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Where are the particles created in Black Hole evaporation?

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The origin of the particles emitted in black hole evaporation is a long standing mystery. A naive examination of Hawking's derivation leads one to say that they are created extremely near the horizon (much less than a Planck distance away) at exponentially high energies. This paper presents the numerical investigation of a dumb hole model which suggests that the particles are actually created at very low energies.

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1. Introduction

One of the most mysterious features of black holes is the thermal radiation emitted continuously after a black hole has formed [1]. This radiation represents some sort of quantum instability of the horizon, but exactly what kind of instability is still a mystery. Since the temperature of a black hole is a function of the energy (mass), this implies an entropy for black hole, an entropy which is necessary to preserve the second law of thermodynamics when black holes are involved, but also leads to many mysteries. Where is the radiation created? Is that entropy fundamental or is it, a la Boltzmann, a statistical attribute of the black hole?

An uncomfortable aspect of Hawking's derivation is that the radiation emitted at a time t after the black hole has formed originates, in his calculations, from the zero point fluctuations in the field at frequencies of $Me^{t/4M}$ in the early stages before the black hole has formed. Thus if we consider the radiation emitted, say, 1 second after a solar mass black hole has formed, this radiation originates from quantum fluctuations in the vacuum before the black hole formed, with an energy of e^{10^5} times the mass of the whole universe. For such quantum fluctuations, even if they are vacuum fluctuations, we do not expect a simple quantum field theory in a fixed background spacetime (the arena in which Hawking did his calculation) to be an adequate approximation.

This has been noticed since shortly after the original paper, but since the Hawking's results are so appealing, it was felt that the calculation, though obviously unphysical in detail, was surely right in principle. In 1981 [2], I noticed that sound waves in a background irrotational flow where the fluid velocities exceeded the velocity of sound were mathematically exact analogs of scalar fields in a black hole background spacetime, at least in the hydrodynamic approximation. Furthermore the quantization of those sound waves (phonons) corresponded exactly to the quantization of a scalar field theory in a background black hole spacetime. Thus one expects and, if one naively carries out the analog of Hawking's calculation, one gets that the horizon would emit sound quanta with some temperature. In this sonic case, that temperature is given by

$$T = \frac{1}{4\pi c(x)} \frac{d(c(x)^2 - v(x)^2)}{dx}\bigg|_{v(x)=c(x)} \tag{1.1}$$

where $c(x)$ is the (perhaps spatially dependent) velocity of sound, and the expression is evaluated at the point where the velocity of the fluid equals the velocity of sound. This sonic analog of a black hole I called a dumb hole– from the phrase "deaf and dumb"– since such structures were incapable of emitting sound. And the calculation of this temperature suffers from the same exponential dependence with time of the initial vacuum fluctuations. However, unlike the case for the black hole, we know that hydrodynamics breaks down at short wavelengths. A fluid has a natural high frequency and wave-number cutoff– at a minimum at the inter-atomic spacing. I.e., such a system has no ultra-high frequency problem.

After Jacobson [4] pointed out that the dominant effect of the inter-atomic spacing was to change the dispersion relation (relation between frequency and wave-number) away from linear, I [5] was able to numerically model the modes in such a background flow, and show, through the positive and negative norm mixing of the modes, that these dumb holes would emit thermal radiation even though those exponentially high frequencies were absent. The thermal effect was robust against high frequency changes in the theory. For a review of the status of the use of such dumb holes to understand black hole evaporation, see for example Carlos Barceló et al. [6]

These dumb holes also present us with the opportunity of asking where the particles are actually created. Let us imagine a fluid containing sound waves with a dispersion relation as in Figure 1 and 2. This dispersion relation is such that there exists some common group and phase velocity at the lowest frequencies, as $k \to 0$. The dispersion relation is crafted so that at intermediate frequencies, the group and phase velocities are again equal, but at a common velocity that is smaller than the low frequency one is. Finally at the highest wave-numbers, the dispersion relation drops so that the group velocity goes to zero. While such a dispersion relation may be artificial, it does allow us to ask which of the various velocities determine the temperature of the emitted radiation.

