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Lattice Formulations of Two Dimensional Topological Field Theories

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We propose a non-perturbative criterion to examine whether supersymmetric lattice gauge theories with preserved supercharges can have the desired continuum limit or not. Since the target continuum theories of the lattice models are extended supersymmetric gauge theories including the topological field theory as a special subsector, the continuum limits of them should reproduce the topological properties. Therefore, whether the topological quantities can be recovered at the continuum limit becomes a non-perturbative criterion. Then we propose it as the criterion. In this work,we investigate the BRST cohomology on the two dimensional $\mathcal{N} = (4,4)$ CKKU lattice model without moduli fixing mass term. We show that the BRST cohomology in the target continuum theory cannot be realized from the BRST cohomology on the lattice. From this result, we obtain the possible implication that the $\mathcal{N} = (4,4)$ CKKU model cannot realize the target continuum theory when the non-perturbative effects are taken into account.

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1. Introduction

Recently, several lattice gauge theories which preserve partial supersymmetry on the lattice are proposed [1, 2, 3, 4]. The main purpose in these models is to solve the fine-tuning problem. The fine-tuning problem is the difficulty to recover the target continuum theory including the quantum effects. To construct such formulations, they utilize the topological twisting. Topological twisting is picking up a set of supersymmetry generators which does not include the infinitesimal translation in its algebra. In this way, partial supersymmetry can be preserved on the lattice which breaks the translational invariance.

It is very important to investigate whether the models really solve the fine-tuning problem or not. To do it, we should examine whether they recover the target continuum theories or not. In the perturbative level, such investigations have been done well [5]. But, on the other hand, there is not a sufficient study which takes the non-perturbative effects into consideration. Then we will non-perturbatively investigate whether the models really solve the fine-tuning problem or not.

2. The non-perturbative criteria

Note that the models can be regarded as the lattice regularizations of the topological field theory (TFT). This is because preserved supercharges on the lattice are equivalent to the BRST charge in the TFT obtained by the topological twisting. The target continuum theories of these lattice models are extended supersymmetric gauge theories including the TFT as a special subsector. Therefore the topological field theory in the continuum theory must be recovered in the continuum limits if the lattice models really recover the target continuum theories.

In this work, among the several properties of the TFT, we investigate the behavior of the BRST cohomology [6]. The BRST cohomology is defined with the vacuum expectation value $\langle \mathscr{O} \rangle$ of an operator $\mathscr{O}$ vanishing under the operation of the BRST charge Q (BRST closed) but not BRST exact. The BRST exact is a quantity written by the Q-operation of a gauge invariant quantity. We can obtain the $\langle \mathscr{O} \rangle$ exactly by the semi-classical approximation since the quantity $\langle \mathscr{O} \rangle$ is independent of the gauge coupling due to the property of the Hilbert space of the TFT. Namely, $\langle \mathscr{O} \rangle$ can be regarded as one of the non-perturbative quantities. Therefore, by examining whether the BRST cohomology in the continuum theory can be recovered at the continuum limit or not, we can non-perturbatively investigate whether a lattice model can recover the continuum theory or not.

In this paper, we consider whether $\mathscr{N} = (4,4)$ two dimensional CKKU model [1] really have the desired continuum limit or not. To do it, we study the BRST cohomology on the lattice. Then we compare the BRST cohomology on the lattice with the BRST cohomology in the continuum theory, and we consider whether the BRST cohomologies in the target theory are really recovered in the continuum limit. From this study, we consider whether the target theory is recovered in the continuum limit or not.

3. The BRST cohomology in the target continuum theory.

To make a comparison between the BRST cohomologies in the target continuum theory and the ones on the lattice, we explain the BRST cohomology in the target continuum theory. The action of the continuum theory is written by the BRST exact form as described at the eq. (5.1) in the paper [7]. The BRST transformation laws of the continuum theory are given at the eq. (5.2) in the paper [7]. Among the transformation laws in the eq. (5.2), we pick up following transformation laws

$$\begin{aligned} Q\phi &= 0, \\ Qv_\mu &= \psi_\mu, \\ Q\psi_\mu &= iD_\mu\phi, \end{aligned} \tag{3.1}$$

here, since we use these to create the BRST cohomologies in the continuum theory. In eq. (3.1), v_μ denotes the gauge field and the ψ_μ denotes the BRST partner of the gauge field.

