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The Kaon B-Parameter from 2+1-Flavor Domain-Wall Fermion Lattices

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We present the final results of the RBC/UKQCD calculation of the kaon B-parameter on 2+1-flavor domain-wall fermion lattices at $a^{-1} = 1.73(3)$ GeV. We simulate on two lattice volumes of about $(1.8 \text{ fm})^3$ and $(2.7 \text{ fm})^3$, with the lightest valence pion about on the large volume approximately 250 MeV. The light pion masses and our chiral fermion action allow us to compare lattice data to NLO chiral perturbation theory, facilitating a controlled extrapolation to the physical point. We present a final result including nonperturbative renormalization and detailed systematic errors. Our final result is $B_K^{\overline{\text{MS}}}(2 \text{ GeV}) = 0.524(10)(28)$.

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1. Introduction

Lattice QCD continues to forge links between the first principles of quantum chromodynamics and the experimental measurement of hadronic observables. For example, when coupled with the experimentally measured indirect *CP*-violation ε parameter, the kaon bag parameter B_K helps to constrain the apex of the unitarity triangle.

The kaon B-parameter is an ideal setting for lattice QCD to provide input to experimental measurements that otherwise would not be able to constrain the Standard Model. Although measurements of the indirect ε parameter have been made since the 1960's, the connection between this parameter and the fundamental parameters of the CKM quark mixing matrix requires calculation of a weak matrix element between hadronic states proportional to B_K . Such matrix elements strongly depend on low-energy nonperturbative QCD effects, and so are inaccessible to perturbation theory. Although phenomenological models estimate B_K at the 10% level, a first-principles QCD calculation on the lattice should be able to do much better.

For a broad class of observables such as the weak matrix elements (including the ε parameter), it is necessary to use lattice simulations that retain chiral symmetry, which is broken by the most economical lattice actions. The domain-wall fermion lattices produced by the RBC and UKQCD collaborations are ideal for the study of weak matrix elements like the kaon mixing matrix element (proportional to the kaon B-parameter B_K). Several publications have previously addressed calculations of B_K using domain wall fermion lattices[1, 2, 3].

Once a calculation is made using lattice QCD it is important that the systematic errors are correctly calculated or estimated, in order to present a phenomenologically relevant number to the wider community. Lattice calculations are made on a femto-torus, with a finite lattice spacing and with quarks that are heavier than their real-world counterparts. Such limitations introduce systematic errors: finite volume effects, discretization errors and chiral extrapolation errors, respectively. Systematic errors also arise from the process of renormalization. It is therefore important that lattice calculations are made with good control over systematic and statistical errors. Domain-wall fermions are attractive in this sense because they have exact vector symmetry and mildly broken chiral symmetry. This leads to $O(a)$ improvement of the discretisation errors and a dramatic simplification of the renormalization procedure due to the fact that chiral symmetry is approximately satisfied [2].

In this proceedings, we will describe a calculation at inverse lattice-spacing $a^{-1} = 1.73(3)$ GeV to determine B_K using-domain wall fermions. The calculation is also described in [4].

In Section 2 we describe the simulation and measurement parameters, and give the approximate range of pseudoscalar masses in our calculation. We discuss the setting of the lattice scale and present the values for the physical light and strange quark masses. These were calculated in [5] and [6] on the 16^3 and 24^3 volumes respectively.

In Section 3 we present the numerical results from the calculation of the bare pseudoscalar B-parameter, B_P.

We discuss the extrapolation of the lattice results to the physical point in Section 4. We present the extrapolations of each volume separately, reflecting the chronological order. In each case we compare the lattice data to functional forms from next-to-leading order (NLO) chiral perturbation theory (XPT). For the larger volume we also consider next-to-NLO (NNLO) forms. In both cases we examine whether NLO XPT is applicable at the kaon scale.

In Section 5 we discuss the systematic errors in our calculation and finally present a renormalized value for $B_K^{\overline{\mathrm{MS}}}(2\ \mathrm{GeV})$.

2. Lattices

We simulated domain-wall fermions on lattice volumes of $16^3 \times 32$ and $24^3 \times 64$ with the Iwasaki gauge action and bare gauge coupling $\beta = 2.13$. The extent of the fifth dimension was $L_s = 16$ and the domain wall height was fixed to $aM_5 = 1.8$. We have three ensembles on the small volume with input light quark mass $am_l \in \{0.01, 0.02, 0.03\}$ and input strange quark mass $am_s = 0.04$, approximately its physical value; these ensembles are more fully described in [5]. On the larger volume, we use bare quark masses $am_s = 0.04$ and $am_l \in \{0.005, 0.01\}$. Table 1 shows the approximate unitary pseudoscalar masses in physical units for the two ensembles. We select for analysis configurations separated by 20 MD time units on the 16^3 and 40 MD time units on the 24^3 simulations. This means that we measure weak matrix elements for each light sea quark mass on 150 (small-volume) or 90 (large-volume) lattices. We bin the data using up to 80 trajectories per bin to reduce the correlations between our samples. This leaves ~ 50 measurements for the standard jackknife analysis.

		$16^3 \times 32$		$24^3 \times 64$	
am_l	am_s	M_π	M_K	M_π	M_K
0.005	0.04	–	–	330	580
0.01	0.04	420	610	420	600
0.02	0.04	560	670	–	–
0.03	0.04	670	710	–	–

Table 1: Approximate physical unitary pseudoscalar masses (in MeV) on the three 16^3 and two 24^3 ensembles. Calculated using lattice spacing $a^{-1} = 1.73$ GeV. The two sea quarks are m_l and m_s.