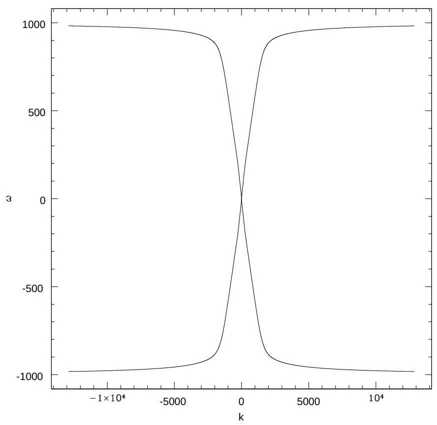

Figure 1: The dispersion relation as a function of k and ω. At high wave-numbers the relation flattens out, but this regime is never probed by the wave-packets.

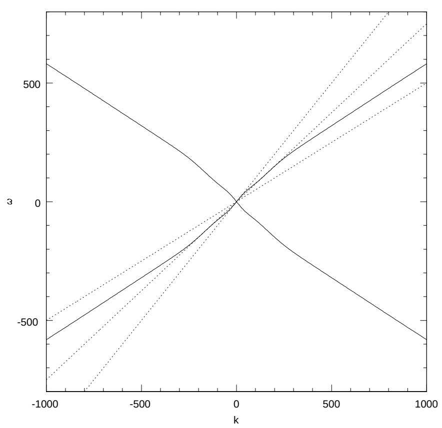

Figure 2: The central part of the dispersion relation of Figure 1. Included are dotted lines with slope of 1, 0.75 and 0.5.

We now design the background flow of the fluid so that the derivative of the velocity, i.e., the temperature, differs at the different possible horizons (the places where the fluid velocity equals the group, or phase velocity at different frequencies). I.e., at the point where the fluid flow equals that intermediate common group and phase velocity, the background flow is taken to have a smaller derivative than it has at the point in the flow where the velocity equals the low wave-number group and phase velocity. To be definite, I choose the fluid flow so that the the derivative is given by

$$\frac{dv}{dx} \propto \left(1 - \left(\frac{v-1}{\Delta v}\right)^2\right) \tag{1.2}$$

where Δv is half the difference between the maximum (inside the horizons) and minimum (outside) of the velocity of the fluid, and the velocity of sound (at the lowest wave-numbers) has been taken equal to 1.

Let us follow the evolution of a wave-packet which eventually becomes a thermal wave-packet emitted by the dumb hole. At early times it is high frequency packet whose group velocity is much smaller than the fluid flow. It is an outgoing packet, but because of its low group velocity it is dragged by the fluid flow toward the horizon. Nearing the horizon, it is stretched by the differential flow of the fluid, and its wave number decreases. It hugs the horizon, its wave-number steadily decreasing until finally its wave-length is of the order of the size of the dumb hole, the group velocity is greater than the velocity of the fluid, and it escapes from the hole. But which horizon? Is it the point where the velocity of the fluid equals the phase, the group, or some other velocity of the wave? And, since the phase and group velocities depend on the wave-number, at which of the ever changing wave-numbers is it the phase/group/other velocity of the packet which determines the horizon location?

Since the fluid flow has a changing velocity and a changing derivative of velocity near the $v = 1$ horizon, each of those possible horizons has a different rate of expansion – of stretching the wave-packet. Each in principle therefore has a different temperature. Which of the temperatures determines the temperature of the escaping radiation? Is it an average of all the temperatures that the wave-packet has encountered at the different horizons it has met before escaping? Is it the horizon with its temperature (proportional to $\frac{dv}{dr}$) where the high frequency vacuum fluctuations first encounter the horizon and begin to be red-shifted? Is it some intermediate horizon that the wave-packet encounters during its red-shifting? Or is it the horizon for the wave when it finally leaves the horizon? If it is the temperature at the lowest frequencies when the wave finally escapes the dumb hole, then one can argue that the properties of the horizon, of the dumb hole, at higher frequencies, are irrelevant. The independence of the temperature on the prior horizons the wave-packet met would then indicate that they had no effect on the particle creation rate. The thermal particles are created at those lowest frequencies. The particle creation process is definitely a low-frequency process.