In the continuum theory, at least, the BRST cohomologies are composed by ϕ, v_μ and ψ_μ. To compose the BRST cohomologies by these fields, we can utilize the 'descent relation' proposed by Witten [8]. Let us prepare the differential 0-form, 1-form and 2-form operator set

$$\begin{aligned} \mathscr{W}_0 &= Tr\phi^2, \\ \mathscr{W}_1 &= Tr\phi\psi, \\ \mathscr{W}_2 &= Tr\phi(dv + v\wedge v) + \psi\wedge\psi, \end{aligned} \tag{3.2}$$

where ψ and v are differential 1-form denoted by $\psi = \psi_\mu dx^\mu$ and $v = v_\mu dx^\mu$. Here d denotes the exterior derivative. The set satisfies the following the 'descent relation'

$$Q\mathscr{W}_0 = 0, \tag{3.3}$$

$$Q\mathscr{W}_k = d\mathscr{W}_{k-1} \qquad (k = 1,2). \tag{3.4}$$

Utilizing this property, the BRST closed operators $\mathscr{O}_k$ can be constructed by the integral of $\mathscr{W}_k$ $(k = 1,2)$ over the k dimensional homology cycle γ_k,

$$\mathscr{O}_k \equiv \int_{\gamma_k} \mathscr{W}_k. \tag{3.5}$$

We can confirm that these operators are BRST closed by the explicit calculation,

$$Q\mathscr{O}_k = Q\int_{\gamma_k}\mathscr{W}_k = \int_{\gamma_k} d\mathscr{W}_{k-1} = \int_{\partial\gamma_k}\mathscr{W}_{k-1} = 0, \tag{3.6}$$

since any homology cycle does not have boundaries. Also the $\mathscr{W}_0$ is the BRST closed operator due to the transformation law $Q\phi = 0$.

These $\mathscr{O}_k$ are BRST cohomologies although they are *formally* written by the BRST exact form,

$$\mathscr{O}_1 = \int Q\mathrm{Tr}\,\phi v, \qquad \mathscr{O}_2 = \int Q\mathrm{Tr}\,\psi\wedge v. \tag{3.7}$$

The operators $\mathrm{Tr}\,\phi v$ and $\mathrm{Tr}\,\psi\wedge v$ are not gauge invariant. The BRST exact quantities are defined by the Q-operation of gauge invariant quantities. Therefore these $\mathscr{O}_1$ and $\mathscr{O}_2$ are not BRST exact but BRST closed quantities, namely these are BRST cohomologies. Here, please note that the Q-operation changes the gauge transformation laws as

$$v_\mu \to g^{-1}v_\mu g + g^{-1}\partial_\mu g, \tag{3.8}$$

$$Qv_\mu = \psi_\mu \to g^{-1}\psi_\mu g. \tag{3.9}$$

This property plays an important role to create the gauge invariant BRST cohomology from the gauge variant quantities $\mathrm{Tr}\,\phi v$ and $\mathrm{Tr}\,\psi\wedge v$.

4. The BRST cohomology on the two dimensional $\mathscr{N}=(4,4)$ CKKU lattice model.

Next, let us consider the BRST cohomology on the two dimensional $\mathscr{N}=(4,4)$ CKKU lattice model without moduli fixing mass term. The action of the lattice model is written at eq. (3.14) in [1], and the preserved supercharges and their transformation laws are given by eqs. (3.2),(3.3),(3.5) and (3.6) in [1]. The action can be written by the equivalent BRST exact form described in eq. (2.14),(2.15) in [7], where the BRST charge is given by the the linear combination of the original supercharges as described at eq. (2.11) in [7]. In fact, also the BRST exact action eq. (3.6) in [6] is completely equivalent to eq. (2.11) in [7]. One can check the equivalence by identifying the fields as follows