Each data set includes two- and three-point correlators from the all nondegenerate combinations of valence quark masses $am_{x,y} \in \{0.01, 0.02, 0.03, 0.04, 0.05\}$ on the small volume and $am_{x,y} \in \{0.001, 0.005, 0.01, 0.02, 0.03, 0.04\}$ on the large volume. Zero-momentum kaon states are created and annihilated using Coulomb-gauge–fixed wall sources located at $t = 5$ (both volumes) and $t = 27$ (small) or $t = 59$ (large) with the operator inserted at all intervening points, such that the entire volume is sampled. The available time-length on the lattice is doubled by summing the propagators found using periodic and antiperiodic boundary conditions at the t-boundary.

The extent of chiral symmetry breaking in domain-wall fermion simulations is quantified by the residual mass, am_{res}, which is an additive quark mass renormalization. The residual mass is determined from the pseudoscalar correlator having a sink at the midpoint of the fifth dimension, which agrees with the (negative) quark mass at which the pseudoscalar mass extrapolates to zero. We find $am_{\mathrm{res}} = 0.00315(2) \approx 5$ MeV, which is quite small compared to the strange quark mass.

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We use data from the large volume to set the scale. We use the mass of the Ω^- baryon, linearly extrapolated to $m_l = (m_u + m_d)/2$, and M_K and M_π treated in SU(2) $\times$ SU(2) chiral perturbation theory to determine $a^{-1} = 1.73(3)$ GeV [6]. For the physical quark masses we find

$$am_l^{\text{phys}} = 0.001300(58) \qquad \text{and} \qquad am_s^{\text{phys}} = 0.0375(16). \tag{2.1}$$

The physical strange quark mass turns out to miss our sea input $(0.04 + am_{\text{res}}) = 0.0432$ by around 15%; since the kaon bag parameter is quite sensitive to the valence strange mass, we will need to interpolate between $am_y = 0.03$ and 0.04 determine the value at the physical point.

3. Numerical Results

The kaon B-parameter is defined by the ratio of the kaon-mixing matrix element to its vacuum saturation value:

$$B_K = \frac{\langle \overline{K^0} | \mathscr{O}_{LL}^{\Delta S=2} | K^0 \rangle}{\frac{8}{3} f_K^2 M_K^2}, \tag{3.1}$$

where M_K is the mass of the neutral kaon, f_K is the decay constant of the kaon (given by its coupling to the axial current), and $O_{LL}^{\Delta S=2} = (\bar{s}(1-\gamma_5)\gamma_\mu d)(\bar{s}(1-\gamma_5)\gamma_\mu d)$ is a four-quark operator coupling to left-handed quarks that changes strangeness by 2.

We determine the pseudoscalar B-parameter by a clever ratio of pseudoscalar-axial wall-point correlators with the figure-eight diagram sandwiching $O_{LL}^{\Delta S=2}$ between operators to create a kaon and annihilate an anti-kaon:

$$B_P^{\text{bare}} = \frac{2V}{\frac{8}{3}} \frac{\mathscr{C}_{wpw}^{P\mathscr{O}P}(t_{\text{src}}, t, t_{\text{snk}})}{\mathscr{C}_{wp}^{PA^4}(t_{\text{src}}, t)\mathscr{C}_{pw}^{A^4P}(t, t_{\text{snk}})}, \tag{3.2}$$

where V is volume, and the $\mathscr{C}$'s are correlators with superscripts denoting the source, insertion (for three-point correlators) and sink operators (one of P pseudoscalar, A^4 time-component axial or the $\mathscr{O}_{VV+AA}^{\Delta S=2}$ mixing operator in this case) and subscripts denoting source, insertion and sink shapes (p point or w wall). For each three-point correlator, the point insertion is summed over all space in its timeslice. This formulation has advantages over a more naive determination of the matrix element alone, since it avoids introduction of noisy wall-wall correlators and the axial renormalization Z_A.

Thanks to the large time extent of our lattices and the use of periodic-plus-antiperiodic propagators, the plateau for B_K is very long. We fit to a constant over the range $t \in [12, 22]$ (small lattices) and $t \in [12, 52]$ (large). A sample of data from each volume can be seen in Figure 1. The fitted values of bare B_P for all combinations of quark masses on both volumes are given in Table 2. Note that the length of the plateau on the 24^3 ensembles is long enough that correlated fits are unreliable so uncorrelated fits are used.

4. Chiral Extrapolations

Since the cost of making calculations directly at the physical up and down quark masses is beyond the present capabilities of computers devoted to lattice QCD, it is necessary to extrapolate from higher masses. The most commonly used analytical tool for this purpose is chiral perturbation

