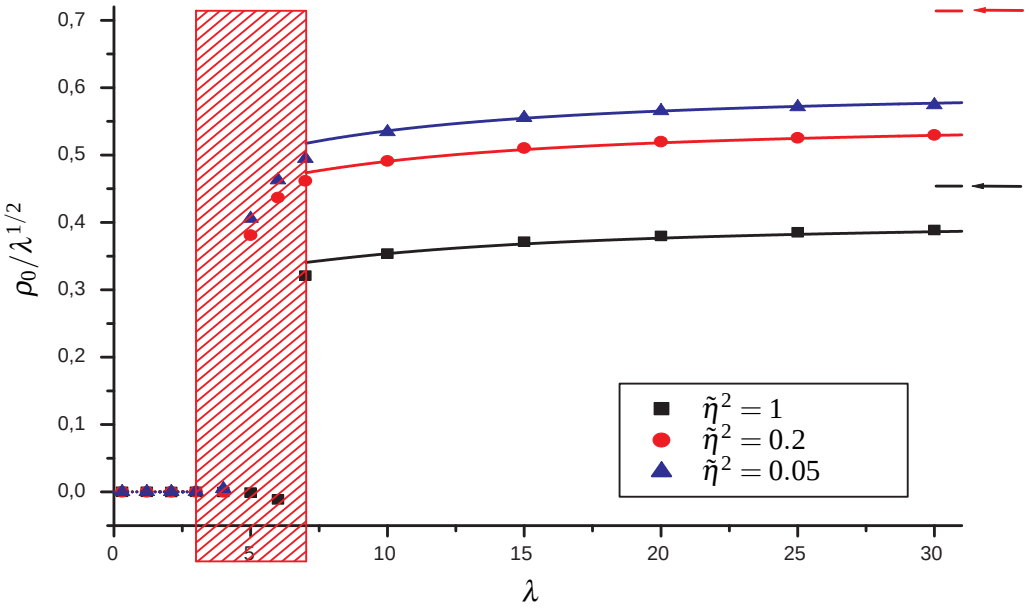

Figure 1: Optimal wave functional parameter $\rho_0(\lambda,\tilde{\eta})$ as a function of λ obtained from the simulation on a $N_\perp^2 \times N_- = 16^3$ lattice for three different values of $\tilde{\eta}^2$. The red shaded area corresponds to the phase transition region for all values of $\tilde{\eta}^2$. The dotted lines show the predicted analytical strong coupling behavior. The arrows indicate the expected asymptotic behavior for weak coupling which is proportional to $\sqrt{\lambda}$, i.e. a constant independent of λ in the plot. The solid lines show the actual analytic parameterizations in the weak coupling regime (c. f. eqn. (3.3)).

coupling region, i.e. $\lambda \to \infty$. The statistical uncertainty on the variational parameters is typically 5%. It becomes larger in the region where the Hamiltonian with the $(-k)$-plaquette in the adjoint representation induces a phase transition related to the $Z(2)$ symmetry. This region is indicated by the red shaded area in the figures. In principle only couplings in the weak coupling region above $\lambda = 7$ are physically meaningful where the artificial $Z(2)$ symmetry in eqn. (2.10) is spontaneously broken.

The discussed analytical strong and weak coupling solutions [11] yield the following estimates for ρ_0 and δ_0 in these limits

$$\rho_0 = \begin{cases} 0 & \text{for } \lambda << 1 \\ \sqrt{\lambda}\ \gamma_{\tilde{\eta}}(\vec{0}) & \text{for } \lambda >> 1 \end{cases}$$

$$\delta_0 = \begin{cases} \frac{1}{3}\ \lambda\ \tilde{\eta}^2 & \text{for } \lambda << 1 \\ \sqrt{\lambda}\ \tilde{\eta}^2\ \gamma_{\tilde{\eta}}(\vec{0}) & \text{for } \lambda >> 1 \end{cases}$$

$$\gamma_{\tilde{\eta}}(\vec{0}) \to \frac{0.038}{\tilde{\eta}}\ , \text{ for } \tilde{\eta} \to 0\ . \qquad (3.2)$$