If the group and phase velocities change, they cannot be equal to each other at all wave-numbers. If

$$\frac{d\omega}{dk} = \frac{\omega}{k} \tag{1.3}$$

then $\omega = vk$ for some constant value of v, and the group and phase velocities are constant.

Thus the lowest wave-number wave-packets escaping the hole must have gone though a frequency regime where the group and phase velocities were not equal. For these frequencies, what place in the fluid flow corresponds to the horizon? Is the effective horizon which determines the temperature of the emitted radiation located where the group velocity equals the fluid velocity, where the phase velocity equals the fluid velocity, or some other place entirely?

2. Numerical model

In order to investigate this I used the same numerical procedure outlined in my 1995 paper [5]. The fluid flow is assumed to be cyclic –i.e., the flow and all modes are assumed to have periodic boundary conditions. Thus the flow which establishes a black hole horizon in some place, must also establish a white hole horizon (a horizon out of which modes can flow, but into which no mode can penetrate.) Such cyclic flows are known to have instabilities, but I never carry the calculations far enough to trigger those instabilities [7] (which arise from modes which circumnavigate the cyclic system, and interfere with themselves or bounce back and forth between the white and black hole horizons.) The procedure I follow is to start with a low wave number wave-packet entirely confined to the region in which the fluid flow is constant. That packet is taken to be an outgoing packet (i.e., travelling away from the black hole horizon) and to have a sufficiently low wave-number that its group velocity is larger than the fluid velocity in the region of the packet. The equations are now solved backwards in time. At earlier times, that packet came from near the horizon. As we go further back in time it is squeezed up against the horizon, and its wave-number becomes higher and higher. Eventually, the wave-number becomes sufficiently high that the modes making up the packet have a group velocity much less than the fluid flow velocity. Going even further back in time, that packet now moves away from the horizon (corresponding to that mode being dragged toward the horizon by the fluid flow if one looks at the mode forward in time) until it is again entirely contained in the region again where the velocity of the fluid is a constant.

The equation obeyed by these modes is assumed to be of the form

$$(\partial_t - \partial_x v(x))\,(\partial_t - v(x)\partial_x)\,\phi(t,x) + F^2(i\partial_x)\phi(t,x) = 0 \qquad (2.1)$$

where all derivatives operate on everything to their right in a term, and $\omega = \pm F(k)$ is the dispersion relation of the wave if the background velocity is zero. The functional form for F that I take has both the group ($\frac{dF}{dk}$) and phase velocity ($\frac{F(k)}{k}$) equal to 1 at the lowest frequencies and wave numbers. There is then a transition to a regime where the group and phase velocities are again almost equal with a value of both 0.75. Finally, the dispersion relation then makes a transition to a regime where the slope is 0.5 and then at very large wave-numbers (which are never achieved in the evolution of the wave-packet), the group velocity drops to zero.

Note that the equation is time independent, and ω is thus conserved. In principle, one could solve this by separation of variables, and have a single ordinary differential equation in x, but for generic F it would be of infinite order. Thus the resulting equation in x is in general impossible to solve, especially for the dispersion relation I use. Corley and Jacobson [3] solved this ODE numerically for a simple dispersion relation where F^2 was a low order (quadratic) polynomial in ∂_x.

In a regime in which $v(x)$ is a constant, there is a definite relation between ω, the temporal frequency, and k the wave number. I.e., if the wave is entirely in a regime where $v(x)$ is constant,

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ω and k obey

$$(\omega - vk)^2 = F^2(k) \tag{2.2}$$

Thus in such a regime one can determine ω without solving the equations for all time and taking the temporal Fourier transform.