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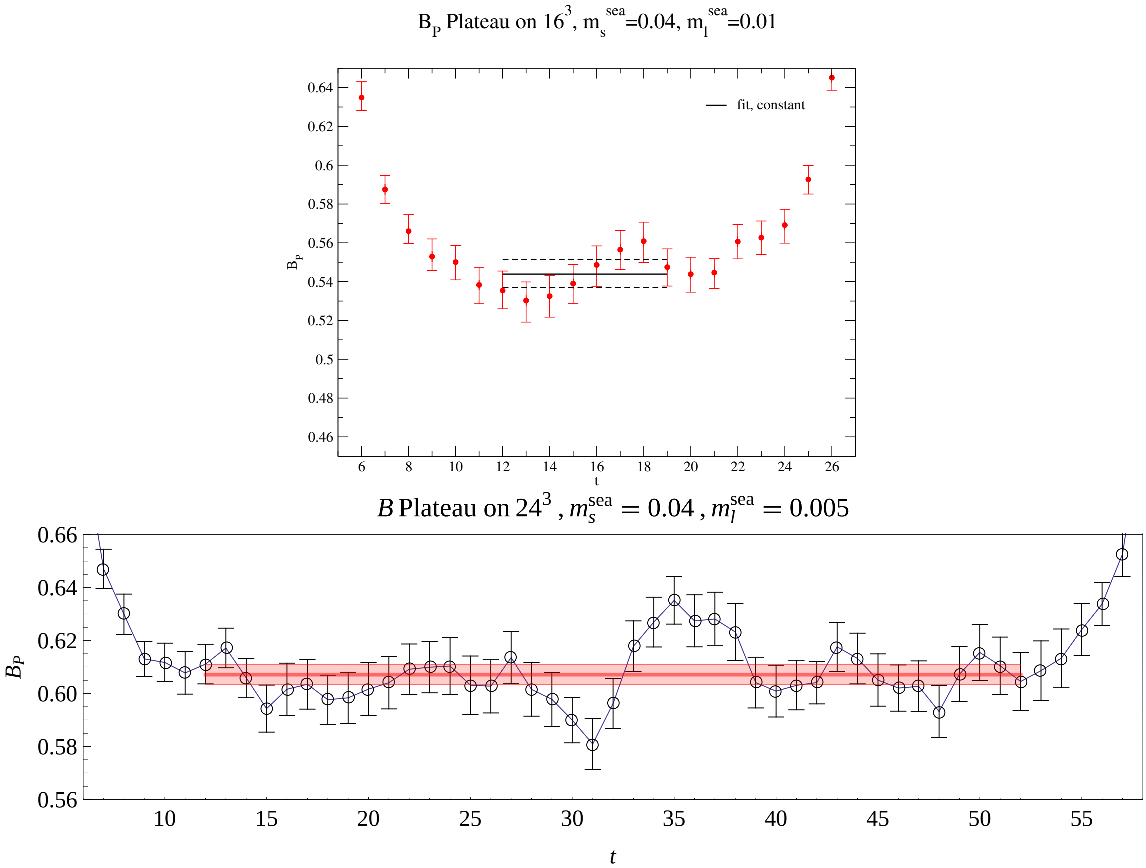

Figure 1: Above: An example of the B_P ratio on the 16^3 $am_l = 0.01$ ensemble with $am_x = am_y = 0.01$; correlated $\chi^2/\mathrm{dof} = 1.37$ for this fit. **Below:** Plateau of B_P for valence masses equal to sea masses on 24^3 $am_s = 0.04, am_l = 0.01$, using an uncorrelated fit. Fitting range, value and uncertainty are indicated by the red band.

theory, a low-energy effective field theory which employs the chiral symmetry of QCD to make predictions about the dependence of observables on quark masses and momenta. However, since chiral symmetry is broken by the quark masses, one expects XPT to work well only at light masses. We need to determine whether it works well enough at the kaon mass, to be useful for extrapolating B_K.

Since domain-wall fermions have good chiral symmetry, we can use the next-to-leading order SU(3) $\times$ SU(3) partially quenched (PQ) 2+1-flavor continuum form given by [7]:

$$\frac{B_K}{B_0} = 1 + \frac{1}{48\pi^2 f^2 M_K^2}\left[I_{\mathrm{conn}} + I_{\mathrm{disc}} + bM_K^4 + c(M_X^2 - M_Y^2)^2 + dM_K^2(2M_D^2 + M_S^2)\right], \tag{4.1}$$

where m_Q is the mass of the pseudoscalar meson with the indicated composition, and the I's are known chiral logs. The function itself has six free parameters: μ and f_0 (the usual leading order dimensionful constants from chiral perturbation theory), B_0 (the chiral limit of the bag parameter), and dimensionless constants arising at NLO b, c and d. We fix the values of μ and f_0 to the values obtained in [6]: $B_0 = 2.35(16)$ and $f_0 = 0.0541(40)$ (large volume). To use the continuum form

		B_P				
		$16^3 \times 32$			$24^3 \times 64$	
am_y	am_x	$am_l = 0.01$	$am_l = 0.02$	$am_l = 0.03$	$am_l = 0.005$	$am_l = 0.01$
0.001	0.001	–	–	–	0.470(7)	0.470(5)
0.005	0.001	–	–	–	0.491(6)	0.495(4)
0.005	0.005	–	–	–	0.507(4)	0.511(3)
0.01	0.001	–	–	–	0.513(5)	0.521(4)
0.01	0.005	–	–	–	0.526(4)	0.5307(28)
0.01	0.01	0.546(8)	0.539(8)	0.527(7)	0.542(3)	0.5456(25)
0.02	0.001	–	–	–	0.549(5)	0.559(4)
0.02	0.005	–	–	–	0.558(3)	0.5626(26)
0.02	0.01	0.577(6)	0.569(6)	0.556(6)	0.5697(26)	0.5729(23)
0.02	0.02	0.598(5)	0.589(5)	0.580(5)	0.5915(24)	0.5941(21)
0.03	0.001	–	–	–	0.577(5)	0.588(5)
0.03	0.005	–	–	–	0.585(3)	0.5883(28)
0.03	0.01	0.600(6)	0.594(6)	0.579(5)	0.5937(25)	0.5961(23)
0.03	0.02	0.617(4)	0.609(5)	0.600(4)	0.6111(23)	0.6132(20)
0.03	0.03	0.633(4)	0.626(4)	0.618(3)	0.6274(22)	0.6291(18)
0.04	0.001	–	–	–	0.601(7)	0.612(7)
0.04	0.005	–	–	–	0.607(4)	0.610(3)
0.04	0.01	0.620(5)	0.616(6)	0.599(5)	0.6143(27)	0.6163(25)
0.04	0.02	0.633(4)	0.627(4)	0.618(4)	0.6286(23)	0.6304(20)
0.04	0.03	0.647(4)	0.641(4)	0.634(3)	0.6423(21)	0.6438(17)
0.04	0.04	0.659(3)	0.655(3)	0.648(3)	0.6553(20)	0.6567(16)
0.05	0.01	0.636(5)	0.634(6)	0.616(4)	–	–
0.05	0.02	0.648(4)	0.643(4)	0.634(3)	–	–
0.05	0.03	0.660(3)	0.655(3)	0.648(3)	–	–
0.05	0.04	0.671(3)	0.667(3)	0.661(3)	–	–
0.05	0.05	0.682(3)	0.679(3)	0.673(3)	–	–

