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# $K_{l3}$ form factor with $N_f = 2+1$ dynamical domain wall fermions

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**RBC + UKQCD Collaborations**

We present the latest results from the UKQCD/RBC collaborations for the $K_{l3}$ form factor from simulations with $2+1$ flavours of dynamical domain wall quarks. Simulations are performed on lattices with two different volumes and four values of the light quark mass, allowing for an extrapolation to the chiral limit. The analysis includes a thorough investigation into the sources of systematic error in our fits. After interpolating to zero momentum transfer, we obtain $f_+(0) = 0.964(5)$ (or $\Delta f = -0.013(5)$) which, when combined with the latest experimental results for $K_{l3}$ decays, leads to $|V_{us}| = 0.2249(14)$.

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## 1. Introduction

$K \to \pi l \nu$ ($K_{l3}$) decays provide an excellent avenue for an accurate determination of the Cabibbo-Kobayashi-Maskawa (CKM) [1] quark mixing matrix element, $|V_{us}|$. This is done by observing that the experimental rate for $K_{\ell 3}$ decays is proportional to $|V_{us}|^2|f_+(0)|^2$,

$$\Gamma_{K\to\pi l\nu} = C_K^2 \frac{G_F^2 m_K^5}{192\pi^3} I\, S_{EW} \left[1 + 2\Delta_{SU(2)} + 2\Delta_{EM}\right] |V_{us}|^2 |f_+(0)|^2 , \tag{1.1}$$

where $I$ is the phase space integral which can be evaluated from the shape of the experimental form factor, and $\Delta_{SU(2)}, S_{EW}, \Delta_{EM}$ contain the isospin breaking, short distance electroweak and long distance electromagnetic corrections, respectively. $f_+(0)$ is the form factor defined from the $K \to \pi$ matrix element of the weak vector current, $V_\mu = \bar{s}\gamma_\mu u$, evaluated at zero momentum transfer

$$\langle \pi(p')\big|V_\mu\big|K(p)\rangle = (p_\mu + p'_\mu) f_+(q^2) + (p_\mu - p'_\mu) f_-(q^2) , \tag{1.2}$$

where $q^2 = (p-p')^2$. PDG(2006) quotes [2][1]

$$|V_{us} f_+(0)| = 0.2169(9) , \tag{1.3}$$

hence in order to obtain $|V_{us}|$ at a precision commensurate with current experiments, we need to determine $f_+(0)$ with an error of less than 1%.

In chiral perturbation theory (ChPT), $f_+(0)$ is expanded in terms of the light pseudoscalar meson masses

$$f_+(0) = 1 + f_2 + f_4 + \ldots , \quad (f_n = \mathscr{O}(m^n_{\pi,K,\eta})) . \tag{1.4}$$

Current conservation ensures that in the $SU(3)_{\text{flavour}}$ limit $f_+(0) = 1$, hence $f_2$ and $f_4$ are *small*. Additionally, as a result of the Ademollo-Gatto Theorem [4], which states that $f_2$ receives no contribution from local operators appearing in the effective theory, $f_2$ is determined unambiguously in terms of $m_\pi$, $m_K$ and $f_\pi$, and takes the value $f_2 = -0.023$ at the physical values of the meson masses [5]. Our task is now reduced to one of finding

$$\Delta f = f_+(0) - (1 + f_2) . \tag{1.5}$$

Until recently, the canonical estimate of $\Delta f = -0.016(8)$ was due to Leutwyler & Roos (LR) [5], whereas more recent ChPT based phenomenological analyses favour a value consistent with zero $\Delta f = 0.001(10)$ [6], $\Delta f = 0.007(12)$ [7], $\Delta f = -0.003(11)$ [8]. These determinations, however, require model input; the 50% error in the LR result, for example, was estimated within the context of a simple quark model. Hence a lattice determination of $\Delta f$ is essential.

