

Fixed Points of Quantum Gravity and the Renormalisation Group

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We review the asymptotic safety scenario for quantum gravity and the role and implications of an underlying ultraviolet fixed point. We discuss renormalisation group techniques employed in the fixed point search, analyse the main picture at the example of the Einstein-Hilbert theory, and provide an overview of the key results in four and higher dimensions. We also compare findings with recent lattice simulations and evaluate phenomenological implications for collider experiments.

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1. Introduction

It is commonly believed that an understanding of the dynamics of gravity and the structure of space-time at shortest distances requires an explicit quantum theory for gravity. The well-known fact that the perturbative quantisation program for gravity in four dimensions faces problems has raised the suspicion that a consistent formulation of the theory may require a radical deviation from the concepts of local quantum field theory, *e.g.* string theory. It remains an interesting and open challenge to prove, or falsify, that a consistent quantum theory of gravity cannot be accommodated for within the otherwise very successful framework of local quantum field theories.

Some time ago Steven Weinberg added a new perspective to this problem by pointing out that a quantum theory of gravity in terms of the metric field may very well exist, and be renormalisable on a non-perturbative level, despite its notorious perturbative non-renormalisability [1]. This scenario, since then known as “asymptotic safety”, necessitates an interacting ultraviolet fixed point for gravity under the renormalisation group (RG) [1]-[4]. If so, the high energy behaviour of gravity is governed by near-conformal scaling in the vicinity of the fixed point in a way which circumnavigates the virulent ultraviolet (UV) divergences encountered within standard perturbation theory. Indications in favour of an ultraviolet fixed point are based on renormalisation group studies in four and higher dimensions [5]-[18], dimensional reduction techniques [19, 20], renormalisation group studies in lower dimensions [1], [21]-[24], 4d perturbation theory in higher derivative gravity [25], large- N expansions in the matter fields [26], and lattice simulations [27]-[29].

In this contribution, we review the key elements of the asymptotic safety scenario (Sec. 2) and introduce renormalisation group techniques (Sec. 3) which are at the root of fixed point searches in quantum gravity. The fixed point structure in four (Sec. 4) and higher dimensions (Sec. 5), and the phase diagram of gravity (Sec. 6) are discussed and evaluated in the light of the underlying approximations. Results are compared with recent lattice simulations (Sec. 7), and phenomenological implications are indicated (Sec. 8). We close with some conclusions (Sec. 9).

2. Asymptotic Safety

We summarise the basic set of ideas and assumptions of asymptotic safety as first laid out in [1] (see [2]-[4] for reviews). The aim of the asymptotic safety scenario for gravity is to provide for a path-integral based framework in which the metric field is the carrier of the fundamental degrees of freedom, both in the classical and in the quantum regimes of the theory. This is similar in spirit to effective field theory approaches to quantum gravity [30]. There, a systematic study of quantum effects is possible without an explicit knowledge of the ultraviolet completion as long as the relevant energy scales are much lower than the ultraviolet cutoff Λ of the effective theory, with Λ of the order of the Planck scale (see [31] for a recent review).

The asymptotic safety scenario goes one step further and assumes that the cutoff Λ can in fact be removed, $\Lambda \rightarrow \infty$, and that the high-energy behaviour of gravity, in this limit, is characterised by an interacting fixed point. It is expected that the relevant field configurations dominating the gravitational path integral at high energies are predominantly “anti-screening” to allow for this limit to become feasible. If so, it is conceivable that a non-trivial high-energy fixed point of gravity may exist and should be visible within *e.g.* renormalisation group or lattice implementations of the

theory, analogous to the well-known perturbative high-energy fixed point of QCD. Then the high-energy behaviour of the relevant gravitational couplings is “asymptotically safe” and connected with the low-energy behaviour by finite renormalisation group flows. The existence of a fixed point together with finite renormalisation group trajectories provides for a definition of the theory at arbitrary energy scales.

The fixed point implies that the high-energy behaviour of gravity is characterised by universal scaling laws, dictated by the residual high-energy interactions. No a priori assumptions are made about which invariants are the relevant operators at the fixed point. In fact, although the low-energy physics is dominated by the Einstein-Hilbert action, it is expected that (a finite number of) further invariants will become relevant, in the renormalisation group sense, at the ultraviolet fixed point.¹ Then, in order to connect the ultraviolet with the infrared physics along some renormalisation group trajectory, a finite number of initial parameters have to be fixed, ideally taken from experiment. In this light, classical general relativity would emerge as a “low-energy phenomenon” of a fundamental quantum field theory in the metric field.

We illustrate this scenario with a discussion of the renormalisation group equation for the gravitational coupling G , following [15] (see also [2, 3]). Its canonical dimension is $[G] = 2 - d$ in d dimensions and hence negative for $d > 2$. It is commonly believed that a negative mass dimension for the relevant coupling is responsible for the perturbative non-renormalisability of the theory. We introduce the renormalised coupling as $G(\mu) = Z_G^{-1}(\mu) G$, and the dimensionless coupling as $g(\mu) = \mu^{d-2} G(\mu)$; the momentum scale μ denotes the renormalisation scale. The graviton wave function renormalisation factor $Z_G(\mu)$ is normalised as $Z_G(\mu_0) = 1$ at $\mu = \mu_0$ with $G(\mu_0)$ given by Newton’s coupling constant $G_N = 6.67428 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg s}^2}$. The graviton anomalous dimension η related to $Z_G(\mu)$ is given by $\eta = -\mu \frac{d}{d\mu} \ln Z_G$. Then the Callan-Symanzik equation for $g(\mu)$ reads

$$\beta_g \equiv \mu \frac{dg(\mu)}{d\mu} = (d - 2 + \eta)g(\mu). \quad (2.1)$$

Here we have assumed a fundamental action for gravity which is local in the metric field. In general, the graviton anomalous dimension $\eta(g, \dots)$ is a function of all couplings of the theory including matter fields. The RG equation (2.1) displays two qualitatively different types of fixed points. The non-interacting (gaussian) fixed point corresponds to $g_* = 0$ which also entails $\eta = 0$. In its vicinity with $g(\mu_0) \ll 1$, we have canonical scaling since $\beta_g = (d - 2)g$, and

$$G(\mu) = G(\mu_0) \quad (2.2)$$

for all $\mu < \mu_0$. Consequently, the gaussian regime corresponds to the domain of classical general relativity. In turn, (2.1) can display an interacting fixed point $g_* \neq 0$ in $d > 2$ if the anomalous dimension takes the value $\eta(g_*, \dots) = 2 - d$; the dots denoting further gravitational and matter couplings. Hence, the anomalous dimension precisely counter-balances the canonical dimension of Newton’s coupling G . This structure is at the root for the non-perturbative renormalisability of

