On the ratio of string tensions in the 3D $\mathbb{Z}_4$ lattice gauge theory

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It was recently pointed out that simple scaling properties of Polyakov correlation functions of gauge systems in the confining phase suggest that the ratios of k-string tensions in the low temperature region is constant up to terms of order T^3. Here we argue that, at least in a three-dimensional $\mathbb{Z}_4$ gauge model, the above ratios are constant in the whole confining phase. This result is obtained by combining numerical experiments with known exact results on the mass spectrum of an integrable two-dimensional spin model describing the infrared behaviour of the gauge system near the deconfining transition.

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1. Introduction

It was recently pointed out that in most confining gauge theories, besides the fundamental string (of tension σ) which is formed between a pair of static sources in the fundamental representation f, there is the freedom of taking the sources in any representation $\mathcal{R}$. If, for instance, the gauge group is SU(N) there are infinitely many irreducible representations at our disposal. However, as the sources are pulled apart, no matter what representation is chosen, the asymptotically stable string tension $\sigma_{\mathcal{R}}$ depends only on the $N-$ality k of $\mathcal{R}$. As a consequence the heavier strings decay into the string of smallest string tension σ_k. The corresponding string is referred to as a k-string. This kind of confining object can be defined whenever the gauge group admits more than one non trivial irreducible representation.

In a previous work [1], some of us have proposed an expression for the low temperature asymptotic expansion for these string tensions. An interesting consequence of such an expansion is that their ratios are expected to be constant up to T^3 terms. The low temperature data presented in support of this expectation were taken from Monte Carlo simulations on a particular system, namely a (2+1)-dimensional $\mathbb{Z}_4$ gauge model.

The main conjecture we want to verify in this work is that $\sigma_k(T)/\sigma(T)$, at least in that $\mathbb{Z}_4$ gauge system, is in fact independent of the temperature in the *whole* of the confining regime. To check this idea we used the fact that the Svetitsky-Yaffe (SY) conjecture [2] allows to reformulate the system in a totally different perspective, based on a two-dimensional integrable theory.

It turns out that the deconfinement transition of the 3D $\mathbb{Z}_4$ gauge model is second order and, according to the SY conjecture, belongs to the same universality class of the 2D symmetric Ashkin-Teller (AT) model.

The two-dimensional AT model can be seen in the continuum limit as a bosonic conformal field theory plus a massive perturbation driving the system away from the critical line (i. e. a sine-Gordon theory). Thus, a map between the AT critical line and the sine-Gordon phase space is provided. This theory is integrable, and the masses of its lightest physical states (first soliton and first breather mode, of masses M and M_1) correspond to the tensions $\sigma(T)$ and $\sigma_2(T)$ near T_c, whose ratio, in this context, can be analytically evaluated and turns out to be

$$\lim_{T \to T_c} \frac{\sigma_2(T)}{\sigma(T)} = \frac{M_1}{M} = 2\sin\frac{\pi}{2}(2\nu - 1) \; , \tag{1.1}$$

where ν is the thermal exponent.

1.1 The (2+1)d $\mathbb{Z}_4$ gauge model and its dual reformulation

The most general form of $\mathbb{Z}_4$ lattice gauge model admits two independent coupling constants, with partition function

$$\mathbb{Z}(\beta_f, \beta_{ff}) = \prod_l \sum_{\xi_l = \pm 1, \pm i} e^{\Sigma_p(\beta_f \mathcal{U}_p + \beta_{ff} \mathcal{U}_p^2/2 + \text{c.c.})} \; , \tag{1.2}$$

in which the gauge field U_l on the links on a cubic lattice is valued among the fourth roots of the identity and the sum in the exponent is taken over the elementary plaquettes of the lattice. Such a

theory can be reformulated as two coupled $\mathbb{Z}_2$ gauge systems (see [1] for details):

$$\mathbb{Z}(\beta_f, \beta_{ff}) = \prod_l \sum_{\{U_l = \pm 1, V_l = \pm 1\}} e^{\Sigma_p [\beta_f (U_p + V_p) + \beta_{ff} U_p V_p]} \quad, (U_p = \prod_{l \in p} U_l \; ; V_p = \prod_{l \in p} V_l \,). \tag{1.3}$$