The last few years have seen an improvement in the accuracy of lattice calculations of this quantity [9, 10, 11, 12, 13] (see [14] for a review), with the results favouring a negative value for $\Delta f$ in agreement with Leutwyler & Roos.

The UKQCD and RBC collaborations have recently completed the first unitary (i.e. $N_f = 2+1$ flavour) lattice calculation of the $K \to \pi$ form factor using dynamical domain wall fermions at light quark masses and on large volumes [15]. In this paper we summarise these findings and discuss a recent development which promises to provide further improvement by removing the extrapolation to $q^2 = 0$ [16].

[1] A more recent analysis finds $|V_{us} f_+(0)| = 0.21673(46)$ [3].

## 2. Simulation Details

We simulate with $N_f = 2+1$ dynamical flavours generated with the Iwasaki gauge action [17] at $\beta = 2.13$, which corresponds to an inverse lattice spacing $a^{-1} = 1.73(3)\,\mathrm{GeV}$ ($a = 0.114(2)\,\mathrm{fm}$) [18, 19], and the domain wall fermion action [20] with domain wall height $M_5 = 1.8$ and fifth dimension length $L_s = 16$. This results in a residual mass of $am_{\rm res} = 0.00315(2)$ [18, 19]. The simulated strange quark mass, $am_s = 0.04$, is close to its physical value [19], and we choose four values for the light quark masses, $am_{ud} = 0.03, 0.02, 0.01, 0.005$, which correspond to pion masses $m_\pi \approx 670$, 560, 420, 330 MeV [18, 19]. The calculations are performed on two volumes, $16^3$ ($(1.83)^3\,\mathrm{fm}^3$) and $24^3$ ($(2.74)^3\,\mathrm{fm}^3$), at each quark mass, except the lightest mass which is only simulated on the larger volume. Further simulation details can be found in [18, 19].

## 3. Lattice Techniques

We start by rewriting the vector form factors given in (1.2) to define the scalar form factor

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_K^2 - m_\pi^2} f_-(q^2)\ , \tag{3.1}$$

which can be obtained on the lattice at $q^2_{\rm max} = (m_K - m_\pi)^2$ with high statistical accuracy [9, 21].

For each quark mass, in addition to evaluating $f_0(q^2)$ at $q^2 = q^2_{\rm max}$, we determine the form factor at several negative values of $q^2$, allowing us to interpolate the results to $q^2 = 0$. Specifically, in the notation of (1.2), we evaluate the form factor with $|\vec{p}\,'| = 0$, $|\vec{p}| = p_L$ or $|\vec{p}| = \sqrt{2}\,p_L$ where $p_L = 2\pi/L$ and $L$ is the spatial extent of the lattice, and also with $|\vec{p}| = 0$, $|\vec{p}\,'| = p_L$ or $|\vec{p}\,'| = \sqrt{2}\,p_L$. To obtain the $f_0(q^2)$ we use standard ratio techniques [21, 9, 12], which do not require normalisation of the vector current.

In order to gain the maximum amount of information from limited data, we perform a simultaneous fit to both the $q^2$ and quark mass dependencies using the ansatz [15]

$$f_0(q^2, m_\pi^2, m_K^2) = \frac{1 + f_2 + (m_K^2 - m_\pi^2)^2 (A_0 + A_1(m_K^2 + m_\pi^2))}{1 - q^2/(M_0 + M_1(m_K^2 + m_\pi^2))^2}\ , \tag{3.2}$$

with four fit parameters $A_0, A_1, M_0, M_1$, and is simply a modification of the standard pole dominance form

$$f_0(q^2) = f_0(0)/(1 - q^2/M^2)\ , \tag{3.3}$$

where $M$ is a pole mass, which has been shown to describe the $q^2$-dependence of lattice results of $f_0(q^2)$ very well [12, 9].

We find that fitting the $24^3$ data using (3.2) provides excellent agreement with the traditional approach of sequentially interpolating in $q^2$ via (3.3) followed by chiral extrapolation of $f_+(0)$ to the physical quark masses.