¹For infrared fixed points, universality considerations often simplify the task of identifying the set of relevant, marginal and irrelevant operators. This is not applicable for interacting ultraviolet fixed points.

quantum gravity within a fixed point scenario.² Consequently, at an interacting fixed point where $g_* \neq 0$, the anomalous dimension implies the scaling

$$G(\mu) = \frac{g_*}{\mu^{d-2}} \quad (2.3)$$

for the dimensionful gravitational coupling. In the case of an ultraviolet fixed point $g_* \neq 0$ for large μ , the dimensionful coupling G becomes arbitrarily small in its vicinity. This is in marked contrast to (2.2). Hence, (2.3) indicates that gravity weakens at the onset of fixed point scaling. Nevertheless, at the fixed point the theory remains non-trivially coupled because of $g_* \neq 0$. The weakness of the coupling in (2.3) is a dimensional effect, and should be contrasted with *e.g.* asymptotic freedom of QCD in four dimensions where the dimensionless non-abelian gauge coupling becomes weak because of a non-interacting ultraviolet fixed point. In turn, if (2.3) corresponds to a non-trivial infrared fixed point for $\mu \rightarrow 0$, the dimensionful coupling $G(\mu)$ grows large. A strong coupling behaviour of this type would imply interesting long distance modifications of gravity.

As a final comment, we point out that asymptotically safe gravity is expected to become, in an essential way, two-dimensional at high energies. Heuristically, this can be seen from the dressed graviton propagator whose scalar part, neglecting the tensorial structure, scales as $\mathcal{G}(p^2) \sim p^{-2(1-\eta/2)}$ in momentum space. Here we have evaluated the anomalous dimension at $\mu^2 \approx p^2$. Then, for small η , we have the standard perturbative behaviour $\sim p^{-2}$. In turn, for large anomalous dimension $\eta \rightarrow 2 - d$ in the vicinity of a fixed point the propagator is additionally suppressed $\sim (p^2)^{-d/2}$ possibly modulo logarithmic corrections. After Fourier transform to position space, this corresponds to a logarithmic behaviour for the propagator $\mathcal{G}(x, y) \sim \ln(|x - y|\mu)$, characteristic for bosonic fields in two-dimensional systems.

3. Renormalisation Group

Whether or not a non-trivial fixed point is realised in quantum gravity can be assessed once explicit renormalisation group equations for the scale-dependent gravitational couplings are available. To that end, we recall the set-up of Wilson's (functional) renormalisation group, which is used below for the case of quantum gravity (see [37]-[42] for reviews). Wilsonian flows are based on the notion of a cutoff effective action Γ_k , where the propagation of fields ϕ with momenta smaller than k is suppressed. A Wilsonian cutoff is realised by adding $\Delta S_k = \frac{1}{2} \int \phi(-q) R_k(q) \phi(q)$ within the Schwinger functional

$$\ln Z_k[J] = \ln \int [D\phi]_{\text{ren.}} \exp \left(-S[\phi] - \Delta S_k[\phi] + \int J \cdot \phi \right) \quad (3.1)$$

and the requirement that R_k obeys (i) $R_k(q) \rightarrow 0$ for $k^2/q^2 \rightarrow 0$, (ii) $R_k(q) > 0$ for $q^2/k^2 \rightarrow 0$, and (iii) $R_k(q) \rightarrow \infty$ for $k \rightarrow \Lambda$ (for examples and plots of R_k , see [43]). Note that the Wilsonian

²Integer values for anomalous dimensions are well-known from other gauge theories at criticality and away from their canonical dimension. In the d -dimensional $U(1)$ +Higgs theory, the abelian charge e^2 has mass dimension $[e^2] = 4 - d$, with $\beta_{e^2} = (d - 4 + \eta) e^2$. In three dimensions, a non-perturbative infrared fixed point at $e_*^2 \neq 0$ leads to $\eta_* = 1$ [32]. The fixed point belongs to the universality class of conventional superconductors with the charged scalar field describing the Cooper pair. The integer value $\eta_* = 1$ implies that the magnetic field penetration depth and the Cooper pair correlation length scale with the same universal exponent at the phase transition [32, 33]. In Yang-Mills theories above four dimensions, ultraviolet fixed points with $\eta = 4 - d$ and implications thereof have been discussed in [34, 35, 36].

momentum scale k takes the role of the renormalisation group scale μ introduced in the previous section. Under infinitesimal changes $k \rightarrow k - \Delta k$, the Schwinger functional obeys $\partial_t \ln Z_k = -\langle \partial_t \Delta S_k \rangle_J$; $t = \ln k$. We also introduce its Legendre transform, the scale-dependent effective action $\Gamma_k[\phi] = \sup_J (\int J \cdot \phi - \ln Z_k[J]) - \frac{1}{2} \int \phi R_k \phi$, $\phi = \langle \phi \rangle_J$. It obeys an exact functional differential equation introduced by Wetterich [46]

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \left(\Gamma_k^{(2)} + R_k \right)^{-1} \partial_t R_k, \quad (3.2)$$

which relates the change in Γ_k with a one-loop type integral over the full field-dependent cutoff propagator. Here, the trace Tr denotes an integration over all momenta and summation over all fields, and $\Gamma_k^{(2)}[\phi](p, q) \equiv \delta^2 \Gamma_k / \delta \phi(p) \delta \phi(q)$. A number of comments are in order:

- **Finiteness and interpolation property.** By construction, the flow equation (3.2) is well-defined and finite, and interpolates between an initial condition Γ_Λ for $k \rightarrow \Lambda$ and the full effective action $\Gamma \equiv \Gamma_{k=0}$. This is illustrated in Fig. 1. The endpoint is independent of the regularisation, whereas the trajectories $k \rightarrow \Gamma_k$ depends on it.
- **Locality.** The integrand of (3.2) is peaked for field configurations with momentum squared $q^2 \approx k^2$, and suppressed for large momenta [due to condition (i) on R_k] and for small momenta [due to condition (ii)]. Therefore, the flow equation is essentially local in momentum and field space [43]-[45].
- **Approximations.** Systematic approximations for Γ_k and $\partial_t \Gamma_k$ are required to integrate (3.2). These include (a) perturbation theory, (b) expansions in powers of the fields (vertex functions), (c) expansion in powers of derivative operators (derivative expansion), and (d) combinations thereof. The iterative structure of perturbation theory is fully reproduced to all orders, independently of R_k [47, 48]. The expansions (b)-(d) are genuinely non-perturbative and lead, via (3.2), to coupled flow equations for the coefficient functions. Convergence is then checked by extending the approximation to higher order.
- **Stability.** The stability and convergence of approximations is, additionally, controlled by R_k [43, 49]. Here, powerful optimisation techniques are available to maximise the physics content and the reliability of the result through well-adapted choices of R_k [43]-[51],[42]. These ideas have been explicitly tested in scalar [50] and gauge theories [51].
- **Symmetries.** Global or local (gauge) symmetries of the underlying theory can be expressed as Ward-Takahashi identities for n -point functions of Γ . Ward-Takahashi identities are maintained for all k if the insertion ΔS_k is compatible with the symmetry. In general, this is not the case for non-linear symmetries such as in non-Abelian gauge theories or gravity. Then the requirements of gauge symmetry for Γ are preserved by either (a) imposing modified Ward identities which ensure that standard Ward identities are obeyed in the physical limit when $k \rightarrow 0$, or by (b) introducing background fields into the regulator R_k and taking advantage of the background field method, or by (c) using gauge-covariant variables rather than the gauge fields or the metric field. For a detailed discussion of the benefits and shortcomings of these options, see [37]. For gravity, most implementations presently employ option (b).

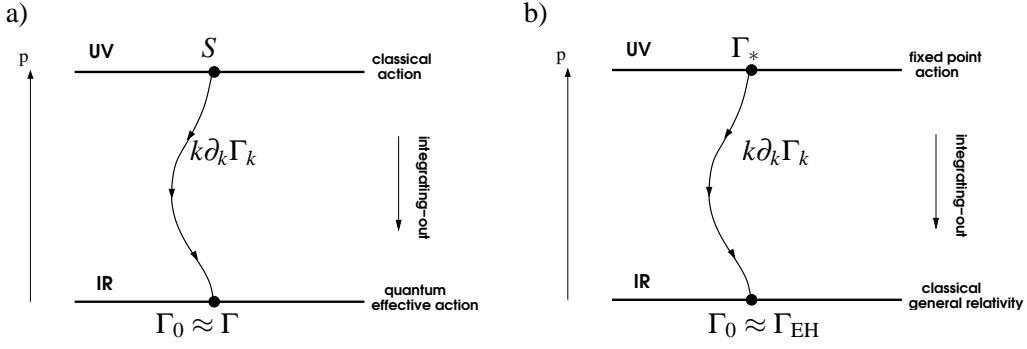


Figure 1: Wilsonian flows for scale-dependent effective actions Γ_k in the space of all action functionals (schematically); arrows point towards smaller momentum scales and lower energies $k \rightarrow 0$. a) Flow connecting a fundamental classical action S at high energies in the ultraviolet with the full quantum effective action Γ at low energies in the infrared (“top-down”). b) Flow connecting the Einstein-Hilbert action at low energies with a fundamental fixed point action Γ_* at high energies (“bottom-up”).

- **Integral representation.** The physical theory described by Γ can be defined without explicit reference to an underlying path integral representation, using only the (finite) initial condition Γ_Λ , and the (finite) flow equation (3.2)

$$\Gamma = \Gamma_\Lambda + \int_\Lambda^0 \frac{dk}{k} \frac{1}{2} \text{Tr} \left(\Gamma_k^{(2)} + R_k \right)^{-1} \partial_t R_k. \quad (3.3)$$

This provides an implicit regularisation of the path integral underlying (3.1). It should be compared with the standard representation for Γ via a functional integro-differential equation

$$e^{-\Gamma} = \int [D\varphi]_{\text{ren.}} \exp \left(-S[\phi + \varphi] + \int \frac{\delta\Gamma[\phi]}{\delta\phi} \cdot \varphi \right) \quad (3.4)$$

which is at the basis of *e.g.* the hierarchy of Dyson-Schwinger equations.

- **Renormalisability.** In renormalisable theories, the cutoff Λ in (3.3) can be removed, $\Lambda \rightarrow \infty$, and $\Gamma_\Lambda \rightarrow \Gamma_*$ remains well-defined for arbitrarily short distances. In perturbatively renormalisable theories, Γ_* is given by the classical action S , such as in QCD. In this case, illustrated in Fig. 1a), the high energy behaviour of the theory is simple, given mainly by the classical action, and the challenge consists in deriving the physics of the strongly coupled low energy limit. In perturbatively non-renormalisable theories such as quantum gravity, proving the existence (or non-existence) of a short distance limit Γ_* is more difficult. For gravity, illustrated in Fig. 1b), experiments indicate that the low energy theory is simple, mainly given by the Einstein Hilbert theory. The challenge consists in identifying a possible high energy fixed point action Γ_* , which upon integration matches with the known physics at low energies. In principle, any Γ_* with the above properties qualifies as fundamental action for quantum gravity. In non-renormalisable theories the cutoff Λ cannot be removed. Still, the flow equation allows to access the physics at all scales $k < \Lambda$ analogous to standard reasoning within effective field theory [31].