From the data in [1], obtained by means of finite-temperature measurements of Polyakov-Polyakov correlation functions, and particularly from those referring to the point P identified by $(\alpha, \beta) = (0.050, 0.207)$, the string tensions σ and σ_2 can be evaluated in the $T \to 0$ limit as temperature-independent quantities:

$$\sigma a^2 = 0.02085(10)\,, \quad \sigma_2 a^2 = 0.03356(22)\,,$$

where a is the lattice spacing. Their ratio, which has been argued to equate the central charge of the CFT related to the 2-string, is then given by

$$\frac{\sigma_2}{\sigma} = 1.610(13)\,. \tag{1.4}$$

2. The Svetitsky-Yaffe conjecture and the Sine-Gordon model

The mapping induced by the Svetitsky-Yaffe conjecture leads to a substantial simplification in the study of the critical properties of the deconfining transition, allowing to study it as a standard symmetry-breaking transition which takes place in a spin model. In the present case we deal with the symmetric Ashkin-Teller model in two-dimensions.

The action for this model is given by:

$$\mathscr{S}_{AT} = -\sum_{\langle xy \rangle} [J(\sigma_x^1 \sigma_y^1 + \sigma_x^2 \sigma_y^2) + J_4(\sigma_x^1 \sigma_y^1 \sigma_x^2 \sigma_y^2)]\,. \tag{2.1}$$

Such a model has been extensively studied in the past, and a number of exact results have been derived [3]. It is useful to note that it can be seen as a perturbation of the Gaussian model, and in such a bosonic language the thermal perturbation can be written as $\cos \beta \varphi$, where β is a marginal parameter equivalent to J_4. Hence we are left with

$$\mathscr{A}_{AT} = \int d^2 x \left(\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \tau \cos \beta \varphi \right)\,, \tag{2.2}$$

which is the action of the Sine-Gordon model. Furthermore, since the confined phase of the gauge theory is mapped in the high-T phase of the Ashkin-Teller, we will only consider the case $\tau > 0$.

Such a QFT is of particular interest because it is integrable, and this is the main reason for rewriting the action of the model near the critical point in a bosonic form. Integrability means that an infinite number of integrals of motion exists. The main consequence in (1+1) dimensions is the fact that the scattering theory is very constrained, because the S-matrix is factorised in products of two-body interactions, and inelastic processes are forbidden. It follows that the latter can be computed exactly together with the mass spectrum.

2.1 Operator correspondence, mass spectrum and correlation functions

We already know, from the Svetitsky-Yaffe original work, that the Polyakov loop in the fundamental representation corresponds to the spin operator. Then, following the same reasoning used in [4], it is possible to deduce that the Polyakov loop in the double fundamental representation is related to the so-called *polarisation* operator $\mathscr{P} = \sigma^1 \sigma^2$, where σ^1 and σ^2 are the spin variables defined in (2.1). Its bosonic form and the corresponding anomalous dimensions are given by

$$\mathscr{P} = \sin\frac{\beta}{2}\varphi, \quad X_{\mathscr{P}} = \frac{\beta^2}{8\pi} ; \tag{2.3}$$

we also notice that $\langle \mathscr{P} \rangle = 0$ in the high-T phase of the model.

Sine-Gordon mass spectrum [5]: The exact knowledge of the S-matrix allows to access to the exact mass spectrum of the theory. Without entering the details, the spectrum of the SG model is given by a soliton/anti-soliton doublet of fundamental particles of mass M, and a number of soliton/anti-soliton bound states, called breathers B_n, whose number is a function of β^2. By defining the coupling constant ξ in the following way

$$\xi = \frac{\pi\beta^2}{8\pi - \beta^2} , \tag{2.4}$$

we have that for $\xi \geq \pi$, i.e. $\beta^2 \geq 4\pi$, no bound states are present and hence the spectrum is given by the soliton/anti-soliton doublet only (repulsive regime). For $\xi < \pi$, i.e. $\beta^2 < 4\pi$, we are in the attractive regime and the breathers B_n appear as simple poles of the S-matrix (see for example [5]). The next step is to associate particle states to operators in the high temperature phase. It has been done in [6] by taking into account their properties of symmetry and locality. The consequence is that the spin operator is naturally associated to the mass of the soliton, and the polarisation operator is associated to the mass M_1 of the breather B_1. Hence, following the Svetitsky-Yaffe conjecture, the ratio of string tensions in the confining phase near the transition is given by

$$\frac{M_1}{M} = 2\sin\frac{\xi}{2} . \tag{2.5}$$

This result, being a dimensionless ratio, is expected to be universal in the limit $\tau \to 0$. This fact can be explicitly seen by expressing the coupling ξ in terms of some critical exponent. It is possible to work out the following relation between ξ and the thermal critical exponent ν

$$\xi = \pi(2\nu - 1) \quad \to \quad \frac{M_1}{M} = 2\sin\frac{\pi}{2}(2\nu - 1) . \tag{2.6}$$

Correlators at large distance: The previous analysis of the mass spectrum allows to compute the leading behaviour of the correlators $\langle \sigma\sigma \rangle$ and $\langle \mathscr{P}\mathscr{P} \rangle$ at large distance by means of their spectral expansion over form factors (the interested reader can refer to [7, 8] for the details).

