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Semileptonic D-decays and Lattice QCD

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We explore four different strategies to extract the D-meson semileptonic decay form factors from the Green functions computed in QCD numerically on the lattice. From our numerical tests we find that two such strategies, based on the use of double ratios of 3-point correlation functions, lead to an appreciable reduction of systematic uncertainties. This is an important step in reducing the overall uncertainty in the lattice QCD results for the D-decay form factors, which are needed to determine the CKM entries $|V_{cd}|$ and $|V_{cs}|$ experimentally, and thus to check the CKM unitarity.

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1. Introduction

An accurate determination of the Cabibbo–Kobayashi–Maskawa (CKM) matrix elements, V_{ij}, is an essential step in testing the Standard Model (SM). Like the quark masses, the couplings V_{ij} are free parameters of the SM and therefore cannot be predicted. Instead they are extracted after confronting the experimental measurements to the SM theoretical expressions. The simplest processes in that respect are the leptonic and semileptonic decays of pseudoscalar mesons. In this note we consider D decays, namely

$$\begin{aligned}\frac{d\Gamma}{dq^2}(D_q \to P_{qq}\ell\nu_\ell) &= |V_{cq}|^2 \frac{G_F^2}{192\pi^2 m_{D_q}^3}\lambda^{3/2}(q^2)|F_+(q^2)|^2\,,\\ \Gamma(D_q^+ \to \ell\nu_\ell) &= |V_{cq}|^2 \frac{G_F^2}{8\pi} f_{D_q}^2 m_{D_q} m_\mu^2 \left(1-\frac{m_\mu^2}{m_{D_q}}\right)\,,\end{aligned} \tag{1.1}$$

where ℓ is μ or e. The left-hand-side in the above expressions is measured experimentally, while the computation of hadronic form factor, $F_+(q^2)$, and/or the meson decay constant, f_{D_q}, requires a first principle description of non-perturbative QCD effects. We now restrain our attention to $F_+(q^2)$, one of the two form factors which parameterise the SM weak matrix element $\langle\pi(\vec{k})|(V-A)_\mu|D(\vec{p})\rangle \equiv \langle\pi(\vec{k})|V_\mu|D(\vec{p})\rangle$, i.e.,

$$\langle\pi(\vec{k})|V_\mu|D(\vec{p})\rangle = \left(p+k-q\frac{m_D^2-m_\pi^2}{q^2}\right)_\mu F_+(q^2) + q_\mu\frac{m_D^2-m_\pi^2}{q^2}F_0(q^2)\,, \tag{1.2}$$

both depending on $q^2=(p-k)^2$ only, with $q^2\in(0,(m_D-m_\pi)^2]$.

Lattice QCD is the only currently available method which allows us to compute this matrix element without introducing any extra parameter and, at least in principle, with an accuracy that can be matched to the experimental one. In practice, however, there is still quite a room for improvement on systematic errors. Here we want to address those that arise from the extraction of the matrix element (1.2) from the correlation functions.

1.1 An abridged description of the standard procedure

The standard method consists in computing the 2- and 3-point functions, namely,

$$\begin{aligned}C_2^{\pi\pi}(\vec{k},t) &= \sum_{\vec{x}}\langle(\bar{q}\gamma_5 q)_{\vec{x},t}(\bar{q}\gamma_5 q)_{\vec{0},0}e^{i\vec{k}\vec{x}}\rangle \xrightarrow{t\gg 0} \frac{\mathscr{Z}_\pi}{2E_\pi}e^{-E_\pi t}\,,\\ C_2^{DD}(\vec{p},t) &= \sum_{\vec{x}}\langle(\bar{c}\gamma_5 q)_{\vec{x},t}(\bar{q}\gamma_5 c)_{\vec{0},0}e^{i\vec{p}\vec{x}}\rangle \xrightarrow{t\gg 0} \frac{\mathscr{Z}_D}{2E_D}e^{-E_D t}\,,\\ C_3^{\pi V_\mu D}(q,t;t_{source}) &= \sum_{\vec{x},\vec{z}}\langle(\bar{q}\gamma_5 q)_{\vec{x},t_{source}}(\bar{q}\gamma_\mu c)_{\vec{z},t}(\bar{c}\gamma_\mu q)_{\vec{0},0}\rangle e^{-i(\vec{q}\vec{z}-\vec{k}\vec{x})}\\ &\xrightarrow{0\ll t\ll t_{source}} \frac{\mathscr{Z}_\pi{}^{1/2}}{2E_\pi}e^{-E_\pi t}\langle\pi(\vec{k})|V_\mu|D(\vec{p})\rangle\frac{\mathscr{Z}_D{}^{1/2}}{2E_D}e^{-E_D(t_{source}-t)}\,,\end{aligned} \tag{1.3}$$

