

Study of the topological vacuum structure of $SU(2)$ gluodynamics at $T > 0$ with overlap fermions and improved action *

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We study $SU(2)$ gluodynamics at finite temperature near the deconfining phase transition. We create the lattice ensembles using the tadpole improved Lüscher-Weisz action. The overlap Dirac operator is used to determine the following three aspects of vacuum structure: (i) The topological susceptibility is evaluated at various temperatures across the phase transition, (ii) the overlap fermion spectral density is determined and found to depend on the Polyakov loop above the phase transition and (iii) the corresponding localization properties of low lying eigenmodes are investigated.

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1. Introduction

More than ten years ago, using a model generalizing random matrix theory, M.A. Stephanov [1] predicted, that in $SU(3)$ gluodynamics above T_c the different Polyakov loop sectors behave differently. In the complex-valued Polyakov loop sectors the chiral condensate should turn to zero at T substantially above T_c . For $SU(2)$ lattice gluodynamics, where the Polyakov loop is real, it was predicted that the chiral condensate stays non-zero, $\langle \bar{\psi}\psi \rangle \neq 0$, for all temperatures $T > T_c$ in the sector with a negative averaged Polyakov loop $L < 0$.

As for $SU(3)$, Gattringer *et al.* [2] came to a different conclusion. They defined a new observable, the gap in the Dirac spectrum, and used it as an order parameter for the restoration of chiral symmetry. It was found that the spectral gap opens up at one single temperature $T = T_c$ in all three Z_3 sectors. Here we examine whether Stephanov's prediction for the Dirac spectrum remains valid in the case of $SU(2)$ gluodynamics in the deconfined phase.

2. Improved action

Ensembles of $O(100)$ statistically independent quenched $SU(2)$ configurations are generated with the tadpole improved Lüscher-Weisz action on $20^3 \times 6$ lattices. This action is known to suppress dislocations. The form of the action is :

$$S = \beta_{imp} \sum_{pl} S_{pl} - \frac{\beta_{imp}}{20u_0^2} \sum_{rt} S_{rt} , \quad (2.1)$$

where S_{pl} and S_{rt} denote the plaquette and 1×2 rectangular loop terms in the action, $S_{pl,rt} = \frac{1}{2} Tr(1 - U_{pl,rt})$. The factor $u_0 = (W_{1 \times 1})^{1/4}$ is the *input* tadpole factor. It is determined from $W_{1 \times 1} = \langle (1/2) Tr U_{pl} \rangle$ computed at zero temperature [3]. The deconfining phase transition occurs at $\beta_{imp} = \beta_c = 3.248(2)$ for $N_\tau = 6$, which corresponds to $T_c/\sqrt{\sigma} = 0.71(2)$ [4].

3. Massless overlap Dirac operator

The massless overlap Dirac operator has the form [5]

$$D_{ov} = \frac{\rho}{a} \left(1 + D_W / \sqrt{D_W^\dagger D_W} \right) , \quad (3.1)$$

where $D_W = M - \rho/a$ is the Wilson Dirac operator with a negative mass term, M is the Wilson hopping term, a is the lattice spacing. The optimal value of the ρ parameter is found to be 1.4 also for the lattice ensembles under investigation. Anti-periodic (periodic) boundary conditions in time (space) directions are imposed to the fermionic field.

In order to compute the sign function

$$D_W / \sqrt{D_W^\dagger D_W} = \gamma_5 \operatorname{sgn}(H_W) , \quad (3.2)$$

where $H_W = \gamma_5 D_W$ is the *hermitian* Wilson Dirac operator, we use the minmax polynomial approximation. The overlap Dirac operator constructed this way preserves the chiral symmetry even on the lattice and allows to study the properties of the Dirac modes from first principles. It will be called D in the following and replaces the continuum Dirac operator $D = D_\mu \gamma_\mu$ where $D_\mu = \partial_\mu - igA_\mu$ is the covariant partial derivative with the gauge field background A_μ .

