

Recurrence relations in the large space-time dimension limit

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The technique, recently used for calculations of the 4- and 5-loop QCD radiative corrections, is reformulated in terms of the expansion of the recurrence relations in the large space-time dimension limit.

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Recurrence relations [1] are widely used tool for calculations in perturbative quantum field theory. Integration by parts provide us with some difference relations

$$R(I^-, I^+, d)F(n_1, \dots, n_k, d) = 0$$

where $F(n_1, \dots, n_k, d)$ are Feynman integrals with propagators raised to integer (maybe negative) powers n_i and d is space-time dimension. In many cases these relations can be rebuilt in the recurrence form, thus allowing reduction procedure with the result

$$F(\underline{n}, d) = C_1(\underline{n}, d)F_1 + \dots + C_k(\underline{n}, d)F_k$$

where F_i are “basic” or “irreducible” integrals with fixed values of n_i . Lets assume that we know how to calculate universal set F_i and concentrate on C_i which consist of specific for some physical problem information.

Unfortunately, direct reduction mentioned above not always possible. Rebuilding in recurrent form is nontrivial work in many-parametric case: hundreds of difficult sub-cases should be considered. Moreover, even constructed, reduction procedure involves long substitutions which can easily overflow any computer available.

The idea is to calculate C_i directly, without reduction. Note that $C_i(\underline{n}, d)$ are rational in d and obey the same difference relations:

$$R(I^-, I^+)C_i(\underline{n}, d) = 0$$

with boundary conditions $C_i(\underline{n}, d) = 0$ if $n_l \leq 0$ for at least one line of the basic integral associated with C_i . In the following we will call such associated lines or indexes “hard”, and others “soft”. Lets expand in $1/d \rightarrow 0$:

$$\begin{aligned} R(I^-, I^+) &= R^{(0)} + \frac{1}{d}R^{(1)} \\ C_i &= C_i^{(0)} + \frac{1}{d}C_i^{(1)} + \frac{1}{d^2}C_i^{(2)} + \dots \\ 0 = R(I^-, I^+)C_i &\Rightarrow \\ 0 &= R^{(0)}C_i^{(0)} \\ 0 &= R^{(0)}C_i^{(1)} + R^{(1)}C_i^{(0)} \\ &\dots \\ 0 &= R^{(0)}C_i^{(k)} + R^{(1)}C_i^{(k-1)} \\ &\dots \end{aligned}$$

The hope is that these equations are simpler so one can calculate sufficiently many coefficients and then reconstruct exact rational d dependence.

Lets check it with 1-dimensional example:

$$f_n = \int_{-\infty}^{\infty} (x^2 + 2x + 2)^d / x^n dx$$

$$nf_{n+1} = (d + 1 - n)f_n + (d + 1 - n/2)f_{n-1}$$

Direct reduction is trivial, but let's try $1/d$:

$$0 = \underbrace{(f_n + f_{n-1})}_{R^{(0)}f} + 1/d \underbrace{((1 - n/2)f_{n-1} + (1 - n)f_n - nf_{n+1})}_{R^{(1)}f}$$

$$0 = R^{(0)}f^{(0)} = f_n^{(0)} + f_{n-1}^{(0)} \Rightarrow f_n^{(0)} = (-1)^n$$

$$\begin{aligned} 0 &= R^{(0)}f^{(1)} + R^{(1)}f^{(0)} \\ &= f_n^{(1)} + f_{n-1}^{(1)} + n/2(-1)^n \Rightarrow f_n^{(1)} = 1/4(n^2 + n)(-1)^n \end{aligned}$$

For simplicity we skip contributions which reflect $f_n \rightarrow c(d)f_n$ uncertainty. In this example leading order equation involves only I^- operators (no explicit n_i coefficients). Such equations can be solved in multi-dimensional case:

$$R(I^-)C_i = 0 \Rightarrow C_i(\underline{n}) = \Pi_a r_a^{-n_a}, \text{ where } R(r_a) = 0.$$

Indeed: $R(I_a^-)C_i = R(r_a)C_i = 0$. Shortly speaking equations without explicit n_i coefficients simpler then general ones like algebraic equations simpler then differential ones.