where we also indicate their asymptotic behavior, using the standard notation, $\mathscr{Z}_D = |\langle 0|\bar{c}\gamma_5 q|D(\vec{p})\rangle|^2$, and similar for $\mathscr{Z}_\pi$. The matrix element (1.2) corresponds to a plateau of the ratio

$$R = \frac{C_3^{\pi\gamma_\mu D}(q,t;t_{source})}{C_2^{DD}(\vec{k},t_{source}-t)C_2^{\pi\pi}(\vec{p},t)}\times\sqrt{\mathscr{Z}_\pi}\sqrt{\mathscr{Z}_D} \xrightarrow{0\ll t\ll t_{source}} \langle\pi(\vec{k})|V_\mu|D(\vec{p})\rangle\,. \tag{1.4}$$

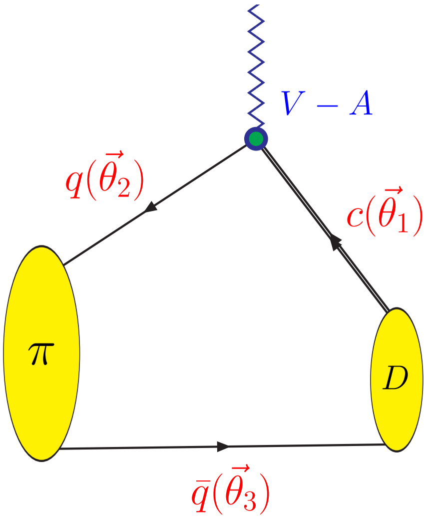

Figure 1: The valence quark diagram of the 3-point function, $C_3^{PV_\mu D}(\vec{k},t)$. We indicate the twisting angles, $\vec{\theta}_{1,2,3}$, which are discussed in the text.

In case of D-decays the plateaus are known not to be long enough to guarantee a percent accuracy (especially when the momenta are given to either of the two mesons). Furthermore, $\sqrt{\mathscr{Z}_\pi}$ and $\sqrt{\mathscr{Z}_D}$ should be computed from a separate study of the 2-point functions, the errors of which are carried over to the ratio R. Finally, a non-negligible statistical error is introduced by the multiplicative renormalization of the vector current, especially when consistently implementing the $\mathscr{O}(a)$ improvement of the Wilson quark operators on the realistic lattices and one of the quark being charmed (heavy). All these difficulties can be avoided by considering various double ratios of 3-point correlation functions. The efficiency of the double ratios was introduced and tested first in *heavy-to-heavy* [1] and then in heavy-to-light decays [2]. In D-decays to a light meson an extra problem is related to the available kinematics on the lattice. More specifically, with the periodic boundary conditions the minimal momentum is $(2\pi)/L$ which is too large ($L = N_L a$ not large enough) if one is to keep the lattice spacing, a, sufficiently small in order to accommodate the charm quark mass. To get around this problem, we adopted the twisted boundary conditions (twBC) recently proposed in ref. [3]. In such a way the momenta $\vec{p}$ and/or $\vec{k}$ become $\vec{\theta}/L = (\theta_0, \theta_0, \theta_0)/L$, where the components θ_0 are chosen anywhere between $0 \leq \theta_0 < \pi$. In this way we are able to compute the form factor $F_+(q^2)$ at several $q^2 > 0$.