- Link with Callan-Symanzik equation.** The well-known Callan-Symanzik equation describes a flow $k \frac{d}{dk}$ driven by a mass insertion $\sim k^2 \phi^2$. In (3.2), this corresponds to the choice $R_k(q^2) = k^2$, which does not fulfill condition (i). Consequently, the corresponding flow is no longer local in momentum space, and requires an additional UV regularisation. This highlights a crucial difference between the Callan-Symanzik equation and functional flows (3.2). In this light, the flow equation (3.2) could be interpreted as a functional Callan-Symanzik equation with momentum-dependent mass term insertion [52].

Now we are in a position to implement these ideas for quantum gravity [5]. A Wilsonian effective action for gravity Γ_k should contain the Ricci scalar $R(g_{\mu\nu})$ with a running gravitational coupling G_k , a running cosmological constant Λ_k (with canonical mass dimension $[\Lambda_k] = 2$), possibly higher order interactions in the metric field such as powers, derivatives, or functions of *e.g.* the Ricci scalar, the Ricci tensor, the Riemann tensor, and, possibly, non-local operators in the metric field. The effective action should also contain a standard gauge-fixing term S_{gf} , a ghost term S_{gh} and matter interactions S_{matter} . Altogether,

$$\Gamma_k = \frac{1}{16\pi G_k} \int d^d x \sqrt{\det g_{\mu\nu}} [-R(g) + 2\Lambda_k + \dots + S_{\text{gf}} + S_{\text{gh}} + S_{\text{matter}}], \quad (3.5)$$

and explicit flow equations for the coefficient functions such as G_k , Λ_k or vertex functions, are obtained by appropriate projections after inserting (3.5) into (3.2). All couplings in (3.5) become running couplings as functions of the momentum scale k . For k much smaller than the d -dimensional Planck scale M_* , the gravitational sector is well approximated by the Einstein-Hilbert action with $G_k \approx G_{k=0}$, and similarly for the gravity-matter couplings. At $k \approx M_*$ and above, the RG running of gravitational couplings becomes important. This is the topic of the following sections.

A few technical comments are in order: To ensure gauge symmetry within this set-up, we take advantage of the background field formalism and add a non-propagating background field $\bar{g}_{\mu\nu}$ [5, 37], [53]-[56]. This way, the extended effective action $\Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}]$ becomes gauge-invariant under the combined symmetry transformations of the physical and the background field. A second benefit of this is that the background field can be used to construct a covariant Laplacean $-\bar{D}^2$, or similar, to define a mode cutoff at momentum scale $k^2 = -\bar{D}^2$. This implies that the mode cutoff R_k will depend on the background fields. The background field is then eliminated from the final equations by identifying it with the physical mean field. This procedure, which dynamically readjusts the background field, implements the requirements of “background independence” for quantum gravity. For a detailed evaluation of Wilsonian background field flows, see [56]. Finally, we note that the operator traces Tr in (3.2) are evaluated using heat kernel techniques. Here, well-adapted choices for R_k [43, 49] lead to substantial algebraic simplifications, and open a door for systematic fixed point searches, which we discuss next.

4. Fixed Points

In this section, we discuss the main picture in a simple approximation which captures the salient features of an asymptotic safety scenario for gravity, and give an overview of extensions. We consider the Einstein-Hilbert theory with a cosmological constant term and employ a momentum cutoff R_k with the tensorial structure of [7] and variants thereof, an optimised scalar cutoff

	θ'	θ''	ref.
a)	1.1 – 2.3	2.5 – 7.0	[7]
b)	1.4 – 2.0	2.4 – 4.3	[11]
c)	1.5 – 1.7	3.0 – 3.2	[13]

Table 1: The variation of $4d$ scaling exponents $\theta_{1,2} = \theta' \pm i\theta''$ in the Einstein-Hilbert theory with the gauge fixing parameter α and the cutoff function R_k . Results indicate the range covered under a) partial variation of both α and R_k , b) full α -variation with optimised R_k , and c) full R_k -variation and optimisation in Feynman gauge ($\alpha = 1$). In all cases the fixed point is stable. The variation with R_k , amended by stability considerations [43, 49], is weaker than the α -variation.

$R_k(q^2) \sim (k^2 - q^2)\theta(k^2 - q^2)$ [43, 49], and a harmonic background field gauge with parameter α in a specific limit introduced in [11]. The ghost wave function renormalisation is set to $Z_{C,k} = 1$, and the effective action is given by (3.5) with $S_{\text{matter}} = 0$. In the domain of classical scaling G_k and Λ_k are approximately constant, and (3.5) reduces to the conventional Einstein-Hilbert action in d euclidean dimensions. The dimensionless renormalised gravitational and cosmological constants are

$$g = k^{d-2} G_k \equiv k^{d-2} Z_G^{-1}(k) \bar{G}, \quad \lambda = k^{-2} \Lambda_k \quad (4.1)$$

where it is understood that g and λ depend on k . Then the coupled system of β -functions is

$$\partial_t \lambda \equiv \beta_\lambda(\lambda, g) = -2\lambda + \frac{g}{2} d(d+2)(d-5) - d(d+2)g \frac{(d-1)g + \frac{1}{d-2}(1-4\frac{d-1}{d}\lambda)}{2g - \frac{1}{d-2}(1-2\lambda)^2} \quad (4.2)$$

$$\partial_t g \equiv \beta_g(\lambda, g) = (d-2)g + \frac{2(d-2)(d+2)g^2}{2(d-2)g - (1-2\lambda)^2}. \quad (4.3)$$

We have rescaled $g \rightarrow g/c_d$ with $c_d = \Gamma(\frac{d}{2} + 2)(4\pi)^{d/2-1}$ to remove phase space factors. This does not alter the fixed point structure. The scaling $g \rightarrow g/(384\pi^2)$ reproduces the $4d$ classical force law in the non-relativistic limit [57]. For the anomalous dimension, we find

$$\eta(\lambda, g; d) = \frac{(d+2)g}{g - g_{\text{bound}}(\lambda)}, \quad g_{\text{bound}}(\lambda; d) = \frac{(1-2\lambda)^2}{2(d-2)}. \quad (4.4)$$

The anomalous dimension vanishes for vanishing gravitational coupling, and for $d = \pm 2$. At a non-trivial fixed point the vanishing of β_g implies $\eta_* = 2 - d$, and reflects the fact that the gravitational coupling is dimensionless in two dimensions. At $g = g_{\text{bound}}$, the anomalous dimension η diverges. The full flow (3.2) is finite (no poles) and well-defined for all k , as are the full β -functions derived from it. Therefore the curve $g = g_{\text{bound}}(\lambda)$ limits the domain of validity of the approximation.