Table 2: Pseudoscalar bag parameter with valence quark masses $am_{x,y}$ and sea quark masses $am_{l,s}$. For all cases, $am_s = 0.04$.

with domain-wall fermions, we shift each quark mass by $m_{\rm res}$. The resulting chiral form should be accurate up to $O(a)$ uncertainties in $m_{\rm res}$, which are not included, although they could be modeled by NLO terms in the chiral expansion. Using the calculated fit parameters and the physical quark masses we can extrapolate to the physical point $m_x = m_l = m_l^{\rm phys}$, $m_y = m_s = m_s^{\rm phys}$ to yield the unrenormalized B_K. In the following section we discuss the extrapolations in chronological order; first the small volume and then the large volume.

4.1 $16^3 \times 32$ Extrapolation

We extrapolate to the physical point using the SU(3) form Eq. 4.1. We do not expect that all of the data points lie within the region of validity of XPT, so we apply a cut in the valence quark masses such that $(m_x + m_y)/2 \leq m_{\rm cut}$ and also try the fit with two and three ensembles included. Figure 2 shows the variation in B_K at the physical point when these cuts are applied. Changing the number of ensembles included in the fit does not alter the value, but increases the error bar. The value seems to show insensitivity below $m_{\rm cut} = 0.03$. Assuming we are in the chiral regime, this is exactly what one might expect: points above a certain mass threshold are not well described by the NLO XPT function and when they are omitted the form fits the data. Figure 3 shows an example of the NLO SU(3) fit on the $am_l = 0.01$ ensemble. The curves fit the data that survives the cut reasonably well, although they miss the lightest point which may be an indication that we are not yet in the chiral regime.

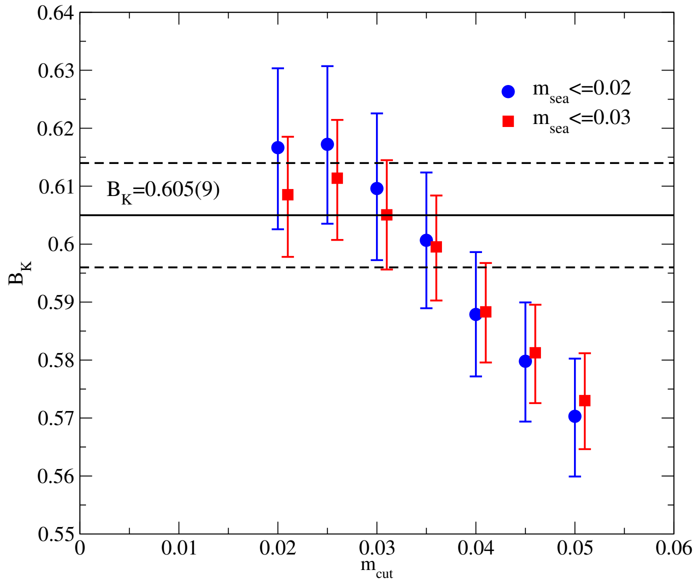

Figure 2: Variation of B_K at the physical point $m_x = m_l = m_l^{\rm phys}$, $m_y = m_s = m_s^{\rm phys}$ with the cut in valence quark mass $m_{\rm cut} = (m_x + m_y)/2$. The blue (circle) and red (square) points are obtained from fits to three and two ensembles respectively.

Given the insensitivity to the valence and sea quark mass cuts we quote a value from the fit using $m_{\rm cut} = 0.03$:

$$B_K^{\rm NLO} = 0.605(9). \tag{4.2}$$

However, a linear fit to the three unitary points gives a comparable value at the physical point:

$$B_K^{\rm linear} = 0.611(8). \tag{4.3}$$

The fact that the chiral and linear extrapolations agree so well, Figure 4, indicates that the NLO fit is not working correctly. We know that in the chiral limit the B_P XPT fitform has a chiral logarithm in M_π of size [8]

$$\frac{1}{2}\frac{M_\pi^2}{(4\pi f)^2}\log\frac{M_\pi^2}{\Lambda_\chi^2}, \tag{4.4}$$

which will make a true chiral extrapolation differ from a linear extrapolation, especially given our data is quite far from the physical point. There will be other log terms in M_K and M_η but these