The variationally determined parameters are in good agreement with the analytic predictions in the strong coupling regime which are represented by the dotted lines in fig. (1) and fig. (2). However,

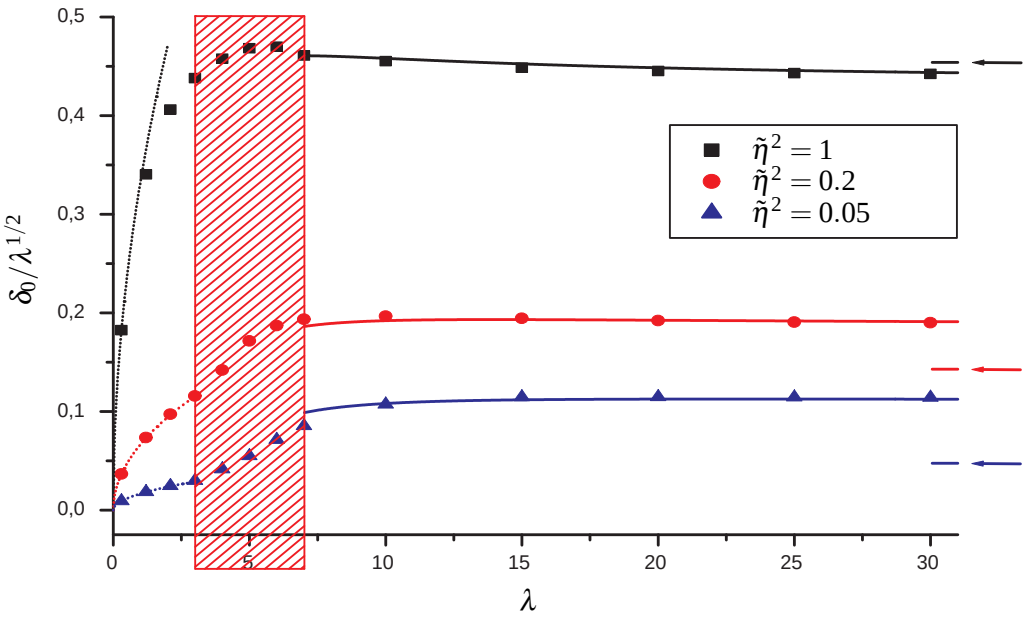

Figure 2: Optimal wave functional parameter $\delta_0(\lambda,\tilde{\eta})$ as a function of λ obtained from the simulation on a $N_\perp^2 \times N_- = 16^3$ lattice for three different values of $\tilde{\eta}^2$. The red shaded area corresponds to the phase transition region for all values of $\tilde{\eta}^2$. The dotted lines show the predicted analytical strong coupling behavior. The arrows indicate the expected asymptotic behavior for weak coupling which is proportional to $\sqrt{\lambda}$, i.e. a constant independent of λ in the plot. The solid lines show the actual analytic parameterizations in the weak coupling regime (c. f. eqn. (3.3)).

in the weak coupling regime the optimal parameters differ from their analytical estimates eqn. (3.2) which are indicated by the arrows in the plots. Both analytic predictions disagree with the optimized values stronger for decreasing values of $\tilde{\eta}$. This is natural, since the light cone limit $\tilde{\eta} \to 0$ builds up correlations among plaquettes separated along the longitudinal direction [11]. The parameters optimizing our product of single plaquette wave functionals can only effectively describe these correlations.

In the physical relevant coupling region beyond $\lambda = 7$, it is possible to fit the variationally optimized wave functional parameters to the following parameterization