We present the results from a fit to the $24^3 \times 64$ data sets using (3.2) in the left plot of Fig. 1. Here the curve shows the fit function at the physical meson masses, while the difference $f_0(q^2, m_\pi^{\rm latt}, m_K^{\rm latt}) - f_0(q^2, m_\pi^{\rm phys}, m_K^{\rm phys})$ has been subtracted from our raw data points and the small scatter is indicative of the quality of our fit.



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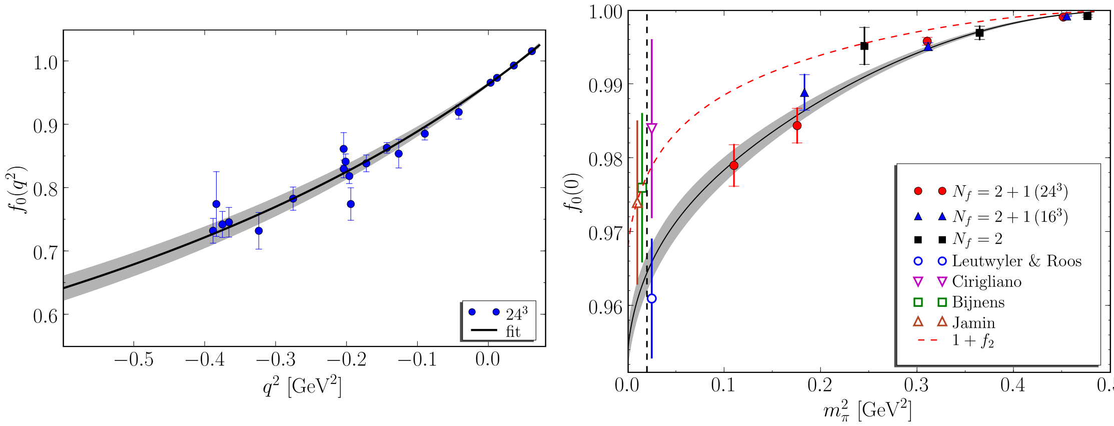


**Figure 1:** Scalar form factor, $f_0(0)$, together with the simultaneous fit (solid line) on the $24^3$ data (red circles) using (3.2).

The quark mass dependence of (3.2) is presented in the right plot of Fig. 1. The solid line represents the fit function evaluated at $q^2 = 0$, plotted as a function of $m_\pi^2$, while the dashed line is the contribution coming from the $\mathscr{O}(p^4)$ terms in the chiral expansion, $1+f_2$. Our results clearly indicate a sizeable, negative value for $\Delta f = -0.013(3)$, in contrast to the recent ChPT based results of [6, 7, 8]. In right side of Fig. 1 we also overlay the results obtained from individual pole fits on each of our ensembles and earlier $N_f = 2$ results [12].

So far, we have assumed a pole dominance behaviour in our lattice data. In order to estimate the systematic error due to this choice, we fit $f_0(q^2)$ at each quark mass with a linear form, a quadratic form, and a parameterisation proposed in Ref. [22]. In Fig. 2 we compare these different fit forms for bare quark mass $am_{ud} = 0.005$ since this is the dataset that requires the largest extrapolation from $q^2_{\max}$. In the inset, we can see the sensitivity of the resulting value at $q^2 = 0$ to the choice of fit form. We find that all four parameterisations agree reasonably well, except the linear ansatz which anyway has the largest $\chi^2/dof$.

Hence we estimate the systematic error due to the choice of (3.2) as our prefered fit form by using a simultaneous quadratic fit similar to (3.2)

$$\begin{aligned} f_0(q^2, m_\pi^2, m_K^2) \;=\; & 1+f_2+(m_K^2-m_\pi^2)^2(A_0+A_1+A_2(m_K^2+m_\pi^2))+ \\ & (A_3+(2A_0+A_1)(m_K^2+m_\pi^2))\,q^2+(A_4-A_0+A_5(m_K^2+m_\pi^2))\,q^4\;. \end{aligned} \tag{3.4}$$

The form of this ansatz is motivated by the expression obtained in ChPT [6]. We find that the results of the two fits, (3.2) and (3.4), agree within statistical precision and we take the difference (0.0034) as an estimate of the systematic error in choosing (3.2) as our preferred ansatz.