$$
\begin{array}{llll}
X_{\mathbf{n}} \Leftrightarrow \sqrt{2} z_{1,\mathbf{n}}, & \lambda_{\mathbf{n}} \Leftrightarrow \sqrt{2}\psi_{1,\mathbf{n}}, & X_{\mathbf{n}}^{\dagger} \Leftrightarrow \sqrt{2}\bar{z}_{1,\mathbf{n}}, & \lambda_{\mathbf{n}}^{\dagger} \Leftrightarrow -\sqrt{2}\xi_{2,\mathbf{n}} \\
Y_{\mathbf{n}} \Leftrightarrow \sqrt{2} z_{2,\mathbf{n}}, & \tilde{\lambda}_{\mathbf{n}} \Leftrightarrow \sqrt{2}\psi_{2,\mathbf{n}} & Y_{\mathbf{n}}^{\dagger} \Leftrightarrow \sqrt{2}\bar{z}_{2,\mathbf{n}}, & \tilde{\lambda}_{\mathbf{n}}^{\dagger} \Leftrightarrow \sqrt{2}\xi_{1,\mathbf{n}}, \\
\bar{\Phi}_{\mathbf{n}} \Leftrightarrow \sqrt{2} z_{3,\mathbf{n}}, & \eta_{\mathbf{n}} \Leftrightarrow \sqrt{2}(\psi_{3,\mathbf{n}}-\lambda_{\mathbf{n}}), & \chi_{\mathbf{n}}^{\mathbb{C}} \Leftrightarrow \sqrt{2}\chi_{\mathbf{n}}, & \chi_{\mathbf{n}}^{\mathbb{C}\dagger} \Leftrightarrow \sqrt{2}\xi_{3,\mathbf{n}}, \\
H_{\mathbf{n}}^{\mathbb{C}} \Leftrightarrow \sqrt{2}\tilde{\bar{G}}_{\mathbf{n}}, & H_{\mathbf{n}}^{\mathbb{C}\dagger} \Leftrightarrow \sqrt{2}\tilde{G}_{\mathbf{n}}, & \chi_{\mathbf{n}}^{\mathbb{R}} \Leftrightarrow -i\sqrt{2}(\psi_{3,\mathbf{n}}+\lambda_{\mathbf{n}}), & H_{\mathbf{n}}^{\mathbb{R}} \Leftrightarrow -\tilde{d}_{\mathbf{n}} \\
\Phi_{\mathbf{n}} \Leftrightarrow \sqrt{2}\bar{z}_{3,\mathbf{n}}. & & &
\end{array}
\tag{4.1}
$$

In this paper, we use the BRST exact form eq. (3.7) in [6] of the CKKU lattice action,

$$
\begin{aligned}
S &= Q\Xi \\
\Xi &= \mathrm{Tr}\Bigg[\frac{1}{4}\eta_{\mathbf{n}}[\Phi_{\mathbf{n}},\bar{\Phi}_{\mathbf{n}}] + \vec{\chi}_{\mathbf{n}}\cdot(\vec{H}_{\mathbf{n}} - i\vec{\mathscr{E}}_{\mathbf{n}}) \\
&\qquad + \frac{1}{2}\Big\{\lambda_{\mathbf{n}}(X_{\mathbf{n}}^{\dagger}\bar{\Phi}_{\mathbf{n}} - \bar{\Phi}_{\mathbf{n}+\mathbf{i}}X_{\mathbf{n}}^{\dagger}) + \lambda_{\mathbf{n}-\mathbf{i}}^{\dagger}(X_{\mathbf{n}-\mathbf{i}}\bar{\Phi}_{\mathbf{n}} - \bar{\Phi}_{\mathbf{n}-\mathbf{i}}X_{\mathbf{n}-\mathbf{i}}) \\
&\qquad + \tilde{\lambda}_{\mathbf{n}}(Y_{\mathbf{n}}^{\dagger}\bar{\Phi}_{\mathbf{n}} - \bar{\Phi}_{\mathbf{n}+\mathbf{j}}Y_{\mathbf{n}}^{\dagger}) + \tilde{\lambda}_{\mathbf{n}-\mathbf{j}}^{\dagger}(Y_{\mathbf{n}-\mathbf{j}}\bar{\Phi}_{\mathbf{n}} - \bar{\Phi}_{\mathbf{n}-\mathbf{j}}Y_{\mathbf{n}-\mathbf{j}})\Big\}\Bigg], \\
\mathscr{E}_{\mathbf{n}}^{\mathbb{R}} &= -(X_{\mathbf{n}}X_{\mathbf{n}}^{\dagger} - X_{\mathbf{n}-\mathbf{i}}^{\dagger}X_{\mathbf{n}-\mathbf{i}} + Y_{\mathbf{n}}Y_{\mathbf{n}}^{\dagger} - Y_{\mathbf{n}-\mathbf{j}}^{\dagger}Y_{\mathbf{n}-\mathbf{j}}), \\
\mathscr{E}_{\mathbf{n}}^{\mathbb{C}} &= 2i(X_{\mathbf{n}}Y_{\mathbf{n}+\mathbf{i}} - Y_{\mathbf{n}}X_{\mathbf{n}+\mathbf{j}}).
\end{aligned}
\tag{4.2}
$$