$$\begin{aligned}
\rho_0(\lambda,\tilde{\eta}) &= \sqrt{\lambda}\ \gamma_1(\vec{0})\ f_\rho(\lambda,\eta) \\
\delta_0(\lambda,\tilde{\eta}) &= \sqrt{\lambda}\ \gamma_1(\vec{0})\ f_\delta(\lambda,\eta) \\
f_i(\lambda,\tilde{\eta}) &= c_{0,i}\left[1 + \begin{pmatrix} c_{1,i} \\ c_{2,i} \end{pmatrix} \cdot \begin{pmatrix} \lambda^{-1} \\ 1-\tilde{\eta} \end{pmatrix} \right. \\
&\qquad \left. + \frac{1}{2} \begin{pmatrix} \lambda^{-1} \\ 1-\tilde{\eta} \end{pmatrix} \cdot \begin{pmatrix} c_{3,i} & c_{4,i} \\ c_{4,i} & c_{5,i} \end{pmatrix} \cdot \begin{pmatrix} \lambda^{-1} \\ 1-\tilde{\eta} \end{pmatrix} \right] \\
i &= \rho, \delta\ .
\end{aligned} \tag{3.3}$$

A good fit of the parameters $c_{0,i},...,c_{5,i}$ minimizing χ^2 in the range $\lambda \in [10,95]$ and $\tilde{\eta} \in [0.15,1]$ is possible and yields the coefficients tabulated in tab. (1). The result of the fitting procedure eqn. (3.3)

i	$c_{0,i}$	$c_{1,i}$	$c_{2,i}$	$c_{3,i}$	$c_{4,i}$	$c_{5,i}$
ρ	0.90	-1.74	0.72	8.12	-0.40	-0.27
δ	0.95	0.93	-1.21	-6.44	-0.83	0.64

Table 1: Coefficients of eqn. (3.3) obtained from least square minimization.

is shown by the solid lines in fig. (1) and fig. (2). Having a parameterization of the ground state wave functional in dependence of the nlc parameter $\tilde{\eta}$ at hand, we plan to extrapolate to the light cone and finally calculate hadronic cross sections by simulating how a color dipole moving near the light cone hits a neutral hadron localized at $x^- = 0$ [12]. The color dipole can be represented by a longitudinal-transversal Wilson loop elongated in x^- direction and the simplified target by a transverse plaquette. Varying the impact parameter one can sample the correlation function of the two gauge-invariant objects and thereby obtain the profile function. A necessary prerequisite of such a calculation for different velocities of the dipole is that the lattice constant in transverse direction stays constant for different $\tilde{\eta}$ values, in order to have a reliable transverse length scale.

Acknowledgements

We are grateful to the Max-Planck-Institut für Kernphysik Heidelberg for providing us with resources on the Opteron cluster. D. G. acknowledges funding by the European Union project EU RII3-CT-2004-506078 and the GSI Darmstadt. E.V. P. thanks the Russian Foundation RFFI for the support in this work. E.-M. I. is supported by DFG under contract FOR 465 (Forschergruppe Gitter-Hadronen-Phänomenologie).

References

[1] S. J. Brodsky, H. C. Pauli and S. S. Pinsky, Phys. Rept. **301** (1998) 299 [arXiv:hep-ph/9705477].

[2] M. Burkardt and S. Dalley, Prog. Part. Nucl. Phys. **48** (2002) 317 [arXiv:hep-ph/0112007].

[3] W. A. Bardeen, R. B. Pearson and E. Rabinovici, Phys. Rev. D **21** (1980) 1037.

[4] S. Dalley and B. van de Sande, arXiv:hep-ph/0311368.

[5] H. J. Pirner, Prog. Part. Nucl. Phys. **29** (1992) 33.

[6] H. Verlinde and E. Verlinde, arXiv:hep-th/9302104.

[7] I. Y. Arefeva, Phys. Lett. B **328** (1994) 411 [arXiv:hep-th/9306014].

[8] I. Balitsky, arXiv:hep-ph/0101042.

[9] J. Raufeisen and S. J. Brodsky, Phys. Rev. D **70** (2004) 085017 [arXiv:hep-th/0408108].

[10] E. V. Prokhvatilov and V. A. Franke, Sov. J. Nucl. Phys. **49** (1989) 688 [Yad. Fiz. **49** (1989) 1109].

[11] D. Grünewald , E.-M. Ilgenfritz, E.V. Prokhvatilov and H. J. Pirner, "Formulating Light Cone QCD on the Lattice," work in progress.

[12] H. J. Pirner and F. Yuan, Phys. Rev. D **66** (2002) 034020 [arXiv:hep-ph/0203184].