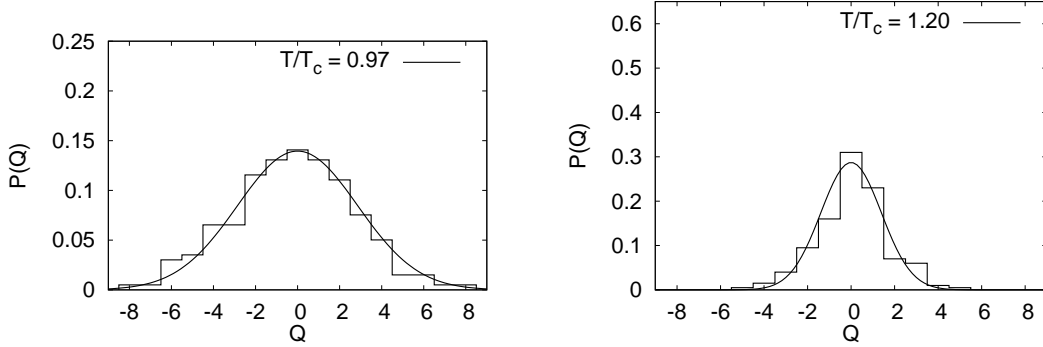


Figure 1: Probability distributions of the topological charge Q for two temperatures below and above T_c .

4. Topological susceptibility $\chi_{top}(T)$

We solved the Dirac equation numerically for its eigensystem

$$D \psi_n = \lambda_n \psi_n \quad (4.1)$$

by diagonalization of D . As a first application we search for the exact zero modes. Their number is related to the total topological charge Q_{top} of the lattice configuration through the Atiyah-Singer index theorem :

$$Q_{top} = Q_{index} = N_- - N_+, \quad (4.2)$$

where N_- and N_+ are the numbers of fermionic modes with negative and positive chirality $\psi^\dagger \gamma_5 \psi$, respectively. For the lattice ensembles the expectation value $\langle Q_{top} \rangle$ should vanish, but $\langle Q_{top}^2 \rangle$ measures the strength of global topological fluctuations. The topological susceptibility is

$$\chi_{top} \equiv \frac{\langle Q_{top}^2 \rangle}{V}, \quad (4.3)$$

where V is the four-dimensional lattice volume in physical units. In Fig. 1 (left) we see a histogram of the topological charge in the confinement phase, close to the transition. Fig. 1 (right) shows the corresponding histogram for a temperature higher up in the deconfinement phase. Both histograms can be approximately fitted by Gaussian distributions.

Let us now discuss the topological susceptibility as function of temperature. In Fig. 2 (left) we show, that the topological susceptibility χ_{top} in the negative Polyakov loop sector ($L < 0$) agrees at all T within two standard deviations with χ_{top} in the positive Polyakov loop sector ($L > 0$). In Fig. 2 (right) we compare our final data for $\chi_{top}(T)$, which are for $T < T_c$ averaged over all configurations and for $T > T_c$ only over the subsample with $L > 0$, with the results of Alles *et al.* [6]. These authors presented the values of $10^{-4} \times \chi_{top} / \Lambda_L^4$ at various values of β for Wilson's action representing different temperatures. We took $\Lambda_L = 14.15(42)$ MeV [6] and extracted their susceptibility $\chi_{top}(T)$ from these data. The topological susceptibility is slowly decreasing with increasing temperature for both sets of data. Notice that the overlap definition of Q results in a systematically higher susceptibility than the improved field theoretic definition employed by the Pisa group.