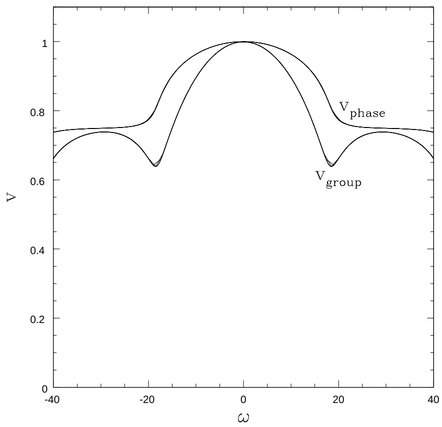

Figure 3: The group and phase velocity of the dispersion relation as a function of the frequency in the constant part of the velocity profile. Note that as expected the phase velocity is monotonic, while the group velocity is not. These curves include the parts where the group and phase velocities are both 1 and both near 0.75

In Figure 3, the group and phase velocities as seen by a stationary observer for a flowing fluid, $\frac{\partial \omega}{\partial k} + v$ and $\frac{\omega}{k} + v$ are plotted as a function of ω for the particular dispersion relation I chose, in the low velocity regime for the fluid.

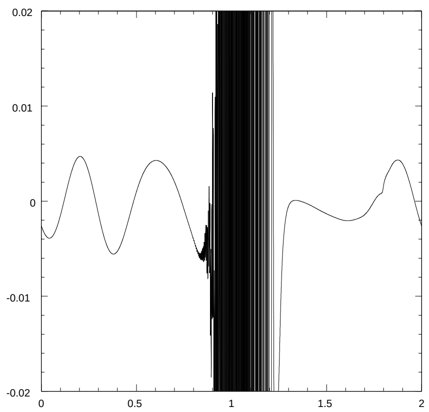

Figure 4: A blow-up by a factor of 100 of of the wave at time $t = -2$ showing the low frequency back-scatter wave coming off the horizon.

Let me define the wave as a left travelling wave (away from the horizon) so that the relation between ω and k in the low velocity regime is

$$\omega - vk = +F(k). \tag{2.3}$$

A right travelling wave would have the opposite sign of F in the constant v region. The packets I start the calculation with are left travellers. What is somewhat surprizing is that the direction of travel of these waves seems to be almost conserved. There is some evidence in the numerical calculations that left movers are turned into right movers by the equation of motion to a very small degree. In Figure 4 we have the plot at the time -2.0. with the y axis expanded by a factor of 100. We see the long wavelength train which is a right-moving wave of relatively long wavelength. This appears to have been generated as the left-going pulse shifted up in frequency at the horizon. The creation of such right-movers here corresponds to the reflection from the region of changing velocity near the dumb hole horizon. This corresponds to the reflection from the curvature of near a black hole, leading to a non-zero albedo for waves incident on a black hole. Some insight into why this occurs can be gleaned if we compare the equation of motion I used with

$$(\partial_t - \partial_x v(x) + F(\partial_x))(\partial_t - v(x)\partial_x - F(\partial_x))\phi(t,x) = 0 \tag{2.4}$$

(where derivatives operate on everything to their right). This equation differs from the above by a term

$$-(F(\partial_x)v(x)\partial_x - \partial_x v(x)F(\partial_x))\phi(t,x) \tag{2.5}$$

where again all derivatives operate on everything to their right. This is non-zero only if $F(\partial_x)$ is not linear in ∂_x, and such that the higher derivatives of $v(x)$ are non-negligible at those wave numbers. I.e., one would expect the scattering of outgoing into ingoing waves to be small, and numerically they are. However, that small scattering does seem to upset the calculation of the temperature at the lowest frequencies.