Lets now consider Feynman integrals

$$\begin{aligned} F(\underline{n}, d) &= \int d^d p_1 \dots d^d p_L / (E_1^{n_1} \dots E_a^{n_a}), \\ E_a &= A_a^{ik}(p_i p_k) + m_a^2. \end{aligned}$$

Integration by parts

$$\begin{aligned} 0 &= \int d^d p_1 \dots d^d p_L \partial_{p_i}(p_k \dots), \\ \partial_{p_i}(p_k \dots) &= d \delta_k^i + p_k(\partial_{p_i} \dots) = d \delta_k^i + (AA)_b^a E_a (\partial_{E_b} \dots), \\ (AA)_b^a &= 2(A^{-1})_{kl}^a A_b^{li} \end{aligned}$$

provide us with the difference equations

$$0 = d \delta_k^i F + (AA)_b^a E_a \partial_{E_b} F.$$

If we now define

$$\begin{aligned} 0 &= d \delta_k^i F + (AA)_b^a E_a \partial_{E_b} F \\ &R^{(0)}? \quad R^{(1)}? \end{aligned}$$

we will face with trivial solution only

$$0 = R^{(0)}C^{(0)} = \delta_k^i C_n^{(0)} \Rightarrow C_n^{(0)} = 0 \quad ??$$

In fact it means that there are no solutions in the simple form $C_i = C_i^{(0)} + \frac{1}{d}C_i^{(1)} + \dots$. Indeed, $C_k \approx d^{-S(\underline{n})}$, where $S(\underline{n})$ is the sum of the “hard” n_i .

In practice we apply $1/d$ expansion to the sub-case when all “hard” $n_i = 1$ ($n_i > 1$ can be reduced to $n_i = 1$ by direct recursion [2, 3, 4]). Lets modify our relations in order to decouple “hard” and “soft” indexes. For that consider integration by parts with additional insertion in numerator (it is equivalent to recombination of original relations with polynomial in E_a factors):

$$0 = \int d^d p_1 \dots d^d p_L \partial_{p_i} (p_k \Pi^{ik}(E_a) \dots).$$

With some polynomials Π^{ik} we come to diagonalized relations

$$0 = \underset{R^{(1)}}{\partial_{E_a}(P(E) F)} - (d - L - 1)/2 \underset{\text{large}}{\partial_{E_a} P(E)} \underset{R^{(0)}}{F},$$

where $P(E) = \det_{ik}(p_i p_k(E_a))$. Equations with “hard” ∂_{E_a} can be used for $n_a \rightarrow 1$ reduction. In the equations with “soft” ∂_{E_a} we can set “hard” $E_a = 0$ (these operators shift n_a from 1 to 0, and by definition $C_i = 0$ if at least one “hard” n_i is non-positive). So for “soft” ∂_{E_a} for leading order in $1/d \rightarrow 0$ limit we got non-trivial equations

$$0 = R^{(0)} F^{(0)} = (\partial_{E_a} P(E)) F^{(0)}.$$

In principle one can proceed further as with toy 1-dimensional example, but we found more efficient to calculate $C_i^{(k)}(\underline{n})$ by expanding auxiliary integrals in $1/d \rightarrow 0$ limit [4]

$$C_i^{(k)}(\underline{n}) = \int dx_1 \dots dx_a / (x_1^{n_1} \dots x_a^{n_a}) P(\underline{x})^{(d-L-1)/2}.$$

In $1/d \rightarrow 0$ they expand to Gaussian type integrals

$$\int dx_1 \dots dx_a x_1^{k_1} \dots x_a^{k_a} \exp(-A^{ik} x_i x_k)$$

which can be efficiently evaluated in gamma-functions.

Let us now briefly discuss possible applications of the method. The ideal case is massless 0-scale problems: $1/d$ coefficients are pure numbers and the set of basic integrals are relatively small. As the result it was successfully applied to the calculation of the variety of 4- and 5-loop quantities in QCD [5].

One can also try 1-mass 0-scale problems, but in this case number of basic integrals increases, which means that more contributions should be calculated and total efforts (setup + CPU demands) became comparable to other approaches [6]; nevertheless cross-checks still will be useful. Finally, the method potentially applicable to the reduction of multi-scale integrals, but in this case the calculation of basic integrals became very complicated (being the key problem).

To summarize, the expansion of the difference equations for Feynman integrals in the limit of large d allows to solve them in systematic way. The mathematical complexity of the original equations is partially transformed to the additional computer resources demands, but the variety of physically relevant 4- and 5-loop problems are reachable.

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