2. Double ratios

In what follows we consider 4 different strategies to increase the accuracy of extraction the form factor $F_+(q^2)$ for different value of q^2 by combining the 3-point functions in several different double ratios. At this point we should emphasize that in each of the strategies discussed in this section, the multiplicative renormalization factors, as well as the source terms $\mathscr{Z}_{\pi,D}$, cancel out. We then numerically test each proposed strategy to check whether or not a plateau region is pronounced enough and the statistical quality of the signal satisfactory to reach a percent accuracy of the extracted form factor.

2.1 First strategy

We first keep the D-meson at rest and inject momenta to the pion only. This is done by imposing $\vec{\theta}_1 = \vec{\theta}_3 = \vec{0}$ (c.f. fig. 1), while for $\vec{\theta}_2$ we choose several different values to explore the kinematics available from this decay, $0 \leq q^2 \leq q^2_{max}$. Similar to what has been proposed by JLQCD in their study of the $K_{\ell 3}$-decay [4], we consider the following double ratios ($\vec{k} = \vec{\theta}_2/L$):

$$\frac{C_3^{\pi V_0 D}\left(\vec{0},t\right) C_3^{DV_0\pi}\left(\vec{0},t\right)}{C_3^{\pi V_0 \pi}\left(\vec{0},t\right) C_3^{DV_0 D}\left(\vec{0},t\right)} \overset{\text{plateau}}{\longrightarrow} R_0\,, \tag{2.1}$$

$$\frac{C_3^{\pi V_0 D}\left(\vec{k},t\right) C_2^{\pi\pi}\left(\vec{0},t\right)}{C_3^{\pi V_0 D}\left(\vec{0},t\right) C_2^{\pi\pi}\left(\vec{k},t\right)} \overset{\text{plateau}}{\longrightarrow} R_1\,, \tag{2.2}$$

$$\frac{C_3^{\pi V_i D}\left(\vec{k},t\right) C_3^{\pi V_0\pi}\left(\vec{k},t\right)}{C_3^{\pi V_0 D}\left(\vec{k},t\right) C_3^{\pi V_i \pi}\left(\vec{k},t\right)} \overset{\text{plateau}}{\longrightarrow} R_2\,. \tag{2.3}$$

These ratios then can be cast into the expressions leading to $F_+(q^2)$, i.e.,

$$F_+(q^2) \;=\; R_1 \times F_0(q^2_{max}) \times \frac{m_D + m_\pi}{m_D + E_\pi}\left[1 + \frac{m_D - E_\pi}{m_D + E_\pi} \times \xi(q^2)\right]^{-1}, \tag{2.4}$$

$$\text{where} \quad F_0(q^2_{max}) \;=\; \frac{2\sqrt{m_D m_\pi}}{m_D + m_\pi}\sqrt{R_0}\,, \tag{2.5}$$

$$\text{and} \quad \xi(q^2) \;=\; 1 - \frac{2 m_D R_2}{(m_\pi + E_\pi) + (m_D - E_\pi) R_2}\,. \tag{2.6}$$

For short we wrote $q^2\xi(q^2) = (m_D^2 - m_\pi^2)[F_0(q^2)/F_+(q^2) - 1]$. The quality of the plateaus is presented in fig. 2. To that end we use the publicly available ensembles of the SU(3) gauge field configurations, produced by the QCDSF collaboration by using the $\mathcal{O}(a)$-improved Wilson quark action with $N_F = 2$ [5]. We computed the quark propagators and correlation functions on the configurations gathered at $\beta = 5.29$, corresponding to $a \simeq 0.08$ fm, on the $24^3 \times 48$ lattice, each time keeping the light valence quark mass equal to that of the sea quark. In fig. 2 we see that the signals for R_0 and R_1 are indeed very good. The fact that the signal for R_2 is not as good does not trouble the whole strategy because it is only needed to compute $\xi(q^2)$, which itself is a correction to 1 in both eq. (2.6) and in eq. (2.4). On the other hand the fact that the quality is less good for $R_2(t)$ is expected as its computation involves the correlation function with with "γ_i"-matrices (mixing the "large" and "small" components of the Dirac spinors), in contrast to $R_{0,1}(t)$ in which we compute the correlation functions with "γ_0"-matrix only.