Recently we have developed a promising method for removing this source of systematic error [16]. On a lattice with periodic boundary conditions, the smallest possible momentum transfer ($q$) available is $2\pi/L$, where $L$ is the lattice spatial extent. However, lower values of $q^2$ can be obtained by altering the boundary conditions (twisted boundary conditions) [23].

By twisting the boundary conditions of the valence quark field before (after) the operator insertion by a twisting angle, $\vec{\theta}_i$ ($\vec{\theta}_f$), it is possible to obtain results with arbitrary momentum

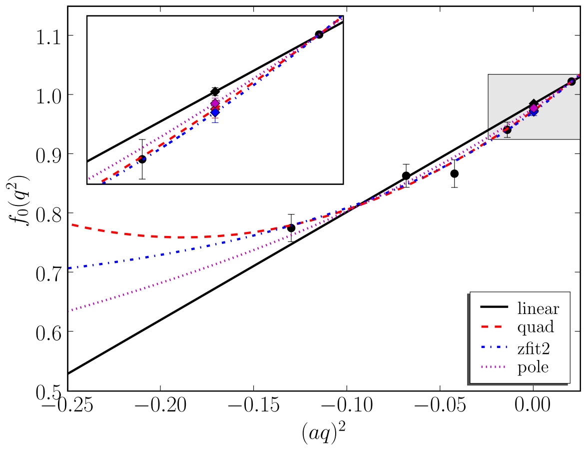


**Figure 2:** Comparison of different fits to the $q^2$-dependence of the scalar form factor $f_0(q^2)$ for bare quark mass $am_{ud} = 0.005$. Solid, dashed, dotted and dot-dashed lines correspond to linear, quadratic, pole dominance 3.3 and zfit[22], respectively.

transfer

$$q^2 = (p_f - p_i)^2 = \left\{ [E_f(\vec{p}_f) - E_i(\vec{p}_i)]^2 - \left[ (\vec{p}_{\mathrm{FT},f} + \vec{\theta}_f/L) - (\vec{p}_{\mathrm{FT},i} + \vec{\theta}_i/L) \right]^2 \right\}. \tag{3.5}$$

where

$$E = \sqrt{m^2 + (\vec{p}_{\mathrm{FT}} + \vec{\theta}/L)^2}, \tag{3.6}$$

$m$ is the mass of the meson and $\vec{p}_{\mathrm{FT}}$ is the meson momentum induced by Fourier summation (the components of $\vec{p}_{\mathrm{FT}}$ are integer multiples of $2\pi/L$). In this way, it is possible to tune $\vec{\theta}_i$ and $\vec{\theta}_f$ such that $q^2 = 0$.

In Fig. 3 we display the results of a proof-of-principle investigation on $16^3 \times 32$ lattices with two values of the light quark masses, $am_l = 0.02$ and $am_l = 0.01$. In the left plot we present results for $f_0(q^2)$ obtained from the standard procedure, together with a pole dominance fit. The right plot displays a zoom into the region around $q^2 = 0$. The two data points at $q^2 > 0$ correspond to the results for $q^2_{\mathrm{max}}$ for which the pion and kaon are both at rest; they can be identified by their strikingly small errors. We also show results for $f_0(q^2 = 0)$ obtained from the pole fit (3.3) at each quark mass. In addition the right plot of Fig. 3 contains the results from the new approach.

The results for $f_0(0)$ as determined from the conventional and the new approaches do not agree exactly but the discrepancy is statistically not significant. The size of the statistical errors is similar in the two approaches, while the new technique avoids the need for an extrapolation and hence is important in removing one source of systematic error.