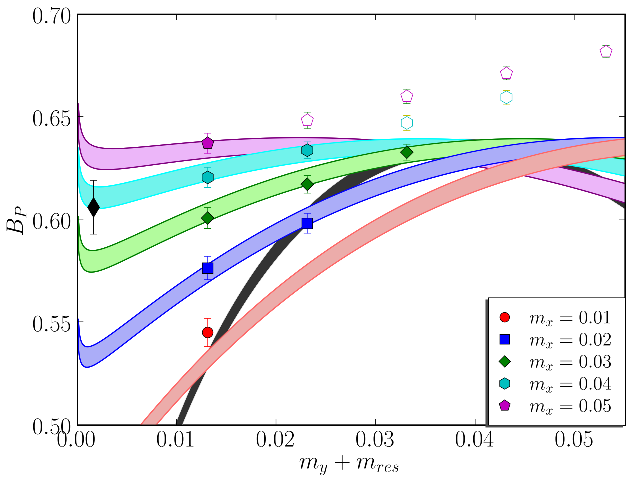

Figure 3: Continuum NLO SU(3) XPT fit applying $m_{\rm cut} = 0.03$. The coloured bands correspond to different values for the valence quark m_x and the grey line is the unitary curve. The fit is to filled points only. The black diamond is the value for B_K at the physical point.

will not affect the deviation from linear so much as they don't become massless in the light quark chiral limit. If we knew that we were in a region where NLO SU(3) XPT could be reasonably

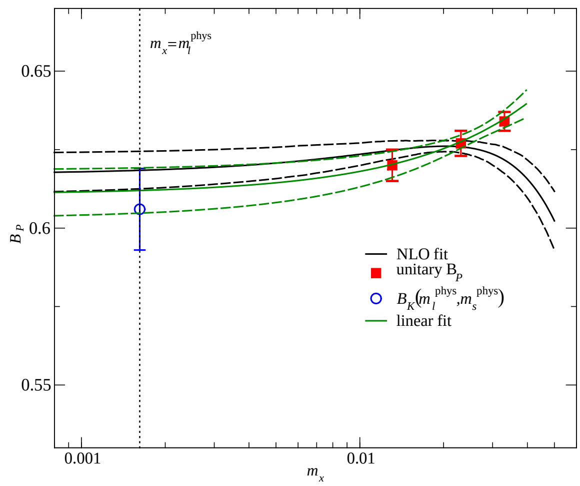

Figure 4: The NLO SU(3) XPT fit agrees well with a linear fit to the unitary points. The black line is the form obtained from fitting all three ensembles to the NLO SU(3) form with $m_{\rm cut} = 0.03$ and the green line from a linear fit to the three unitary points.

applied there would be negligible uncertainty in taking the chiral limit using the method outlined above. However, given the agreement between XPT and the linear fit, coupled with the fact that SU(3) NLO PQXPT did not fit the pseudoscalar data [5] it would appear that the SU(3) NLO fit is simply a smooth interpolating function. We must estimate the error introduced by this unceratin chiral limit. We do this by appealing to NLO SU(2) XPT where we no longer consider the kaon to be a Goldstone boson. Appealing to the XPT form for heavy-light mesons we can derive an SU(2) chiral form in the $M_K \gg M_\pi$ limit [9]

$$B_K = B_0\left[1 - \frac{M_\pi^2}{2(4\pi f)^2}\log\left(\frac{M_\pi^2}{\Lambda_\chi^2}\right) + c_0\frac{M_\pi^2}{\Lambda_\chi^2}\right]. \tag{4.5}$$

Eq. 4.5 gives an exact description of the non-analyticity in M_π for scales $M_\pi \ll M_K$ so it should provide a good estimate of what happens when we extrapolate into a region that is presently inaccessible to us. It should also be stressed that for small $m_\pi, p \ll m_K$ the SU(3) form previously used will tend towards Eq. 4.5.

To estimate the chiral extrapolation error we match the linear fit of the unitary points to Eq. 4.5 at some matching point $m_q^{\rm min}$. This corresponds to the region where XPT is applicable. By varying $m_q^{\rm min}$ we adjust our chiral extrapolation estimate e.g. if $m_q^{\rm min} = 0$ then the linear extrapolation is correct or if $m_q^{\rm min} = 0.01$ our data lies just outside the region where XPT is applicable. We calculate the chiral extrapolation error by measuring the deviation between the linear fit and the matched curve at the physical point, see Figure 5. To be maximally pessimistic we push $m_q^{\rm min}$ as high as possible whilst remaining consistent with the data. This corresponds to $m_q^{\rm min} = 0.02$ and a chiral extrapolation error of 4%. We therefore quote the unrenormalized B_K in the chiral limit

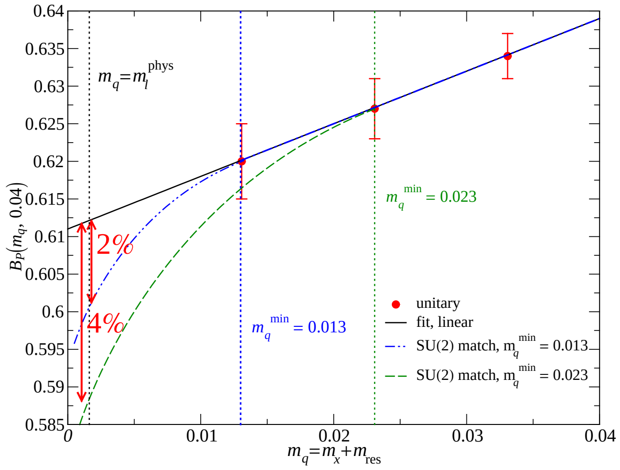

Figure 5: To estimate the chiral extrapolation error we measure the deviation between a linear fit to the unitary points and an SU(2) curve at the physical point. The error depends on where the matching is performed: this is where we expect XPT to "turn on". Using $m_q^{\rm min} = 0.01$ (blue curve) the deviation is ≈2% and using $m_q^{\rm min} = 0.02$ (green curve) the deviation is ≈4%.

$$B_K^{16^3} = 0.605(9)(24), \tag{4.6}$$

where the first error is statistical and the second is the estimated chiral extrapolation error. XPT does not appear to be trustworthy at energy scales close to the kaon mass. This calls into question the applicability of SU(3) $\times$ SU(3) XPT and motivates the use of SU(2) $\times$ SU(2) XPT, where we no longer consider the kaon to be a Goldstone boson. We conclude from this calculation on the small volume that simulations with lighter quark masses are required to reliably extrapolate to the physical point. This is addressed in the $24^3 \times 64$ simulation.