In the tree level, the continuum limit of the eq. (3.7) in [6] becomes the topological field theory action eq. (3.11) in [6] (or eq. (5.1) in [7]), which is equivalent to the two dimensional $\mathscr{N}=(4,4)$ super Yang-Mills theory. In the continuum limit, the lattice field variable Φ becomes the field ϕ in the continuum theory, and the gauge fields v_μ come from the bosonic link fields $X, X^{\dagger}, Y, Y^{\dagger}$. The BRST partners of the gauge fields ψ_μ come from the fermionic link fields, $\lambda, \lambda^{\dagger}, \tilde{\lambda}, \tilde{\lambda}^{\dagger}$. For later use, we distinguish the degree of freedom as the two part $\{\Phi_{\mathbf{n}}\}$ and $\vec{\mathscr{A}}_{\mathbf{n}}$. Here $\{\Phi_{\mathbf{n}}\}$ is the set composed only by the field Φ, and the set $\vec{\mathscr{A}}_{\mathbf{n}}$ is composed by the other fields.

The BRST transformation laws are given in eq. (3.7) in [6],

$$
\begin{array}{ll}
QX_{\mathbf{n}} = \lambda_{\mathbf{n}}, & Q\lambda_{\mathbf{n}} = \Phi_{\mathbf{n}}X_{\mathbf{n}} - X_{\mathbf{n}}\Phi_{\mathbf{n}+\mathbf{i}}, \\
QY_{\mathbf{n}} = \tilde{\lambda}_{\mathbf{n}}, & Q\tilde{\lambda}_{\mathbf{n}} = \Phi_{\mathbf{n}}Y_{\mathbf{n}} - Y_{\mathbf{n}}\Phi_{\mathbf{n}+\mathbf{j}}, \\
QH_{\mathbf{n}}^{\mathbb{R}} = [\Phi_{\mathbf{n}}, \chi_{\mathbf{n}}^{\mathbb{R}}], & Q\chi_{\mathbf{n}}^{\mathbb{R}} = H_{\mathbf{n}}^{\mathbb{R}}, \\
QH_{\mathbf{n}}^{\mathbb{C}} = \Phi_{\mathbf{n}}\chi_{\mathbf{n}}^{\mathbb{C}} - \chi_{\mathbf{n}}^{\mathbb{C}}\Phi_{\mathbf{n}+\mathbf{i}+\mathbf{j}}, & Q\chi_{\mathbf{n}}^{\mathbb{C}} = H_{\mathbf{n}}^{\mathbb{C}}, \\
Q\bar{\Phi}_{\mathbf{n}} = \eta_{\mathbf{n}}, & Q\eta_{\mathbf{n}} = [\Phi_{\mathbf{n}}, \bar{\Phi}_{\mathbf{n}}], \\
Q\Phi_{\mathbf{n}} = 0. &
\end{array}
\tag{4.3}
$$