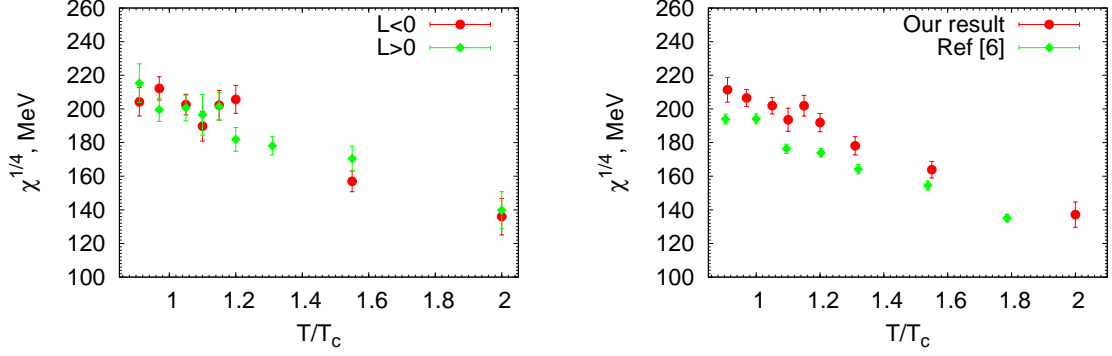


Figure 2: The topological susceptibility χ_{top} as function of T separately for $L > 0$ and $L < 0$ (left), and comparison of our final result with that of Ref. [6] (right).

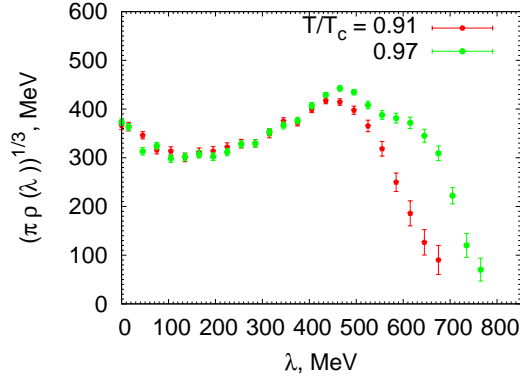


Figure 3: The spectral density of eigenmodes of the overlap Dirac operator for two temperatures $T < T_c$ on the $20^3 \times 6$ lattice.

5. Spectral density, chiral symmetry restoration and different Z_N sectors

The chiral condensate $\langle \bar{\psi}\psi \rangle$ is related to the density $\rho(\lambda)$ of the non-zero eigenvalues λ at $\lambda \rightarrow 0$ via the Banks-Casher [7] relation:

$$\langle \bar{\psi}\psi \rangle = - \lim_{\lambda \rightarrow 0} \lim_{V \rightarrow \infty} \frac{\pi\rho(\lambda)}{V}. \quad (5.1)$$

The non-zero modes are globally non-chiral, but the near-zero ones are still locally chiral and correlated with lumps of the topological charge density. The number of modes belonging to this near-zero band is proportional to the total volume V . In the chirally broken phase the required limit (5.1) of $\rho(\lambda)$ is non-vanishing at $\lambda = 0$ [7]. In the chirally symmetric phase one expects $\rho(\lambda) = 0$ in a finite region around the origin, *i.e.* that the spectrum develops a gap. For the confinement (chirally broken) phase we find indeed that the spectral density in physical units is practically constant (almost T independent), as can be seen in Fig. 3. Comparing results for configurations with $L > 0$ and $L < 0$ we found that at low λ the density $\rho(\lambda)$ is somewhat higher for negative Polyakov loop sector.