In order to determine the particle creation rate, one starts (at late time) with a “positive norm" wave-packet. In this context this will be a wave-packet with low wave-number entirely contained in the regime where $v(x) = constant$. Furthermore, the wave is assumed to be such that $\omega - vk = -F(k)$ and such that $\omega > 0$. These modes have positive norm where the norm is defined via the conserved current for the scalar field given by

$$\langle\phi_2,\phi_1\rangle = \frac{i}{2}\int \phi_2^*\left(\overleftrightarrow{\partial_t - v\partial_x}\right)\phi_1(t,x)dx \tag{2.6}$$

One now evolves the system backwards in time until the wave-packet is again entirely contained within a regions where $v(x)$ is a constant. One can again determine ω in terms of k. However, one will now discover that the constant ω modes which make up the mode are not entirely positive norm. There will be components with $k < 0$, which can easily be seen to also have negative norm. This negative norm component is a measure of how many particles would be produced by the hole in the “final mode" in the quantum regime. I.e., it is directly related to the Bogoliubov coefficients of the transformation from initial to final waves.

Note that since ω is conserved, one can take any initial wave-packet which contains a broad range of values of ω and determine the Bogoliubov coefficient at each frequency contained in that packet.

At each frequency, since the evolution of the field is linear, the density matrix is Gaussian, and thus is thermal. The ratio between the positive and negative norm components at each frequency is thus a measure of the temperature of that mode. In particular the ratio of the norm of the negative to positive norm components is just $e^{-\frac{\omega}{kT}}$ where T is the temperature corresponding to this particular mode.

3. Numerical technique

As in my previous paper [5], the problem is solved by a combination of finite differencing and Fourier transform. At each time the the term with F is evaluated in the Fourier regime, using fast Fourier transforms and multiplication by $F^2(k)$ which is defined to be real. The convective derivatives on the other had are solved by an implicit solver scheme to ensure stability of the evolution. Since the derivative is sufficiently simple, this implicit solver is no more expensive than an explicit scheme. Furthermore the periodic boundary conditions are solved by an algebraic matching, which again is trivial because of the linearity of the equations. I have never seen any hint of an instability due to numerical issues in any of the evolutions I have carried out, no matter what the dispersion relation, and no matter what the form of the velocity field of the fluid.

The dispersion relation I chose was given by

$$F(k) = \left[\left(\frac{k}{(1+k^8)^{\frac{1}{8}}} \right) + .7 \left(\frac{k}{(1+(\frac{k}{5})^8)^{\frac{1}{8}}} \right) \left(1. - \frac{1.}{(1+k^8)^{\frac{1}{8}}} \right) \right] \left(\frac{3}{4} + \frac{1}{4} e^{-16k^2} \right) \tag{3.1}$$

and the velocity was given by

$$v(x) = 1 - .43 \tanh\left(50(\sin(2\pi x) + .6)\right) \tag{3.2}$$

where the coordinate x is defined so that $x \equiv x+1$. The spatial derivative of the velocity as a function of the velocity is, to a very good approximation

$$\frac{dv(x)}{dx} = 100\pi(.8)(.43) \left(1 - \left(\frac{v(x)-1}{.43} \right)^2 \right) \tag{3.3}$$

for $|v(x)-1| < .43$. (This assumes that $\cos(2\pi x)$ does not vary much over the horizon, which is a good approximation since $\frac{dv}{dx} \approx 100$ at the horizon.)

I.e., the temperature, proportional to $\frac{dv(x)}{dx}$ at $v(x) = c$ changes significantly as c changes.

In Figure 5a-d we have four frames of the real part of the field as we propagate it backward in time. Since the mode has a wave-number range which encompasses significant changes in the dispersion relation, we can see the effect of the dispersion on the shape of the wave-packet when it was closer to the horizon. Further back in time, the wave has been processed by the horizon, being squeezed against it, but with the various wave-numbers dropping to the regime in which they were dragged away from the horizon (backwards in time). Finally in the last frame we have essentially the whole of the wave-packet contained in the region where the velocity is again a constant. The imaginary part of the wave-packet behaves very similarly.

