

## CC DIS at $\alpha_s^3$ in Mellin- $N$ and Bjorken- $x$ spaces

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Third-order results for the structure functions of charged-current deep-inelastic scattering are discussed. New results for 11'th Mellin moment for  $F_{2,L}^{vP-\bar{v}P}$  structure functions and 12'th moment for  $F_3^{vP-\bar{v}P}$  are presented as well as corresponding higher Mellin moments of differences between the respective crossing-even and -odd coefficient functions. Approximations in Bjorken- $x$  space for these differences obtained with lowest five moments as well as consistency of new results with these approximations are discussed. The  $1/N_c$  suppression of the differences is shown and the correction to the Paschos-Wolfenstein relation is discussed.

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## 1. Introduction

Structure functions in deep-inelastic scattering (DIS) are among the most extensively measured observables. Today the combined data from fixed-target experiments and the HERA collider spans about four orders of magnitude in both Bjorken- $x$  and the scale  $Q^2 = -q^2$  given by the momentum  $q$  of the exchanged electroweak gauge boson [1]. In this report I focus on the  $W$ -exchange charged-current (CC) case, see Refs. [2, 3, 4] for recent measurements in neutrino DIS and at HERA. I present new Mellin moments for coefficient functions in combination  $\nu p - \bar{\nu} p$  and discuss the results for differences between the corresponding crossing-even and -odd coefficient functions. I show suppression of such differences in large  $1/N_c$  limit and discuss  $\alpha_s^3$  correction to the Paschos-Wolfenstein relation [5].

## 2. Results for the CC coefficient functions and their applications

Recently the first five odd-integer moments have been computed of the third-order coefficient functions for  $F_{2,L}^{\nu p - \bar{\nu} p}$  in charged-current DIS, together with the corresponding moments  $N = 2, \dots, 10$  for  $F_3^{\nu p - \bar{\nu} p}$  [6]. Meanwhile calculation of new results for 11'th moment for the first and 12'th moment for the latter case has been finished. We use scale choice  $\mu_r = \mu_f = Q$  and standard QCD colour factors  $C_A = 3$  and  $C_F = 4/3$  throughout this paper and denote the Mellin- $N$  moments of corresponding coefficient functions as  $C_{a,N}^{\text{ns}}$ ,  $a = 2, 3, L$ . Following the formalism outlined in [6] we find the following numerical results

$$\begin{aligned}
C_{2,11}^{\text{ns}} &= 1 + 21.01295976 a_s + a_s^2 (722.3767644 - 51.01867375 n_f) \\
&\quad + a_s^3 (29020.51723 - 4259.717409 n_f + 89.53420655 n_f^2), \\
C_{L,11}^{\text{ns}} &= 0.4444444444 a_s + a_s^2 (30.42631299 - 1.781422693 n_f) \\
&\quad + a_s^3 (2021.685213 - 266.3750306 n_f + 7.082458684 n_f^2), \\
C_{3,12}^{\text{ns}} &= 1 + 22.20106054 a_s + a_s^2 (774.6238566 - 53.26617873 n_f) \\
&\quad + a_s^3 (31152.95983 - 4483.444700 n_f + 91.41515482 n_f^2), \tag{2.1}
\end{aligned}$$

where the normalized coupling constant  $a_s = \alpha_s/(4\pi)$  and  $n_f$  denotes the number of effectively massless quark flavours. The results in analytical form can be found in App. A.

Unlike fixed- $N$  calculations for the combination  $\nu p - \bar{\nu} p$ , the complete three-loop results for  $F_{2,L}^{\nu p + \bar{\nu} p}$  [7, 8] (the  $\alpha_s^3$  coefficient functions for this process are those of photon-exchange DIS, but without the contributions of the  $fl_{11}$  flavour classes) and  $F_3^{\nu p + \bar{\nu} p}$  [9] facilitate analytic continuations to these values of  $N$ . This continuation has been performed using the  $x$ -space expressions and the Mellin transformation package provided with version 3 of FORM [10]. Thus we are in a position to derive the respective moments of the hitherto unknown third-order contributions to the even-odd differences which are defined as

$$\delta C_{2,L} = C_{2,L}^{\nu p + \bar{\nu} p} - C_{2,L}^{\nu p - \bar{\nu} p}, \quad \delta C_3 = C_3^{\nu p - \bar{\nu} p} - C_3^{\nu p + \bar{\nu} p}. \tag{2.2}$$