4.2 $24^3 \times 64$ Chiral Extrapolation

Measurements on the large-volume ensembles have dynamical quarks and valence quarks as light as $1/5$ and $1/10$ of the dynamical strange, respectively (including the necessary addition of $m_{\rm res}$). This should allow for better overlap with the region of validity of NLO PQXPT than on the small-volume calculation. We begin by attempting to extrapolate the date using the SU(3) from in Eq. 4.1. As before we employ a valence mass cut $m_{\rm cut} = 0.03$, a mass cutoff which compromises with the need to use light masses in XPT and the need to include the physical kaon mass. We perform an uncorrelated, least-squares fit, giving the result shown in Figure 6 and the extrapolated

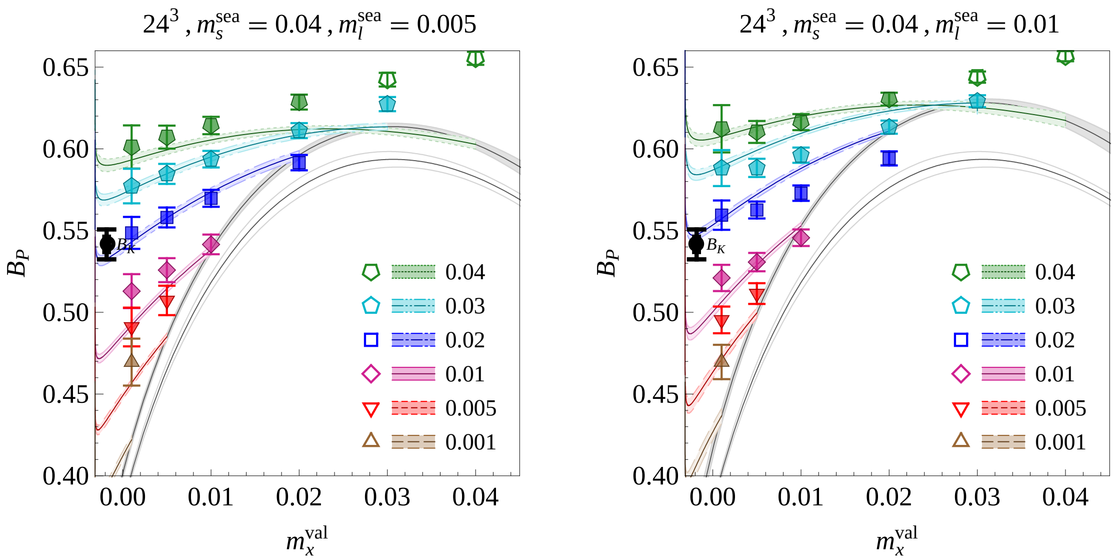

Figure 6: Pseudoscalar bag parameter as a function of light valence quark mass (where the chiral limit is at $m = -m_{\rm res}$). Each colored band indicates a different strange valence quark mass: brown $= 0.001$ through green $= 0.04$. The black point marks the extrapolation to physical B_K.

value $B_K^{\rm NLO} = 0.556(8)$. Although we expect the XPT form to work best at low masses, we see in the figure that the best-fit form does not agree with the data in that region. We therefore conclude that NLO XPT is probably unsuitable at the kaon mass. It is still useful for determining the low-energy constant B_0, which we extract by fitting to a much more restricted data set, $am_{\rm av} \leq 0.01$; this yields $B_0 = 0.30(3)$. Without resorting to chiral perturbation theory, we might hope to be able to simply extrapolate using a linear ansatz including only the unitary points. Of course, due to the mismatch of the physical strange mass, we need to include also the non-unitary $am_y = 0.03$ points. Such a simple fit gives $B_K^{\rm linear} = 0.582(10)$. However, the linear ansatz is probably not well justified, since at low light-quark mass, there is a known chiral logarithm.