The analysis of the previous section allows immediately to write down the leading term for $\langle \sigma\sigma \rangle$ and $\langle \mathscr{P}\mathscr{P} \rangle$ correlators in the high-T phase of the theory, up to an inessential proportionality constant

$$\langle \sigma(x)\sigma(0) \rangle \sim K_0(M|x|), \quad |x| \to \infty ;$$
$$\langle \mathscr{P}(x)\mathscr{P}(0) \rangle \sim K_0(M_1|x|), \quad |x| \to \infty , \tag{2.7}$$

where K_0 denotes the modified Bessel function of order zero, and M, M_1 are the masses of the soliton and the first breather respectively.

2.2 Baryon vertices and mass spectrum

As noticed in [1], the balance of the string tensions for a given vertex gives the following expression for the angles at the center of the junction of three arbitrary k-strings

$$\cos \theta_i = \frac{\sigma_j^2(T) + \sigma_k^2(T) - \sigma_i^2(T)}{2\,\sigma_j(T)\sigma_k(T)}, \quad \text{and cyclic permutations of the indices.} \tag{2.8}$$

The rigidity of the geometry of the vertex is then ensured by requiring that such angles are kept fixed when the temperature varies. As a consequence, all the string tension ratios are constant up to a given order in T, namely as far as the effective string picture is valid.

A similar picture emerges when studying the gauge system near the deconfining transition. The scattering theory describing the system in such a case can exhibit bound states whose mass m_b is given by the following relation

$$m_b^2 = m_1^2 + m_2^2 + 2m_1 m_2 \cos u_{12}^b, \quad \text{triangle of masses} \tag{2.9}$$

where $\theta = i u_{12}^b$ is the purely imaginary value of the rapidity corresponding to the creation of the particle m_b, and m_1, m_2 are the masses of the initial state.

In the present case the process of coalescence of two fundamental strings into a 2-string corresponds to the scattering of a soliton/anti-soliton pair creating the bound state B_1. For such a process we know that $u_{S\bar{S}}^{B_1} = \pi - \xi$ which, once inserted in (2.9), gives

$$M_1^2 = 2M^2(1 - \cos \xi) \quad \rightarrow \quad \frac{M_1}{M} = 2\sin\frac{\xi}{2} \tag{2.10}$$

which is nothing but the mass formula used in the previous Section.

3. Monte Carlo setting and procedure

3.1 Mass ratio by correlators

As introduced in Subsection 2.1, we can determine the ratio M_1/M using the large distance asymptotic behaviour of correlators; actually, exploiting the Svetitsky-Yaffe conjecture, we measured the Polyakov-Polyakov correlators $G_{\mathcal{R}}(R)$ of the (2+1)d $\mathbb{Z}_4$ gauge theory:

$$G_{\mathcal{R}}(R) = \langle P_{\mathcal{R}}(0)P_{\mathcal{R}}^{\dagger}(R)\rangle. \tag{3.1}$$

In Section 1.1 we have explained we can study this theory by means of simulations on the AT model and in [9] the measurement of Polyakov-Polyakov correlators in both the fundamental and double fundamental representations, $G(R)_f$ and $G(R)_{ff}$, is described in detail.

We have taken 10^6 measures on the $64^2 \times 7$ lattice in the phase space point P; $N_\tau = 7$ is chosen because it is the lowest possible value above the deconfinement transition. Simulations have been done for each value of R in the range $[15 \div 44]$. These data are fitted using an expansion of the $K_0(mR)$ Bessel function, truncated to first two terms, in a range $[R_{min}, R_{max}]$, where $R_{max} = 44$; we have verified the results are stable when R_{min} varies in the range $[22 \div 33]$. Therefore, it is possible to determine the two masses:

$$aM_{ff} = 0.0698(15) \quad (\chi^2/\text{d.o.f.} \approx 1.3), \quad aM_f = 0.0433(8) \quad (\chi^2/\text{d.o.f.} \approx 1.2),$$

from which we can determine the ratio:

$$\sigma_2(T \sim T_c)/\sigma(T \sim T_c) = M_{ff}/M_f = 1.612(46) \,.$$ (3.2)

This result, obtained near the critical temperature, is compatible with the zero-temperature value (1.4), providing a strong evidence for our conjecture.