The signs are chosen such that the differences are always ‘even – odd’ in the moments  $N$  accessible by the OPE, and it is understood that the  $d^{abc} d_{abc}$  part of  $C_3^{\nu p + \bar{\nu} p}$  [11, 9] is removed before the

difference is formed. The non-singlet quantities (2.2) have an expansion in powers of  $\alpha_s$ ,

$$\delta C_a = \sum_{l=2} a_s^l \delta c_a^{(l)}. \quad (2.3)$$

There are no first-order contributions to these differences, hence the sums start at  $l = 2$  in Eq. (2.3). Here I present only numerical results for the differences corresponding to the Mellin moments in Eqs. (2.1) whereas the corresponding results in analytical form are shown in App. B. The lower Mellin moments of such differences can be found in Ref. [12]. Using the notation  $\delta C_{a,N}$  for the  $N$ -th moment of  $\delta C_a(x)$  the results for higher Mellin moments read

$$\begin{aligned} \delta C_{2,11} &= -0.004083868756 a_s^2 + a_s^3 (+0.1559414787 - 0.01710053059 n_f), \\ \delta C_{L,11} &= -0.001670175019 a_s^2 + a_s^3 (-0.3317043993 + 0.006939009889 n_f), \\ \delta C_{3,12} &= -0.009709081656 a_s^2 + a_s^3 (-0.6201718804 + 0.01191844785 n_f). \end{aligned} \quad (2.4)$$

The new  $\alpha_s^3$  contributions are rather large if compared to the leading second-order results also included in Eqs. (2.4) with, e.g.,  $a_s = 1/50$  corresponding to  $\alpha_s \simeq 0.25$ . On the other hand, the integer- $N$  differences  $\delta C_{a,N}$  are entirely negligible compared to the  $\nu p \pm \bar{\nu} p$  moments  $C_{a,N}$  of Eqs. (2.1) and Refs. [11, 6].

To discuss the colour structure of the results I present the  $\alpha_s^3$  part for  $\delta C_{2,11}$  with analytical colour factor dependence

$$\delta c_{2,11}^{(3)} = 0.9495866025 C_F C_{FA}^2 - 0.4076041653 C_F^2 C_{FA} + 0.07695238768 C_F C_{FA} n_f. \quad (2.5)$$

One notes here that result contains an overall factor  $C_{FA} = C_F - C_A/2 = -1/(2N_c)$ . This occurrence is typical for all calculated differences  $\delta c_{a,N}^{(3)}$ . Hence the third-order even-odd differences are suppressed in the large- $N_c$  limit as conjectured, to all orders, in Refs. [13, 14]

Let us now consider consequences of the moments of the type Eqs. (2.4) for the  $x$ -space functions  $\delta c_a^{(3)}(x)$ . For moment-based approximations a simple ansatz was chosen, and its free parameters were determined from the first five moments available in Ref. [12]. This ansatz is then varied in order to estimate the remaining uncertainties. Finally two approximations, denoted below by  $A$  and  $B$ , are selected which indicate the widths of the uncertainty bands. For  $F_2$  these functions are, with  $L_0 = \ln x$ ,  $x_1 = 1 - x$  and  $L_1 = \ln x_1$ ,

$$\begin{aligned} \delta c_{2,A}^{(3)}(x) &= (54.478 L_1^2 + 304.6 L_1 + 691.68 x) x_1 + 179.14 L_0 - 0.1826 L_0^3 \\ &\quad + n_f \{ (20.822 x^2 - 282.1 (1 + \frac{x}{2})) x_1 - (285.58 x + 112.3 - 3.587 L_0^2) L_0 \}, \\ \delta c_{2,B}^{(3)}(x) &= -(13.378 L_1^2 + 97.60 L_1 + 118.12 x) x_1 - 91.196 L_0^2 - 0.4644 L_0^5 \\ &\quad + n_f \{ (4.522 L_1 + 447.88 (1 + \frac{x}{2})) x_1 + (514.02 x + 147.05 + 7.386 L_0) L_0 \}. \end{aligned} \quad (2.6)$$

The uncertainty band presented by Eqs. (2.6) does not directly indicate the range of applicability, since the coefficient functions enter observables only via smoothening Mellin convolutions with non-perturbative initial distributions. As result theoretical uncertainties for physical observables become even smaller (see Ref. [12] for details). It is worth to mention that the new Mellin moments in Eqs. (2.1) and (2.4) are consistent with the approximations based on the first five Mellin moments only, thus confirming the reliability of the uncertainty estimates.