Figure 6 displays the squared spectrum (as a function of ω, not k) of the initial wave-packet of Figure 5a, and in Figure 7, the final wave-packet of Figure 5d. I.e., Figure 6 is the exiting low wave-number wave-packet (the one we start with going backward in time). Although the frequency

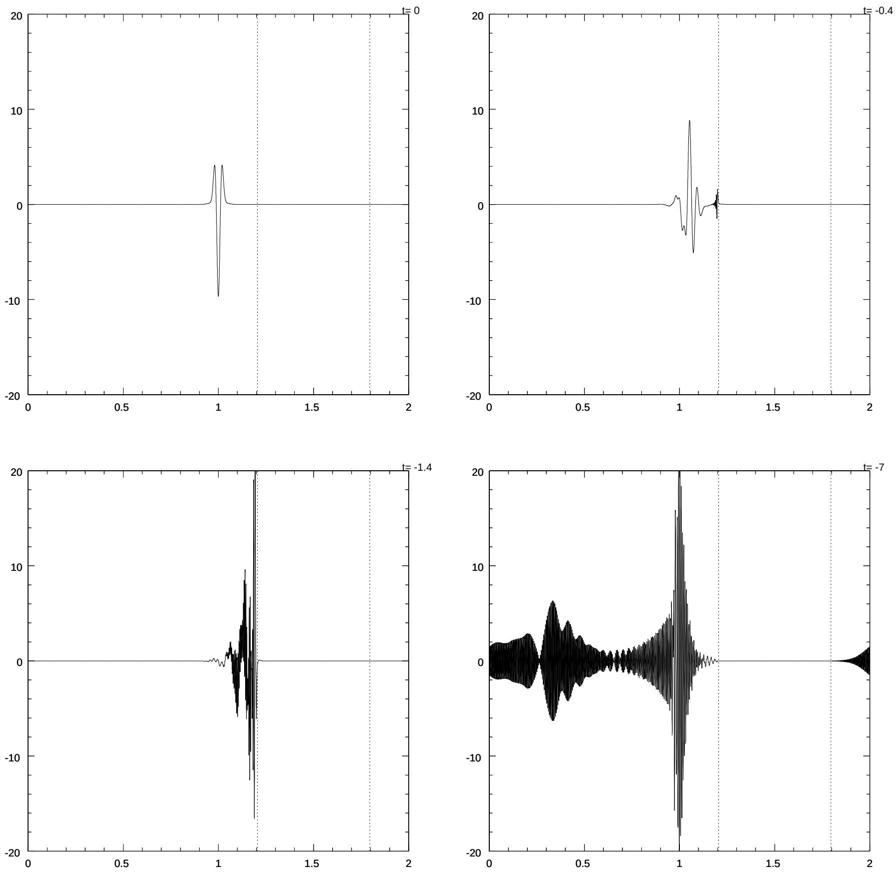

Figure 5: The time development of the pulse This is at time 0, time -0.4, -1.4 and -7 in the natural units were the low frequency velocity of sound is unity. The dotted lines are the black hole and white hole analog horizons. In the last image, the pulse is (almost) entirely contained in the constant velocity region between the horizons.

is conserved and thus the same for both, the wave-numbers of Figure 6 are small, while those of Figure 7 are large.

In Figure 7, the two components are the positive norm part (the larger component) and the negative norm part of the wave-packet for each frequency ω obtained from the initial (5d) wave-packet. Note that if there were no thermal effect, this negative norm component would be zero.

Finally in Figure 8, I have calculated ω divided by the logarithm of the ratio of the negative frequency norm to the positive. This will equal the effective temperature at that frequency. I have also plotted the temperatures predicted if we assume that $\frac{dv}{dx}$ is taken to be that at the horizon corresponding to the phase, group and geometric mean of these velocities for each of the frequencies ω.