Note that this is a homogeneous transformation of $\vec{\mathscr{A}}_{\mathbf{n}}$. Therefore, the transformation can be written as the tangent vector

$$
\begin{aligned}
Q = \sum_{\mathbf{n}} \Bigg[& \lambda_{\mathbf{n}} \frac{\partial}{\partial X_{\mathbf{n}}} + \lambda_{\mathbf{n}}^{\dagger} \frac{\partial}{\partial X_{\mathbf{n}}^{\dagger}} + \tilde{\lambda}_{\mathbf{n}} \frac{\partial}{\partial Y_{\mathbf{n}}} + \tilde{\lambda}_{\mathbf{n}}^{\dagger} \frac{\partial}{\partial Y_{\mathbf{n}}^{\dagger}} \\
& + [\Phi_{\mathbf{n}}, \chi_{\mathbf{n}}^{\mathbb{R}}] \frac{\partial}{\partial H_{\mathbf{n}}^{\mathbb{R}}} + (\Phi_{\mathbf{n}} \chi_{\mathbf{n}}^{\mathbb{C}} - \chi_{\mathbf{n}}^{\mathbb{C}} \Phi_{\mathbf{n}+\mathbf{i}+\mathbf{j}}) \frac{\partial}{\partial H_{\mathbf{n}}^{\mathbb{C}}} + (\Phi_{\mathbf{n}} \chi_{\mathbf{n}}^{\mathbb{C}\dagger} - \chi_{\mathbf{n}}^{\mathbb{C}\dagger} \Phi_{\mathbf{n}-\mathbf{i}-\mathbf{j}}) \frac{\partial}{\partial H_{\mathbf{n}}^{\mathbb{C}\dagger}} + \eta_{\mathbf{n}} \frac{\partial}{\partial \Phi_{\mathbf{n}}} \\
& + (\Phi_{\mathbf{n}} X_{\mathbf{n}} - X_{\mathbf{n}} \Phi_{\mathbf{n}+\mathbf{i}}) \frac{\partial}{\partial \lambda_{\mathbf{n}}} + (\Phi_{\mathbf{n}} X_{\mathbf{n}}^{\dagger} - X_{\mathbf{n}}^{\dagger} \Phi_{\mathbf{n}-\mathbf{i}}) \frac{\partial}{\partial \lambda_{\mathbf{n}}^{\dagger}} \\
& + (\Phi_{\mathbf{n}} Y_{\mathbf{n}} - Y_{\mathbf{n}} \Phi_{\mathbf{n}+\mathbf{j}}) \frac{\partial}{\partial \tilde{\lambda}_{\mathbf{n}}} + (\Phi_{\mathbf{n}} Y_{\mathbf{n}}^{\dagger} - Y_{\mathbf{n}}^{\dagger} \Phi_{\mathbf{n}-\mathbf{j}}) \frac{\partial}{\partial \tilde{\lambda}_{\mathbf{n}}^{\dagger}} \\
& + \vec{H}_{\mathbf{n}} \cdot \frac{\partial}{\partial \vec{\chi}_{\mathbf{n}}} + [\Phi_{\mathbf{n}}, \bar{\Phi}_{\mathbf{n}}] \frac{\partial}{\partial \eta_{\mathbf{n}}} \Bigg].
\end{aligned} \tag{4.4}
$$

In addition to this charges, we introduce another fermionic operator written by the tangent vector

$$
\tilde{Q} = \sum_{\mathbf{n}} X_{\mathbf{n}} \frac{\partial}{\partial \lambda_{\mathbf{n}}} + X_{\mathbf{n}}^{\dagger} \frac{\partial}{\partial \lambda_{\mathbf{n}}^{\dagger}} + Y_{\mathbf{n}} \frac{\partial}{\partial \tilde{\lambda}_{\mathbf{n}}} + Y_{\mathbf{n}}^{\dagger} \frac{\partial}{\partial \tilde{\lambda}_{\mathbf{n}}^{\dagger}} + \bar{\Phi}_{\mathbf{n}} \frac{\partial}{\partial \eta_{\mathbf{n}}} + \vec{\chi}_{\mathbf{n}} \cdot \frac{\partial}{\partial \vec{H}_{\mathbf{n}}}. \tag{4.5}
$$