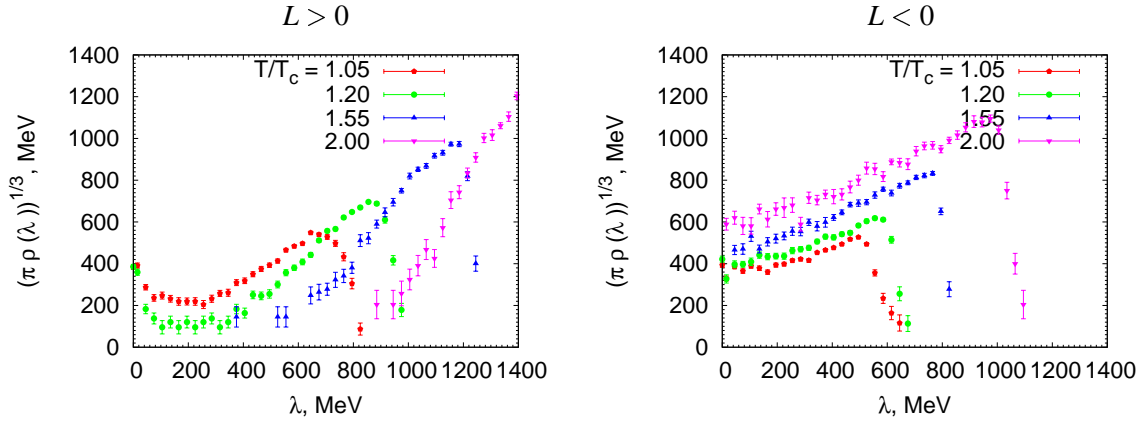


Figure 4: The spectral density of eigenmodes of the overlap Dirac operator for four temperatures $T > T_c$ on a $20^3 \times 6$ lattice, evaluated separately according to the sign of the averaged Polyakov loop.

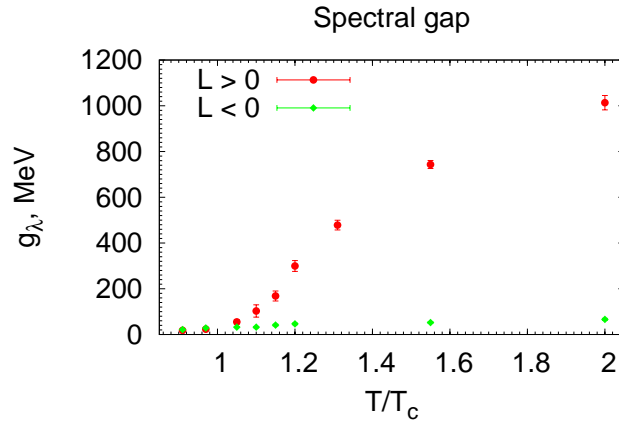


Figure 5: The spectral gap for $SU(2)$ lattice gauge theory as function of temperature, evaluated separately according to the sign of the averaged Polyakov loop.

For the deconfinement phase, when we take only configurations with an average Polyakov loop $L > 0$, Fig. 4 (left) shows that $\rho(\lambda)$ gradually decreases with increasing temperature, indicating the decrease of the chiral condensate until a gap opens and gets wider. For configurations with $L < 0$ Fig. 4 (right) shows that $\rho(\lambda)$ at low lambda is nonzero and even grows with increasing temperature.

6. Spectral gap

The spectral gap g_λ was defined by the smallest eigenvalue, which does not belong to a zero-mode. In Ref. [2] Gattringer *et al.* have shown for $SU(3)$ gluodynamics that the gap, as function of temperature, has a similar behavior for the real and both complex sectors corresponding to the phase of the averaged Polyakov loop. The phase transition occurs at the same T_c , and with increasing lattice volume the gap is decreasing. Analogously, $SU(2)$ gluodynamics has only two sectors in the deconfinement phase, distinguished by the sign of the (real-valued) averaged Polyakov loop. We show in Fig. 5 a clearly defined and rapidly growing gap for configurations with $L > 0$, whereas for

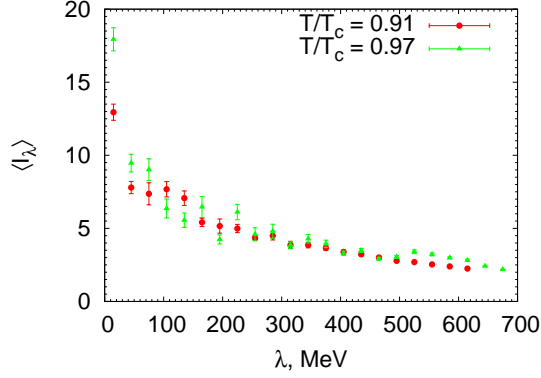


Figure 6: The average IPR within spectral bins for one temperature $T < T_c$.