2.2 Second and third strategies

Next we consider the kinematics in which we either keep the D-meson at rest ($\vec{\theta}_1 = \vec{\theta}_3 = \vec{0}$) and inject momenta to the pion source ($\vec{\theta}_2 = \vec{\theta}_q$), or keep the pion at rest ($\vec{\theta}_1 = \vec{\theta}_3 = \vec{0}$) and inject

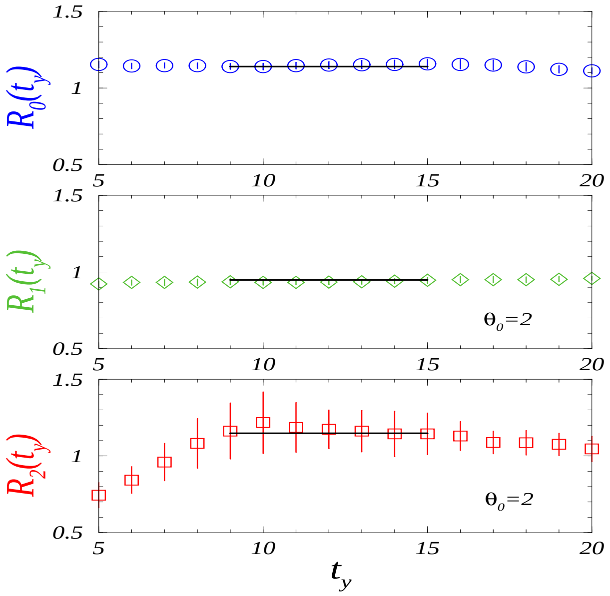

Figure 2: Double ratios, $R_{0,1,2}(t)$, illustrating the first strategy. The lattice data refer to the simulation with Wilson $\mathscr{O}(a)$-improved quarks with $N_F = 2$ ($\beta = 5.29$) and $\kappa_q = \kappa_{\rm sea} = 0.1355$. The shown signals refer to $q^2 \approx 1$ GeV^2.

momenta to D-meson ($\vec{\theta}_1 = \vec{\theta}_c$). Notice that keeping the q^2 fixed requires that $\vec{\theta}_c = (m_D/m_\pi) \times \vec{\theta}_q$. We then build the following four double ratios:

$$\frac{C_3^{\pi V_0 D}\left(\vec{0},\vec{p},t\right) C_3^{D V_0 \pi}\left(\vec{p},\vec{0},t\right)}{C_3^{\pi V_0 \pi}\left(\vec{0},\vec{0},t\right) C_3^{D V_0 D}\left(\vec{p},\vec{p},t\right)} \xrightarrow{\text{plateau}} R_3\,, \qquad \frac{C_3^{D V_0 \pi}\left(\vec{0},\vec{k},t\right) C_3^{\pi V_0 D}\left(\vec{k},\vec{0},t\right)}{C_3^{\pi V_0 \pi}\left(\vec{0},\vec{0},t\right) C_3^{D V_0 D}\left(\vec{k},\vec{k},t\right)} \xrightarrow{\text{plateau}} R_4\,,$$
$$\frac{C_3^{\pi V_i D}\left(\vec{0},\vec{p},t\right) C_3^{D V_i \pi}\left(\vec{p},\vec{0},t\right)}{C_3^{\pi V_0 \pi}\left(\vec{0},\vec{0},t\right) C_3^{D V_i D}\left(\vec{p},\vec{p},t\right)} \xrightarrow{\text{plateau}} R_3'\,, \qquad \frac{C_3^{D V_i \pi}\left(\vec{0},\vec{k},t\right) C_3^{\pi V_i D}\left(\vec{k},\vec{0},t\right)}{C_3^{\pi V_i \pi}\left(\vec{k},\vec{k},t\right) C_3^{D V_0 D}\left(\vec{0},\vec{0},t\right)} \xrightarrow{\text{plateau}} R_4'\,, \qquad (2.7)$$

where $\vec{p} = \vec{\theta}_1/L$, and $\vec{k} = \vec{\theta}_2/L$. In terms of form factors, the above ratios read

$$R_3 = \frac{\left[E_D + m_\pi + (E_D - m_\pi)\xi(q^2)\right]^2}{2m_\pi 2E_D}[F_+(q^2)]^2\,, \quad R_4 = \frac{\left[m_D + E_\pi + (m_D - E_\pi)\xi(q^2)\right]^2}{2E_\pi 2m_D}[F_+(q^2)]^2\,,$$

$$R_3' = p_i \frac{1-\xi^2(q^2)}{4m_\pi}[F_+(q^2)]^2\,, \qquad R_4' = k_i \frac{1-\xi^2(q^2)}{4m_D}[F_+(q^2)]^2\,. \qquad (2.8)$$