At low frequencies the calculated temperature is very noisy. This is to be expected since at those frequencies the wavelength is of the order the whole space. The fact that the wave is not entirely contained in the region where v(x) is constant would be expected to impact these

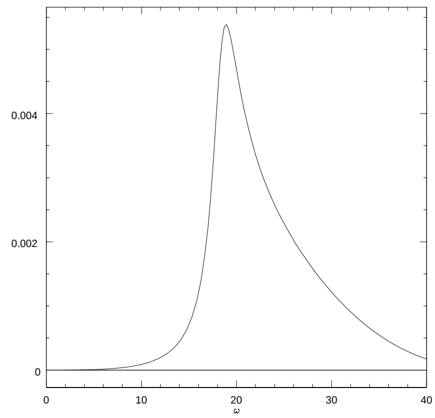

Figure 6: The "initial" spectrum at time 0, before propagating it back in time toward the horizon.

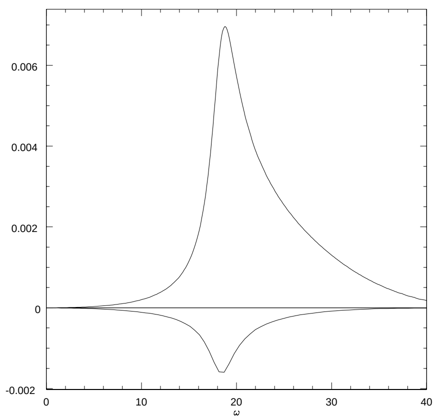

Figure 7: The positive and negative frequency components of the "final" wave (earliest in time) (4d) plotted as a function of the frequency in the constant velocity part of the space. The ration of negative to positive is $e^{\omega/T}$ where T is the temperature of that mode

frequencies most. Whether the small deviation at high frequencies are due to similar numerical problems or are an actual indication that the relevant horizon is not the phase horizon will require further study.

However at low frequencies, all of the various velocities are the same, and on average the fit of the numerical temperature of the modes is well given by that common temperature. At higher frequencies however, the temperature is most closely approximated by the phase velocity.

The comparison between this work and the analytic work in the paper by R. Schuetzhold and me [8], which uses a very different velocity profile for the fluid near the horizon ($v(x) \propto \frac{1}{x}$) suggest that the exact dependence of the temperature of the emitted radiation depends crucially on the form of the velocity profile. In the analytic case, we found that the temperature is exactly given by the rate of change of the velocity at the point where the velocity of the fluid equals the geometric mean of the group and phase velocities of the wave packet far from the horizon. Teasing out the detailed dependence of the temperature on the the fluid flow and the dispersion relation will be the subject of further numerical, and if possible analytic, work.

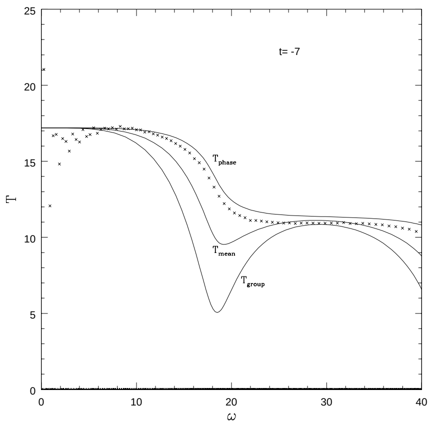

Figure 8: The effective temperature of the components of the packet (5d) as a function of ω. Also plotted are the predicted temperatures based on the hypothesis that the temperature is that given by the $\frac{dv}{dx}$ of the fluid flow at the horizon given by the phase velocity, the group velocity, and the geometric mean of the two velocities as determined in the regime where the fluid flow is constant. The noise in the temperature determined from the wave-packets at low frequencies is probably due to the the wave-packet not quite being in a region of completely constant velocity. The cause of the deviation at high frequencies is unknown.

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