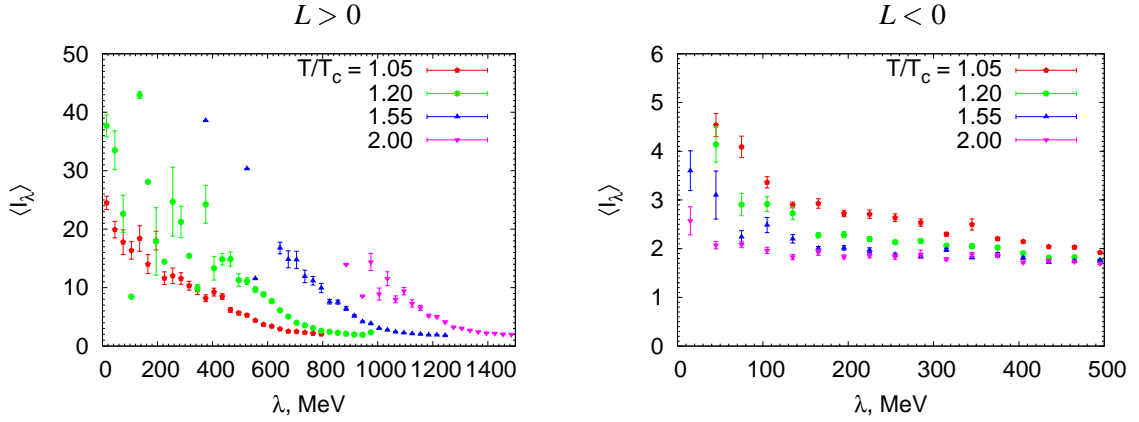


Figure 7: The average IPR within spectral bins for four temperatures $T > T_c$, separated according to the sign of the averaged Polyakov loop L .

configurations with $L < 0$ the gap remains very small up to temperatures several times higher than T_c . The small gap is a finite-volume effect and can be made to vanish in the limit of spatial $V_3 \rightarrow \infty$.

7. Localization in different parts of the spectrum

The scalar density of an eigenmode $\psi_\lambda(x)$ corresponding to an eigenvalue λ is denoted as $\rho_\lambda(x) = \psi_\lambda^\dagger(x)\psi_\lambda(x)$, such that $\sum_x \rho_\lambda(x) = 1$ by virtue of normalization. The inverse participation ratio (IPR) I_λ is the natural measure of the localization. For any finite volume V it is defined by

$$I_\lambda = V \sum_x \rho_\lambda^2(x). \quad (7.1)$$

The IPR characterizes the inverse volume fraction of sites forming the support of $\rho_\lambda(x)$. From Fig. 6 we conclude that for the temperature close but below T_c near but below T_c the IPR (localization) monotonously increases with decreasing eigenvalue. There is no clear mobility edge.¹ The monotony is not perfect among the lowest one or two bins. Thus, out of the low lying modes, the higher ones are continuously less localized. We found that at this temperature for configurations

¹For $T = 0$ the localization of overlap eigenmodes has been investigated in Refs. [8, 9].

with negative Polyakov loop, $L < 0$, the modes are somewhat less localized. In Fig. 7 we show that with increasing temperature the average IPR within the respective eigenvalue bins is increasing. The effect sets in for higher and higher eigenvalues corresponding to the mobility edge moving outward in the deconfinement phase with the gap for $L > 0$. In the negative Polyakov loop sector the IPR is constant at a low level, except for $\lambda < 500$ MeV, where the tendency of the IPR to grow exists but is very weak.

8. Summary

We performed first measurements of the topological susceptibility with the help of the overlap Dirac operator in finite temperature $SU(2)$ gluodynamics. We found that the topological susceptibility in the confinement phase is almost constant and is slowly decreasing in the deconfinement phase, in agreement with previous results [6]. We did not find systematic effects of the sign of the averaged Polyakov loop on the topological susceptibility. The chiral condensate, however, behaves completely different in the $L < 0$ sector. Chiral symmetry remains broken, the spectral gap stays close to zero for all $T > T_c$ in agreement with Stephanov's model predictions. A microscopic explanation in terms of the interplay of holonomy and topology [4] needs to be worked out. This difference is accompanied by a different localization behavior of the lowest fermionic eigenmodes in the two sectors.

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