Then, the anti-commutation relation between these two charges becomes the number operator $\hat{N}_{\mathscr{A}}$, which count the number of fields in the set $\vec{\mathscr{A}}_{\mathbf{n}}$, as follows

$$
\begin{aligned}
\{Q, \tilde{Q}\} &= \sum_{\mathbf{n}} X_{\mathbf{n}} \frac{\partial}{\partial X_{\mathbf{n}}} + X_{\mathbf{n}}^{\dagger} \frac{\partial}{\partial X_{\mathbf{n}}^{\dagger}} + Y_{\mathbf{n}} \frac{\partial}{\partial Y_{\mathbf{n}}} + Y_{\mathbf{n}}^{\dagger} \frac{\partial}{\partial Y_{\mathbf{n}}^{\dagger}} + \bar{\Phi}_{\mathbf{n}} \frac{\partial}{\partial \Phi_{\mathbf{n}}} + \vec{H}_{\mathbf{n}} \cdot \frac{\partial}{\partial \vec{H}_{\mathbf{n}}} \\
&\quad + \lambda_{\mathbf{n}} \frac{\partial}{\partial \lambda_{\mathbf{n}}} + \lambda_{\mathbf{n}}^{\dagger} \frac{\partial}{\partial \lambda_{\mathbf{n}}^{\dagger}} + \tilde{\lambda}_{\mathbf{n}} \frac{\partial}{\partial \tilde{\lambda}_{\mathbf{n}}} + \tilde{\lambda}_{\mathbf{n}}^{\dagger} \frac{\partial}{\partial \tilde{\lambda}_{\mathbf{n}}^{\dagger}} + \eta_{\mathbf{n}} \frac{\partial}{\partial \eta_{\mathbf{n}}} + \vec{\chi}_{\mathbf{n}} \cdot \frac{\partial}{\partial \vec{\chi}_{\mathbf{n}}} \\
&= \hat{N}_{\mathscr{A}}.
\end{aligned} \tag{4.6}
$$

Please note that any function of the field variables h can be written in terms of a sum of eigenfunction of $\hat{N}_{\mathscr{A}}$, namely

$$
h = \sum_{n_{\mathscr{A}}=0}^{\infty} h_{n_{\mathscr{A}}}, \qquad \hat{N}_{\mathscr{A}} h_{n_{\mathscr{A}}} = n_{\mathscr{A}} h_{n_{\mathscr{A}}}, \quad n_{\mathscr{A}} \in \{0\} \cup \mathbb{N}, \tag{4.7}
$$

since any term in the function h has definite number of fields in the set $\vec{\mathscr{A}}_{\mathbf{n}}$. In addition to this homogeneous property of the BRST charge Q, this Q does not change the gauge transformation law opposite to the continuum theory case. One can confirm it by checking that each field variable resides on the same link or site as its BRST partner respectively. Please look at BRST transformation laws eq. (3.7) in [6] and Fig. 1 in [6].

From these properties of BRST charges, we can see that BRST cohomology must be composed only by Φ on the lattice. We will show it. First, let us consider the BRST closed function h_c satisfying $Qh_c = 0$. From the property eq. (4.7), also h_c can be decomposed by the sum of eigenfunctions of the operator $\hat{N}_{\mathscr{A}}$,

$$
h_c = \sum_{n_{\mathscr{A}}=0}^{\infty} h_{c, n_{\mathscr{A}}}. \tag{4.8}
$$

Since the BRST operator is homogeneous transformation which does not change the number of fields in $\vec{\mathscr{A}}_{\mathbf{n}}$, the BRST operator Q commutes with the number operator $\hat{N}_{\mathscr{A}}$, namely

$$[Q, \hat{N}_{\mathscr{A}}] = 0. \tag{4.9}$$

Then, if $Qh_c = 0$, each eigenfunction $h_{c,n_{\mathscr{A}}}$ must be BRST closed,

$$Qh_c = 0 \Leftrightarrow Qh_{c,n_{\mathscr{A}}} = 0, \quad (n_{\mathscr{A}}^{\forall} \in \{0\} \cup \mathbb{N}). \tag{4.10}$$