Note that UKQCD [6] recently considered R_3 and R_3' to probe $F_+^{K\to\pi}(0)$. We now need to combine the two of above four double ratios to obtain $F_+(q^2)$, namely either R_3 with R_3', or R_4 with R_4', or R_3 with R_4', or R_3' with R_4. In fig. 3 we show the quality of the signals corresponding to the same set-up as the signals displayed in fig. 2. We observe that only R_4 and R_4' are reasonably good, whereas $R_3(t)$ and $R_3'(t)$ do not exhibit plateaus, likely due to the fact that m_D/m_π is large, so that θ_c in $\vec{\theta}_c = (m_D/m_\pi) \times \vec{\theta}_q$ is such that the twBC simply destroys the signal. Therefore, we will retain the ratios R_4 and R_4' which can be used to compute the form factor $F_+(q^2)$ for various values of q^2.

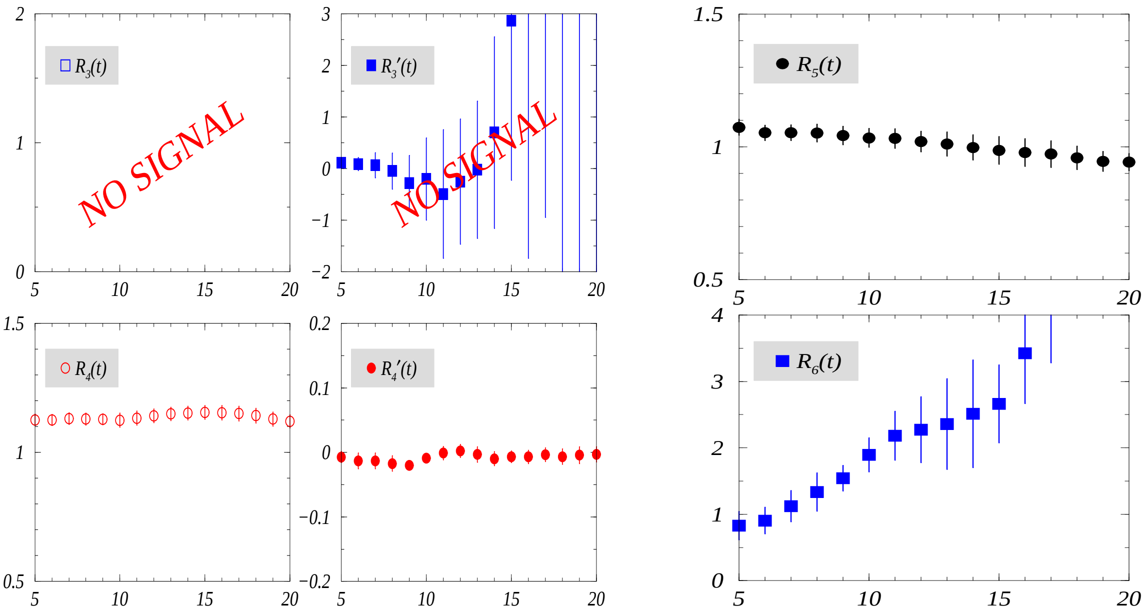

Figure 3: Illustration of the signals for the double ratios used in the second, third (R_3, R_4, R_3', R_4') and fourth strategies (R_5, R_6), defined in the text.