We first consider the case $\lambda = 0$ and find two fixed points, the gaussian one at $g_* = 0$ and a non-gaussian one at $g_* = 1/(4d) < g_{\text{bound}}(0)$, which are connected under the renormalisation group. The universal eigenvalue $\partial \beta_g / \partial g|_* = -\theta$ at the fixed point are $\theta = 2 - d$ at the gaussian, and

$$\theta = 2d \frac{d-2}{d+2} \quad (4.5)$$

at the non-gaussian fixed point.

	i	θ'	θ''	θ_3	θ_4	ref.
a)	2	2.1 – 3.4	3.1 – 4.3	8.4 – 28.8	–	[8]
b)	2	1.4	2.8	25.6	–	[16]
c)	2	1.7	3.1	3.5	–	[16]
d)	2	1.4	2.3	26.9	–	[17]
e)	3	2.7	2.3	2.1	–4.2	[17]
f)	4	2.9	2.5	1.6	–3.9	[17]
g)	5	2.5	2.7	1.8	–4.4	[17]
h)	6	2.4	2.4	1.5	–4.1	[17]

Table 2: The variation of $4d$ scaling exponents with the order i of the expansion, including the invariants $\int \sqrt{\det g_{\mu\nu}}$ and $\int \sqrt{\det g_{\mu\nu}} R^i$ from $i = 2 \dots 6$, using a) Feynman gauge under partial variations of R_k ; b), c) Feynman gauge and optimised R_k ; d)-h) Landau-deWitt background field gauge with optimised R_k .

Next we allow for a non-vanishing dynamical cosmological constant term $\Lambda_k \neq 0$ in (3.5). The coupled system exhibits the gaussian fixed point $(\lambda_*, g_*) = (0, 0)$ with eigenvalues -2 and $d - 2$. Non-trivial fixed points of (4.2) and (4.3) are found as follows: For non-vanishing $\lambda \neq 0$, we find a non-trivially vanishing β_g for $g = g_0(\lambda)$, with $g_0(\lambda) = \frac{1}{4d}(1 - 2\lambda)^2$. Note that $g_0(\lambda) < g_{\text{bound}}(\lambda)$ for all $d > 2$. Evaluating (4.2) for $g = g_0(\lambda)$, we find $\beta_\lambda(\lambda, g_0(\lambda)) = \frac{1}{4}(d - 4)(d + 1)(1 - 2\lambda)^2 - 2d\lambda + \frac{d}{2}$. The first term vanishes in $d = 4$ dimensions. Consequently, we find a unique ultraviolet fixed point $\lambda_* = \frac{1}{4}$ and $g_* = \frac{1}{64}$. In $d > 4$, the vanishing of β_λ leads to two branches of real fixed points with either $\lambda_* > \frac{1}{2}$ or $\frac{1}{2} > \lambda_* > 0$. Only the second branch corresponds to an ultraviolet fixed point which is connected under the renormalisation group with the correct infrared behaviour. This can be seen as follows: At $\lambda = \frac{1}{2}$, we find $\eta = d + 2 > 0$. On a non-gaussian fixed point, however, $\eta < 0$. Furthermore, g cannot change sign under the renormalisation group flow (4.3). Consequently, η cannot change sign either. Hence, to connect a fixed point at $\lambda > \frac{1}{2}$ with the gaussian fixed point at $\lambda = 0$, η would have to change sign at least twice, which is impossible. Therefore, we have a unique physically relevant solution given by

$$\lambda_* = \frac{d^2 - d - 4 - \sqrt{2d(d^2 - d - 4)}}{2(d - 4)(d + 1)}, \quad g_* = \frac{(\sqrt{d^2 - d - 4} - \sqrt{2d})^2}{2(d - 4)^2(d + 1)^2}. \quad (4.6)$$

An interesting property of this system is that the scaling exponents θ_1 and θ_2 – the eigenvalues of the stability matrix $\partial\beta_i/\partial g_j$ ($g_1 \equiv g, g_2 \equiv \lambda$) at the fixed point – are a complex conjugate pair, $\theta_{1,2} = \theta' \pm \theta''$ with $\theta' = \frac{5}{3}$ and $\theta'' = \frac{\sqrt{167}}{3}$ in four dimensions. The reason for this is that the stability matrix, albeit real, is not symmetric. Complex eigenvalues reflect that the interactions at the fixed point have modified the scaling behaviour of the underlying operators $\int \sqrt{\det g_{\mu\nu}} R$ and $\int \sqrt{\det g_{\mu\nu}}$. This pattern changes for lower and higher dimensions, where eigenvalues are real [15].

At this point it is important to check whether the fixed point structure and the scaling exponents depend on technical parameters such as the gauge fixing procedure or the momentum cutoff function R_k , see Tab. 1 and 2. For the Einstein-Hilbert theory in $4d$, results are summarised in Tab. 1. The α -dependence of the β -functions is fairly non-trivial, *e.g.* [5, 7, 11]. It is therefore noteworthy that scaling exponents only depend mildly on variations thereof. Furthermore, the R_k -dependence

d	5	6	7	8	9	10
θ'	2.69 – 3.11	4.26 – 4.78	6.43 – 6.89	8.19 – 9.34	10.5 – 12.1	13.1 – 15.2
θ''	4.54 – 5.16	6.52 – 7.46	8.43 – 9.46	10.3 – 11.4	12.1 – 13.2	13.9 – 15.0
$ \theta $	5.31 – 6.06	7.79 – 8.76	10.4 – 11.6	13.2 – 14.7	16.1 – 17.9	19.1 – 21.3