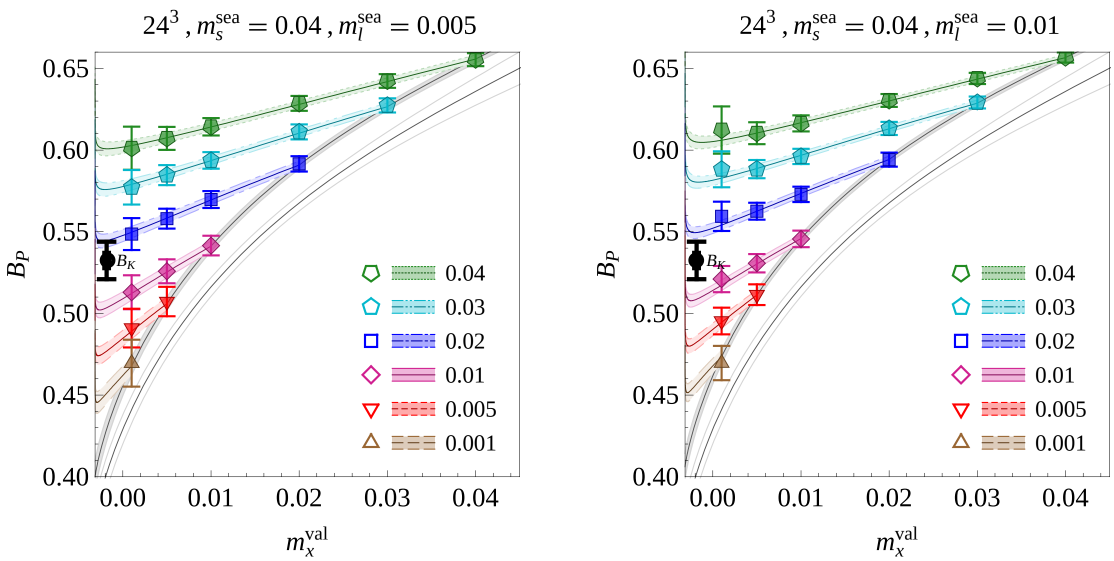

Figure 7: As Figure 3, but using a fit form including analytic NNLO terms from chiral perturbation theory.

Another alternative is to attempt to go to higher orders of chiral perturbation theory; if next-to-leading order is not sufficient at the kaon mass, NNLO might succeed. However, given that NLO terms appear to be large at the kaon mass, one might expect NNLO terms to also be large. Unfortunately, such a calculation is very difficult and does not appear in the literature. However, the analytic terms are known:

$$\begin{aligned}\frac{B_K}{B_0} &= 1 + \frac{1}{48\pi^2 f^2 M_K^2}\big[I_{\rm conn} + I_{\rm disc} + bM_K^4 + c(M_X^2 - M_Y^2)^2 + dM_K^2(2M_D^2 + M_S^2) \\ &\quad + n_1 M_K^6 + n_2 M_K^4(2M_L^2 + M_S^2) + n_3 M_K^2(2M_L^2 + M_S^2)^2\big]\,. \end{aligned} \tag{4.7}$$

The introduction of these three new parameters n allows us to fit the entire range of data, as seen in Figure 7; this gives $B_K^{\rm NNLO} = 0.552(10)$. However, the introduction of only particular known terms at higher orders may be seen as somewhat arbitrary, and the number of parameters needed nearly doubles.

The failure of SU(3) fits at valence quark masses $m_y \approx m_s$ again motivates the use of an SU(2) form. In contrast to the small volume where we *matched* to an SU(2) form we now have data light enough that we can perform a *fit* using Eq. 4.5. We extrapolate the B-parameter to the physical value of the light quark mass at a fixed value of the strange quark mass. We do not have measurements at valence quark masses corresponding to the physical strange quark mass, so we perform the extrapolation with valence strange masses $am_y \in \{0.03, 0.04\}$, and then interpolate to the physical strange mass. We find that a cut in light valence mass $m_{\rm cut} = 0.01$ leads to a good fit, in agreement with similar considerations for the extrapolations of M_K and f_K on the same ensembles [6]. The two extrapolations for $m_y \in \{0.03, 0.04\}$ are shown in Figure 8. Interpolating to the physical strange mass we find

$$B_K^{\rm SU(2)} = 0.565(10). \tag{4.8}$$

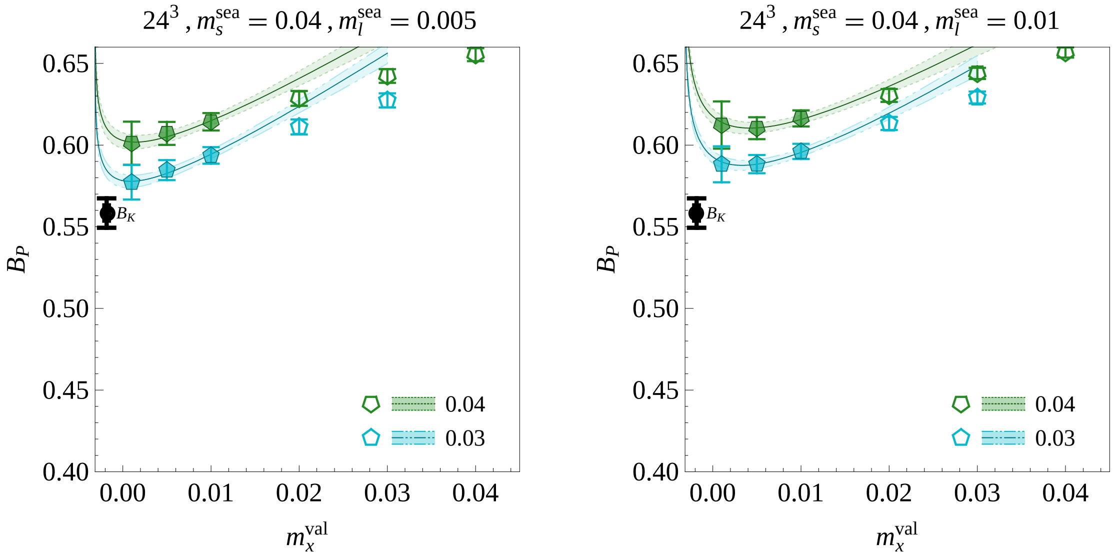

Figure 8: As Figure 6, but using a fit form using NLO SU(2) partially quenched chiral perturbation theory and treating the strange quark as heavy.