2.3 Fourth strategy

If one wants to study the shape of the form factor (i.e., its q^2-dependence), then it is worth trying to impose the twBC on the spectator quark ($\vec{\theta}_3 \neq 0$) without twisting the other two quarks ($\vec{\theta}_1 = \vec{\theta}_2 = \vec{0}$). In such a way we may probe many values of q^2 but never reach $q^2 = 0$, because in this set-up the condition $E_D = E_\pi$ does not allow a real solution in $|\vec{p}| = |\vec{k}| = \vec{\theta}_3/L$. To test this option we consider the following two ratios:

$$\frac{C^{\pi V_0 D}_{3\theta}(t)\,C^{DV_0\pi}_{3\theta}(t)}{C^{\pi V_0\pi}_{3\theta}(t)\,C^{DV_0 D}_{3\theta}(t)} \xrightarrow{\text{plateau}} R_5\,, \qquad \frac{C^{DV_i\pi}_{3\theta}(t)\,C^{\pi V_i D}_{3\theta}(t)}{C^{\pi V_i\pi}_{3}(t)\,C^{DV_i D}_{3}(t)} \xrightarrow{\text{plateau}} R_6\,, \tag{2.9}$$

where index “θ" is used to distinguish that the spectator quark is actually twisted. Expressed in terms of form factors

$$R_5 = \frac{\left[E_D + E_\pi + (E_D - E_\pi)\xi(q^2)\right]^2}{2E_D 2E_\pi}[F_+(q^2)]^2\,, \qquad R_6 = [F_+(q^2)]^2\,. \tag{2.10}$$

The corresponding signals from our numerical study are shown in fig. 3. While the statistical quality of $R_5(t)$ is reasonably good, the signal for $R_6(t)$ is not promising if we are after a strategy that could lead us to a percent accuracy on the extracted form factor. Although we did not try it, we suspect the flatness of R_5 could be achieved by a judicious choice of smearing. We point out, however, that our numerical evaluation of the ratios (2.9) indicate the large statistical errors so that this strategy is not competitive with the first or the third ones, discussed in this section.

3. Summary

It this note we report on the results of our exploratory study in which we use various double ratios and twisted boundary condition on the quark propagators in order to extract the form factor

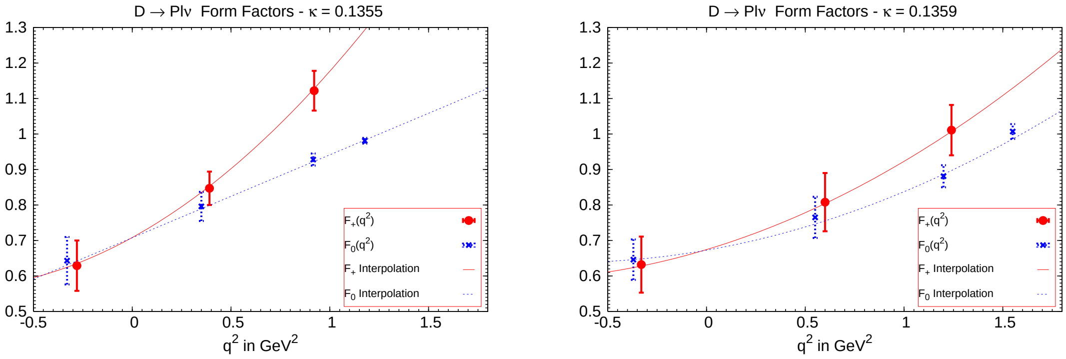

Figure 4: $D \longrightarrow \pi$ semileptonic form factors for the unphysically heavy pions: left panel corresponds to $m_\pi \simeq 750 MeV$, the right one to $m_\pi \simeq 600 MeV$

relevant to the semileptonic heavy-to-light D-decays to a percent accuracy. In total we proposed four different strategies, which we then tested numerically on the set of unquenched ($N_F = 2$) gauge field configurations. On the basis of our analysis we conclude that the first (double ratios $R_{0,1,2}$) and the third strategy (R_4 and R_4'), discussed in the text, can be used to compute the form factor $F_+(q^2)$ to a desired precision. Even though our numerical tests are made by using the Wilson quarks, our conclusions apply to any lattice QCD action.

In fig. 4, we illustrate the results obtained by employing the first strategy at three values of the twisting angle $\vec{\theta}$. Those will be improved and the results discussed in our forthcoming paper.

4. Acknowledgments

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