Finally, since we simulate at a single lattice spacing, we are unable to extrapolate to the continuum limit. However, leading lattice artefacts with domain wall fermions are of $O(a^2\Lambda^2_{QCD})$; assuming $\Lambda_{QCD} \sim 300$ MeV we estimate these to be no larger than $\approx 4\%$ (of $1 - f_+$). Observations that $\mathcal{O}(a^2)$ effects in physical quantities, such as $f_\pi$, are consistent with this [19], and we will explicitly check this for $K_{\ell 3}$ decays on our new ensemble which is being generated on a finer lattice. Note that our current uncertainty is dominated by statistics and the chiral and $q^2$ extrapolations and

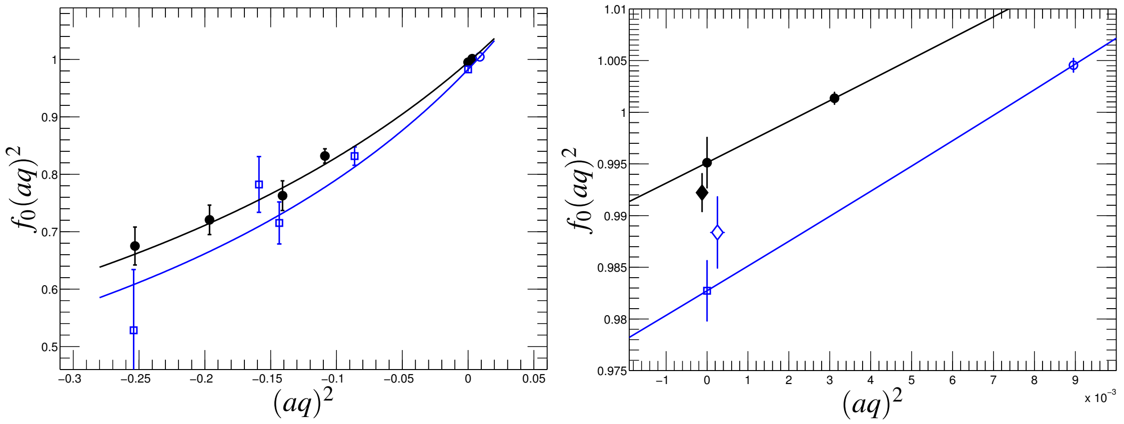


**Figure 3:** Results for the form factor ($am_l = 0.02$ as full black circles and $am_l = 0.01$ as blue squares). Left: All data points entering the conventional approach. Right: Zoom which shows the data points for both the new (diamonds) and the conventional approach at $q^2 = 0$ and the data points at $q^2_{\rm max}$.

not by the discretisation error. Hence our final result is

$$f_+(0) = 0.9644(33)(34)(14)\ , \tag{3.7}$$

where the first error is statistical, and the second and third are estimates of the systematic errors due to our choice of parametrisation (3.2) and lattice artefacts, respectively. Our result agrees very well with the Leutwyler-Roos value [5] and earlier lattice calculations [9, 10, 11, 12]. In particular, we note that our findings prefer a sizeable, negative value for $\Delta f = -0.0129(33)(34)(14)$, in contrast to recent ChPT based phenomenological results [6, 7, 8].

Using $|V_{us} f_+(0)| = 0.2169(9)$ from PDG(2006) [2][2]

$$|V_{us}| = 0.2249(9)_{\rm exp}(11)_{f_+(0)}\ , \tag{3.8}$$

and combined with $|V_{ud}| = 0.97377(27)$ [2] we find

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 - \delta, \quad \delta = 0.0012(8)\,, \tag{3.9}$$

compared with the PDG(2006) [2] result, $\delta = 0.0008(10)$.

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[2]Using the result from [3] gives $|V_{us}| = 0.2247(5)_{\rm exp}(11)_{f_+(0)}$.

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