Table 3: The variation of scaling exponent with dimensionality, gauge fixing parameters (using either Feynman gauge, or harmonic background field gauge with $0 \leq \alpha \leq 1$), and the regulator R_k ; data from [13, 14]. The R_k -variation, covering various classes of cutoff functions, is on the level of a few percent and smaller than the variation with α .

is smaller than the dependence on gauge fixing parameters. We conclude that the fixed point is fully stable and R_k -independent for all technical purposes, with the presently largest uncertainty arising through the gauge fixing sector. In Tab. 2, we discuss the stability of the fixed point under extensions beyond the Einstein-Hilbert approximation, including higher powers of the Ricci scalar both in Feynman gauge [8, 16] and in Landau-DeWitt gauge [17]. Once more, the fixed point and the scaling exponents come out very stable. Furthermore, starting from the operator $\int \sqrt{\det g_{\mu\nu}} R^3$ and higher, couplings become irrelevant with negative scaling exponents [17, 18]. This is an important first indication for the set of relevant operators at the UV fixed point being finite. Finally, we mention that the stability of the fixed point under the addition of non-interacting matter fields has been confirmed in [10].

5. Extra Dimensions

It is interesting to discuss fixed points of quantum gravity specifically in more than four dimensions. The motivation for this is that, first of all, the critical dimension of gravity – the dimension where the gravitational coupling has vanishing canonical mass dimension – is two. For any dimension above the critical one, the canonical dimension is negative. Hence, from a renormalisation group point of view, the four-dimensional theory is not special. Continuity in the dimension suggests that an ultraviolet fixed point, if it exists in four dimensions, should persist towards higher dimensions. More generally, one expects that the local structure of quantum fluctuations, and hence local renormalisation group properties of a quantum theory of gravity, are qualitatively similar for all dimensions above the critical one, modulo topological effects for specific dimensions. Secondly, the dynamics of the metric field depends on the dimensionality of space-time. In four dimensions and above, the metric field is fully dynamical. Hence, once more, we should expect similarities in the ultraviolet behaviour of gravity in four and higher dimensions. Interestingly, this pattern is realised in the results [11], see the analytical fixed point (4.6). An extended systematic search for fixed points in higher-dimensional gravity for general cutoff R_k has been presented in [13, 14], also testing the stability of the result against variations of the gauge fixing parameter (see Tab. 3). The variation with R_k , amended by stability considerations, is smaller than the variation with α . We conclude from the weak variation that the fixed point indeed persists in higher dimensions. Further studies including higher derivative operators confirm this picture [16]. This structural stability also strengthens the results in the four-dimensional case, and supports the view introduced above. A

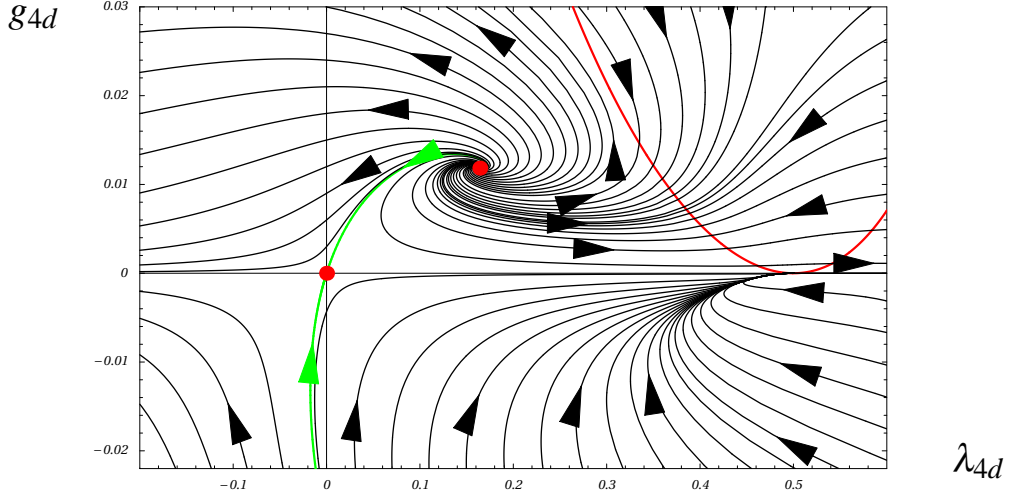


Figure 2: The phase diagram for the running gravitational coupling g_{4d} and the cosmological constant λ_{4d} in four dimensions. The Gaussian and the ultraviolet fixed point are indicated by dots (red). The separatrix connects the two fixed points (full green line). The full (red) line indicates the bound $g_{\text{bound}}(\lambda)$ where $1/\eta = 0$. Arrows indicate the direction of the RG flow with decreasing $k \rightarrow 0$.

phenomenological application of these findings in low-scale quantum gravity is discussed below (see Sec. 8).

6. Phase Diagram

In this section, we discuss the main characteristics of the phase portrait of the Einstein-Hilbert theory [9, 11] (see Fig. 2). Finiteness of the flow (3.2) implies that the line $1/\eta = 0$ cannot be crossed. Slowness of the flow implies that the line $\eta = 0$ can neither be crossed (see Sec. 4). Thus, disconnected regions of renormalisation group trajectories are characterised by whether g is larger or smaller g_{bound} and by the sign of g . Since η changes sign only across the lines $\eta = 0$ or $1/\eta = 0$, we conclude that the graviton anomalous dimension has the same sign along any trajectory. In the physical domain which includes the ultraviolet and the infrared fixed point, the gravitational coupling is positive and the anomalous dimension negative. In turn, the cosmological constant may change sign on trajectories emanating from the ultraviolet fixed point. Some trajectories terminate at the boundary $g_{\text{bound}}(\lambda)$, linked to the present approximation. The two fixed points are connected by a separatrix. The rotation of the separatrix about the ultraviolet fixed point reflects the complex nature of the eigenvalues. At $k \approx M_{\text{Pl}}$, the flow displays a crossover from ultraviolet dominated running to infrared dominated running. The non-vanishing cosmological constant modifies the flow mainly in the crossover region rather than in the ultraviolet. In the infrared limit, the separatrix leads to a vanishing cosmological constant $\Lambda_k = \lambda_k k^2 \rightarrow 0$ and is interpreted as a phase transition boundary between cosmologies with positive or negative cosmological constant at large distances. Trajectories in the vicinity of the separatrix lead to a positive cosmological constant at large scales and are, therefore, candidate trajectories for realistic cosmologies [60]. This

picture agrees very well with numerical results for a sharp cut-off flow [9], except for the location of the line $1/\eta = 0$ which is non-universal. Similar phase diagrams are found in higher dimensions [13, 14].