3.2 Estimating σ_2/σ through the thermal exponent ν with finite-size scaling

To use the formula for the mass ratio, Eq. (2.6), we need a quite precise estimate for the thermal critical exponent ν in the phase space point P. It can be obtained by means of a finite-size scaling analysis of the plaquette operator or some related observable that we denote with $\langle \square \rangle_L$, where L is the spacial size of the lattice. Let us notice that in the SY context the plaquette operator is mapped into a combination of the unity and the energy operator of the corresponding CFT [10].

In order to exploit the computational advantages of the dual transcription of the gauge model, it is convenient to evaluate directly the internal energy of the 3D AT model defined by

$$\langle \square \rangle_L \equiv -\frac{1}{3L^2 L_t} \langle S_{AT} \rangle \,.$$ (3.3)

We decided however to use the corresponding (density of) susceptivity

$$\langle \chi \rangle_L \equiv \langle (\square - \langle \square \rangle_L)^2 \rangle_L \,,$$ (3.4)

whose power-law to compare with has the form

$$\langle \chi \rangle_L = b' \cdot L^{\frac{2}{\nu} - d} \,,$$ (3.5)

with the advantage that no constant additive terms are present, which could largely spoil the stability of the numerical results.

At the practical level, the system at the coupling P turns out to be critical for a temperature T_c such that $6 < \frac{1}{T_c} < 7$, hence, having to work with integer inverse temperatures, it is not possible to avoid some approximate method. In particular we decided to define two new points, P_7 and P_6, at which the system is critical for temperatures $T = 1/7$ and $T = 1/6$ respectively, and then, with a linear interpolation, construct the corresponding quantity for the original point of phase transition P. To perform the simulations, we used a cluster-based nonlocal update algorithm, an adaptation of the Swendsen-Wang prescription, which is described in more detail in [1].

We used $L = 200$ finite-temperature lattices to find the couplings corresponding to P_6 and P_7, and at such critical points we took $\mathcal{O}(10^5)$ measurements of the plaquette at 26 values of spatial side L, ranging from $L = 10$ to $L = 165$. The data fitted very well to the expectation from $L = 70$ already, so we could extract two values of the critical index ν:

$$\nu_{T=1/6} = 0.8004(19)[22] \,, \quad \nu_{T=1/7} = 0.7942(18)[38] \,,$$

in which the first uncertainty refers to the statistical fluctuations while the second is an estimate of the systematic error in the measurement.

By linear interpolation along the couplings, the value of ν and the (coupling-dependent) critical temperature T_c was calculated for the very point P. We found

$$\frac{T_c}{\sqrt{\sigma}} = 1.0393(12), \quad \nu(P) = 0.7984(19)[27]. \tag{3.6}$$

By plugging it into the formula for the mass ratio (2.6), we obtain the following result for the mass ratio:

$$\frac{M_1}{M}(P) = 1.6124(71)[102], \tag{3.7}$$

which is well compatible with the less accurate estimate coming from the quantities in [1] and thus well supports our conjecture.

4. Conclusions

In this paper we studied the ratio of the string tensions $\sigma_2(T)/\sigma(T)$ near the deconfining point T_c of a 3D $\mathbb{Z}_4$ gauge model and compared the result with a general formula which is expected to be true near $T = 0$ for a generic gauge theory in three or four dimensions. In this particular case we have combined numerical experiments with known exact results of an integrable 2D quantum field theory that belongs, according to the Svetitsky-Yaffe conjecture, to the same universality class of the critical gauge system.

An interesting property of the integrable model is that the mass ratio of the two physical states of the theory, which should equate the string tension ratio near T_c, can be expressed as a simple function of the thermal exponent ν (see Eq.(2.6)). We used two different methods to evaluate such a ratio, and both the estimates give compatible results which nicely agree with the ratio σ_2/σ evaluated at $T = 0$ (see Eq.(1.4)). We then conclude that, at least in this model, the k-string tension ratios do not depend on T.

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