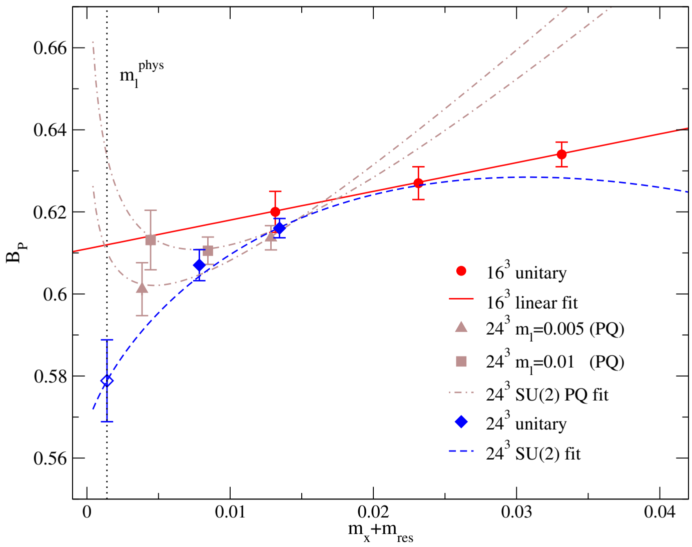

Figure 9: The partially quenched $SU(2) \times SU(2)$ chiral extrapolation of the $24^4 \times 64$ data with $am_y = 0.04$. Also shown is the $16^3 \times 32$ unitary data. The two volumes agree and significant deviation from the 16^3 linear extrapolation can be seen, indicating chiral curvature.

Figure (9) shows a comparison between the extrapolations of the two volumes.

Comparing the extrapolated values from the linear fit and the SU(2) fit we see that the SU(2) result is approximately 3% below the linear result. The difference indicates that we are seeing significant chiral curvature; evidence that we are in a region where XPT can be reliably applied. The SU(2) fit fits well in the region where light quark mass is less than about 0.013 in lattice units and is theoretically sound: we adopt this method for the extrapolation of B_P to the physical point to quote our best value.

5. Systematic Errors

One systematic error enters due to our use of SU(2) XPT, assuming that the kaon is heavy. The analysis is applicable whenever $M_\pi \ll M_K$, whether or not the kaon is heavy or light compared to other scales, and we do not rely on chiral perturbation theory being convergent at kaon masses. The low-energy constants (LECs) will be strange-mass dependent, and since the kaon is somewhat lighter than a typical chiral scale, the convergence of the chiral expansion, controlled by these LECs, may be correspondingly impacted. This will merely reflect the new dynamics that enter at kaon mass scale, and an reasonable estimate suggests a reduced suppression of m_l/m_s relative to NLO may apply to NNLO analytic terms. The NLO correction is of order 6%, so we estimate NNLO contamination as $O(\leq 2\%)$ for our mass cuts.

The 16^3 data shown in Figure 9 is well described by a straight line, and the linear extrapolation is larger than the 24^3 SU(2) extrapolation by approximately 6%. We estimate the extrapolation error as this 6% difference scaled by m_l/m_s for $am_l = 0.01$: 2%. This is an estimate of the size of NNLO terms.

The input dynamical strange quark mass is 15% larger than the measured physical strange quark mass. By examining the change in the B-parameter as the light dynamical quark mass is increased we estimate this to contribute a 1% error.

We can measure the finite volume effects (FVE) by comparing the bare B_P values on the $am_l = 0.01$ ensembles on both volumes. Examining the numbers in Table 2 we see that there are no significant differences beyond statistical errors between the two volumes. Finite volume XPT suggests that FVE are negligible [11] for all masses and volumes in our simulation except for $am_x = 0.001$ on the $am_l = 0.01$ ensemble. However, removing this point from the extrapolation has no significant effect. We estimate the error from FVE from the difference between the B-parameters on the two volumes: 1%.

We estimate the continuum extrapolation error by appealing to the quenched CP-PACS calculation, also done using domain wall fermions and the Iwasaki gauge action [12]. This suggests a scaling error of 3.5% for our slightly coarser lattice spacing and we choose 4% as the most likely estimate for $O(a^2)$ scaling errors. This is in agreement with the 4% difference we see between our calculate decay constants and the experimental values [13, 6].

Including all systematic errors we calculate the unrenormalized kaon B-parameter

$$B_K = 0.565(10)_{\mathrm{stat}}(06)_{\mathrm{FVE}}(11)_{\mathrm{Ch}}(06)_{m_s}(23)_{\mathrm{scale}}, \tag{5.1}$$

where the errors are due to statistics, finite-volume effects, chiral extrapolation, determination of the physical strange quark mass and scaling to the continuum limit, respectively.

6. Renormalization

In order to compare our B_K result, we must determine the renormalization of its operator and the matching to $\overline{\mathrm{MS}}$ scheme. We use nonperturbative renormalization in the RI-MOM scheme, for which domain-wall fermions are well suited; see Ref. [1] for a full description of the method. We find $Z_{B_K}^{\overline{\rm MS}} = 0.928(05)(23)$ [10] where the first error is statistical and the second is a systematic error arising from discretization errors [13]. Combining all the systematic errors in quadrature we quote for our final number

$$B_K^{\overline{\rm MS}}(2\ \mathrm{GeV}) = 0.524(10)(28), \tag{6.1}$$

where the first error is statistical and the second is the estimated systematic uncertainty.

7. Conclusions

Despite the computational cost of domain-wall fermion simulations, we have simulated at light enough dynamical and valence quark masses such that we can make use of NLO XPT to guide extrapolations to the physical point. We find that NLO SU(3) XPT cannot be used up to the kaon mass, and extrapolations in the two light dynamical quarks is much more reliable. Our value for B_K removes the quenching systematic completely and correctly includes the dynamical effects of 2+1 quark flavours. Forthcoming simulations at finer lattice spacings will improve this result and allow better estimates of the systematic errors.

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