7. Lattice

Lattice implementations for gravity in four dimensions have been put forward based on Regge calculus techniques [27]-[28] and causal dynamical triangulations [29]. In the Regge calculus approach, a critical point which allows for a lattice continuum limit has been given in [27] using the Einstein Hilbert action with fixed cosmological constant. A scaling exponent has been measured in the four-dimensional simulation based on varying Newton's coupling to the critical point, with $\partial_g \beta_g|_* = -\frac{1}{\nu}$. The result reads $\nu \approx \frac{1}{3}$, and should be contrasted with the RG result $\nu = 1/\theta = \frac{3}{8}$ [11] as discussed in Sec. 4. In the large-dimensional limit, geometrical considerations on the lattice lead to the estimate $\nu = \frac{1}{d-1}$ [28], a behaviour which is in qualitative agreement with the explicit RG fixed point result $\nu = \frac{1}{2d}$ in the corresponding limit, see (4.5).

Within the causal dynamical triangulation approach, global aspects of quantum space-times have been assessed in [29]. There, the effective dimensionality of space-time has been measured as a function of the length scale by evaluating the return probability of random walks on the triangulated manifolds. The key result is that the measured effective dimensionality displays a cross-over from $d \approx 4$ at large scales to $d \approx 2$ at small scales of the order of the Planck scale. This behaviour compares nicely with the cross-over of the graviton anomalous dimension η under the renormalisation group (see Sec. 2), and with renormalisation group studies of the spectral dimension (see [2, 3, 58]). These findings corroborate the claim that asymptotically safe quantum gravity behaves, in an essential way, two-dimensional at short distances.

8. Phenomenology

The phenomenology of a gravitational fixed point covers the physics of black holes [59], cosmology [60, 61, 62], modified dispersion relations [63], and the physics at particle colliders [64, 65, 66]. In this section we concentrate on the later within low-scale quantum gravity models [67, 68]. There, gravity propagates in $d = 4 + n$ dimensional bulk whereas matter fields are confined to a four-dimensional brane. The four-dimensional Planck scale $M_{\text{Pl}} \approx 10^{19}$ GeV is no longer fundamental as soon as the n extra dimensions are compact with radius $\sim L$. Rather, the $d = 4 + n$ -dimensional Planck mass M_* sets the fundamental scale for gravity, leading to the relation $M_{\text{Pl}}^2 \sim M_*^2 (M_* L)^n$ for the four-dimensional Planck scale. Consequently, M_* can be significantly lower than M_{Pl} provided $1/L \ll M_*$. If M_* is of the order of the electroweak scale, this scenario lifts the hierarchy problem of the standard model and opens the exciting possibility that particle colliders could establish experimental evidence for the quantisation of gravity [69, 70, 71].

The renormalisation group running of the gravitational coupling in this scenario has been studied in [13, 14, 64] and is summarised in Fig. 3. The main effects due to a fixed point at high energies set in at momentum scales $k \approx M_*$, where the gravitational coupling displays a cross-over from perturbative scaling $G(k) \approx \text{const.}$ to fixed point scaling $G(k) \approx g_* k^{2-d}$. Therefore we expect

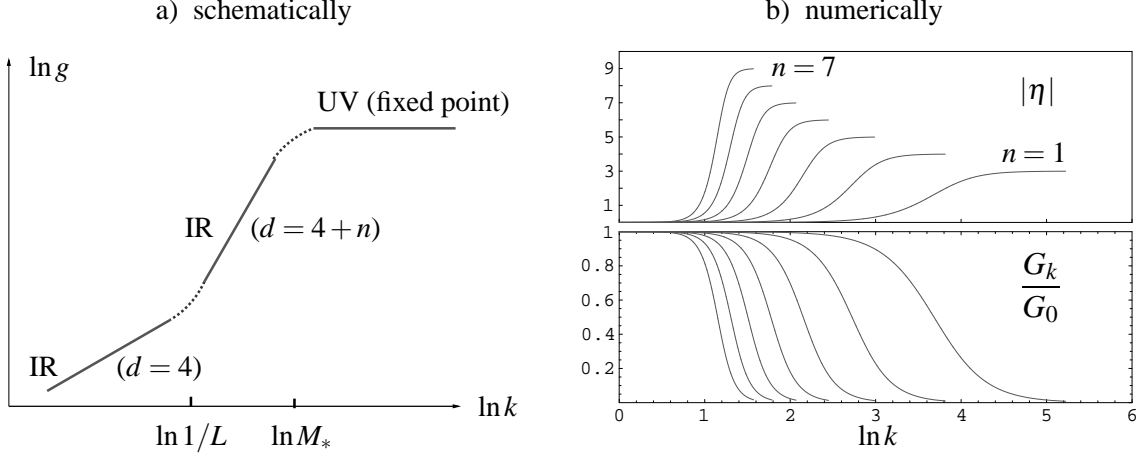


Figure 3: The scale-dependence of the gravitational coupling in a scenario with large extra dimensions of size $\sim L$ with fundamental Planck scale M_* and $M_*L \gg 1$. The fixed point behaviour in the deep ultraviolet enforces a softening of gravitational coupling (see text). a) In the infrared (IR) regime where $|\eta| \ll 1$, the coupling $g = G_k k^{d-2}$ displays a crossover from 4-dimensional to $(4+n)$ -dimensional classical scaling at $k \approx 1/L$. The slope $d \ln g / d \ln k \approx d - 2$ measures the effective number of dimensions. At $k \approx M_*$, a classical-to-quantum crossover takes place from $|\eta| \ll 1$ to $\eta \approx 2 - d$ (schematically). b) Classical-to-quantum crossover at the respective Planck scale for G_k and the anomalous dimensions η from numerical integrations of the flow equation; $d = 4 + n$ dimensions and $n = 1, \dots, 7$ (from right to left).

that signatures of this cross-over should be visible in scattering processes at particle colliders as long as these are sensitive to momentum transfers of the order of M_* .