The BRST closed eigenfunctions $h_{c,n_{\mathscr{A}}}$ with non-zero eigenvalue $n_{\mathscr{A}} \neq 0$ can be formally written as the BRST exact form since

$$h_{c,n_{\mathscr{A}}} = n_{\mathscr{A}}^{-1} \hat{N}_{\mathscr{A}} h_{c,n_{\mathscr{A}}} = n_{\mathscr{A}}^{-1} \{Q, \tilde{Q}\} h_{c,n_{\mathscr{A}}} = n_{\mathscr{A}}^{-1} Q \tilde{Q} h_{c,n_{\mathscr{A}}}. \tag{4.11}$$

In this equation, $\tilde{Q}h_{c,n_{\mathscr{A}}}$ must be gauge invariant if the function $h_{c,n_{\mathscr{A}}}$ is a gauge invariant function, since the Q-operation does not change the gauge transformation law. Therefore, in the BRST closed function h_c, BRST closed non-zero eigenfunction $h_{c,n_{\mathscr{A}}}$ must be BRST exact. Finally, we can see that the only the zero eigenfunction $h_{c,0}$, which is the polynomial composed only by Φ, can be the BRST cohomology among the eigenfunctions. This is the end of proof.

The above situation stands for any lattice spacing. This tells that the BRST cohomology must be composed only by Φ no matter how the lattice spacing is small, namely even in the continuum limit. Therefore the BRST cohomology in the target continuum theory, which are composed not only by ϕ but also by gauge fields v_μ and their partners ψ_μ, cannot be realized from the BRST cohomology on the lattice. Finally, we obtain the possible implication that the $\mathscr{N} = (4,4)$ CKKU lattice model cannot realize the desired target continuum theory.

4.1 A possible reason why the BRST cohomology cannot be realized on the lattice

Among the BRST cohomologies in the target theory, the quantities composed by v_μ and ψ_μ, which are 1-form and 2-form operators $\mathscr{O}_1$ and $\mathscr{O}_2$, are defined by the inner product between the homology cycle and its dual cohomology of the base manifold. Such inner products are topological quantities which are invariant under the infinitesimal transformation of the base manifold. It is generally difficult to construct such topological quantities on the lattice. On the lattice, gauge symmetry is defined with the gauge parameters which are completely independent of the parameters on the neighbor sites. Such a property of the lattice gauge symmetry admits the singular gauge transformation which prevents us from the realization of the topological quantities on the lattice. Therefore we could not obtain the BRST cohomologies which are composed by v_μ and ψ_μ.

Inhomogeneous term $g^{-1}\partial_\mu g$ in the eq. (3.8) are removed from the gauge transformation law of the corresponding link gauge fields due to the property of the lattice gauge symmetry. By this property, Q on the lattice does not change the gauge transformation law. Also it would be the reason of the impossibility to realize the corresponding BRST cohomology on the lattice.

5. Conclusion and discussion.

We investigate whether the supersymmetric lattice model, which is the two dimensional $\mathscr{N} = (4,4)$ CKKU supersymmetric lattice model, really recover the target theory or not through the examining whether the property of the TFT are really recovered in the continuum limit or not. In this paper, as the first step, we estimate the situation by the comparison between the BRST cohomology on the CKKU lattice and the BRST cohomology in the $\mathcal{N}=(4,4)$ target continuum theory. By the investigation, we have understood that the BRST cohomology in the target continuum theory cannot be realize from the BRST cohomology on the lattice. This implies that there is a possibility that the CKKU lattice model cannot realize the desired target continuum theory.

Moreover, we consider the reason of the impossibility. The reason of the impossibility would be that the BRST cohomology is a topological quantity defined by the inner product between the homology cycle and its dual cohomology. It is generally difficult to construct such a topological quantity on the lattice since the gauge symmetry on the lattice admits the singular gauge transformation which prevents us from the realization of the topological quantity on the lattice. From this observation, we can guess that also other models like [3, 4] might be difficult to recover the desired target theories. But, from this, we could obtain the valuable strategy to develop the lattice formulation which can easily recover the desired target theory, namely the formulation applicable to the numerical study. We propose to apply the Admissibility condition [9] etc, which would enables us to define the topological quantity, to define the BRST cohomology on the lattice and to recover the desired target theory.

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