We illustrate this at the example of dilepton production through virtual gravitons at the Large Hadron Collider (LHC) [64]. To lowest order in canonical dimension, the dilepton production amplitude is generated through an effective dimension-8 operator in the effective action, involving four fermions and a graviton [69]. Tree-level graviton exchange is described by an amplitude $A = S \cdot T$, where $T = T_{\mu\nu} T^{\mu\nu} - \frac{1}{2+n} T_\mu^\mu T_\nu^\nu$ is a function of the energy-momentum tensor, and

$$S = \frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{1}{M_*^4} \int_0^\infty \frac{dm}{M_*} \left(\frac{m}{M_*} \right)^{n-1} \mathcal{G}(s, m) \quad (8.1)$$

is a function of the scalar part $\mathcal{G}(s, m)$ of the graviton propagator [69, 72]. The integration over the Kaluza-Klein masses m , which we take as continuous, reflects that gravity propagates in the higher-dimensional bulk. If the graviton anomalous dimension is small, the propagator is well approximated by $\mathcal{G}(s, m) = (s + m^2)^{-1}$. This propagator is used within effective theory settings, and applicable if the relevant momentum transfer is $\ll M_*$. In this case, (8.1) is ultraviolet divergent for $n \geq 2$ due to the Kaluza-Klein modes [69]. Regularisation by an UV cutoff leads to a power-law dependence of the amplitude $S \sim M_*^{-4} (\Lambda/M_*)^{n-2}$ on the cutoff Λ . In a fixed point scenario, the behaviour of S is improved due to the non-trivial anomalous dimension η of the graviton, e.g. (4.4). Evaluating η at momentum scale $k^2 \approx s + m^2$, we are lead to the dressed propagator $\mathcal{G}(s, m) \approx \frac{M_*^{n+2}}{(s+m^2)^{n/2+2}}$ in the vicinity of an UV fixed point. The central observation is that (8.1) becomes finite even in the UV limit of the integration. An alternative matching has been proposed in [65], based on the substitution $G(k) \rightarrow G(\sqrt{s})$ in (8.1), setting $G = M_*^{2-d}$. In that case, however,

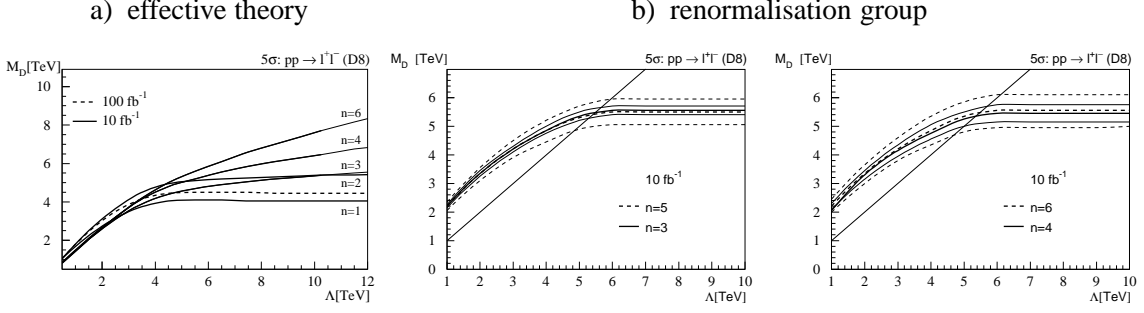


Figure 4: The 5σ discovery contours in M_D at the LHC ($d = 4 + n$), as a function of a cutoff Λ on E_{parton} for an assumed integrated luminosity of 10fb^{-1} (100fb^{-1}). a) Effective theory: the sensitivity to the cutoff Λ is reflected in the M_D contour; plot from [72]. b) Renormalisation group: the limit $\Lambda \rightarrow \infty$ can be performed, and the leveling-off at $M_D \approx \Lambda$ reflects the gravitational fixed point, thin lines show a $\pm 10\%$ variation in the transition scale; plot from [64].

(8.1) remains UV divergent due to the Kaluza-Klein modes. We conclude that the large anomalous dimensions in asymptotically safe gravity provides for a finite dilepton production rate.

In Fig. 4 we show the discovery potential in the fundamental Planck scale at the LHC, and compare effective theory studies [72] with a gravitational fixed point [64]. In either case the minimal signal cross sections have been computed for which a 5σ excess can be observed, taking into account the leading standard model backgrounds and assuming statistical errors. This translates into a maximum reach M_D for the fundamental Planck scale M_* . To estimate uncertainties in the RG set-up, we allow for a 10% variation in the scale where the transition towards fixed point scaling sets in. Consistency is checked by introducing an artificial cutoff Λ on the partonic energy [72], setting the partonic signal cross section to zero for $E_{\text{parton}} > \Lambda$. It is nicely seen that M_D becomes independent of Λ for $\Lambda \rightarrow \infty$ when fixed point scaling is taken into account.

9. Conclusions

The asymptotic safety scenario offers a genuine path towards quantum gravity in which the metric field remains the fundamental carrier of the physics even in the quantum regime. We have reviewed the ideas behind this set-up in the light of recent advances based on renormalisation group and lattice studies. The stability of renormalisation group fixed points and scaling exponents detected in four- and higher-dimensional gravity is remarkable, strongly supporting this scenario. Furthermore, underlying expansions show good numerical convergence, and uncertainties which arise through approximations are moderate. If the fundamental Planck scale is as low as the electroweak scale, signs for the quantisation of gravity and asymptotic safety could even be observed in collider experiments. It is intriguing that key aspects of asymptotic safety are equally seen in lattice studies. It will be interesting to evaluate these links more deeply in the future. Finally, asymptotically safe gravity is a natural set-up which leads to classical general relativity as a “low energy phenomenon” of a fundamental quantum field theory in the metric field.

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