The approximations (2.6) and analogue of it for  $\delta c_L^{(3)}(x)$  can be used to determine  $\alpha_s^3$  corrections to the Paschos-Wolfenstein relations (see Ref. [12] and references therein for details and discussion)

$$R^- = \frac{1}{2} - \sin^2 \theta_W + \frac{u^- - d^- + c^- - s^-}{u^- + d^-} \left\{ 1 - \frac{7}{3} \sin^2 \theta_W + \left( \frac{1}{2} - \sin^2 \theta_W \right) \cdot \frac{8}{9\pi} \left[ \alpha_s + 1.689 \alpha_s^2 + (3.661 \pm 0.002) \alpha_s^3 \right] \right\} + \mathcal{O}((u^- + d^-)^{-2}) + \mathcal{O}(\alpha_s^4). \quad (2.7)$$

The ratio (2.7) is an expansion in  $\alpha_s$  and inverse powers of the dominant isoscalar combination  $u^- + d^-$ , where  $q^- = \int_0^1 dx x(q(x) - \bar{q}(x))$  is the second Mellin moment of the valence quark distributions. The third term in the square brackets is determined with our  $\alpha_s^3$  corrections and the perturbation series appears reasonably well convergent although the correction is not negligible. On the other hand, due to the small prefactor of this expansion, the new third-order term increases the complete curved bracket in Eq. (2.7) by only about a percent, which can therefore be considered as the new uncertainty of this quantity due to the truncation of the perturbative expansion.

### 3. Summary

In this report I have presented new results for 11'th Mellin moment for  $C_2^{\text{ns}}$  and  $C_L^{\text{ns}}$  Wilson coefficient functions as well as 12'th moment for  $C_3^{\text{ns}}$  function. I have discussed the Mellin moments for differences between the corresponding crossing-even and -odd coefficient functions and use of these to obtain approximations in Bjorken- $x$  space which are ready for phenomenology applications (see, e.g., Ref. [15]). It was shown that the differences are suppressed by the number of colours. The third order QCD correction to the Paschos-Wolfenstein relation was obtained with help of the approximations in Bjorken- $x$  space. The correction was found to be small.

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### A. Appendix

In this Appendix I present the analytic expressions up to order  $a_s^3$  for the coefficient functions  $C_{2,11}^{\text{ns}}$ ,  $C_{L,11}^{\text{ns}}$  and  $C_{3,12}^{\text{ns}}$  at the scale  $\mu_r = \mu_f = Q$ . The notation follows Sec. 2 where these functions were presented numerically in Eqs. (2.1).  $C_A$  and  $C_F$  are the standard QCD colour factors,  $C_A \equiv N_c$  and  $C_F = (N_c^2 - 1)/(2N_c)$ , and  $\zeta_i$  stands for Riemann's  $\zeta$ -function. The Wilson coefficients read

$$C_{2,11}^{\text{ns}} = 1 + a_s C_F \frac{13105783}{831600}$$

$$\begin{aligned}
& + a_s^2 C_F n_f \left( -\frac{122253517912789}{3195000547200} \right) \\
& + a_s^2 C_F^2 \left( \frac{98146880716389133}{1845112816008000} + \frac{8462}{105} \zeta_3 \right) \\
& + a_s^2 C_A C_F \left( \frac{236212260301543}{1037337840000} - \frac{109357}{1155} \zeta_3 \right) \\
& + a_s^3 C_F n_f^2 \left( \frac{63689629686066726367}{996360920644320000} + \frac{251264}{93555} \zeta_3 \right) \\
& + a_s^3 C_F^2 n_f \left( -\frac{7118183480913604311036887}{10880261253435974400000} + \frac{125632}{3465} \zeta_4 - \frac{8620003991}{85135050} \zeta_3 \right) \\
& + a_s^3 C_F^3 \left( -\frac{35440239155810867032199497}{3544454339100103968000000} - \frac{81164}{99} \zeta_5 + \frac{151689577}{8004150} \zeta_4 \right. \\
& \quad \left. + \frac{381164463904607}{249609417750} \zeta_3 \right) \\
& + a_s^3 C_A C_F n_f \left( -\frac{206484440760943457173}{203899125830400000} - \frac{125632}{3465} \zeta_4 + \frac{244949367479}{936485550} \zeta_3 \right) \\
& + a_s^3 C_A C_F^2 \left( \frac{130339510428192960678390973}{54401306267179872000000} - \frac{21218}{99} \zeta_5 - \frac{151689577}{5336100} \zeta_4 \right. \\
& \quad \left. + \frac{80821712466167}{576875098800} \zeta_3 \right) \\
& + a_s^3 C_A^2 C_F \left( \frac{6214478354002179611868649}{1884027922672896000000} + \frac{197732}{231} \zeta_5 + \frac{151689577}{16008300} \zeta_4 \right. \\
& \quad \left. - \frac{1039437594401}{416215800} \zeta_3 \right), \tag{A.1}
\end{aligned}$$

$$\begin{aligned}
C_{L,11}^{\text{ns}} &= a_s C_F \frac{1}{3} \\
& + a_s^2 C_F n_f \left( -\frac{15151}{11340} \right) \\
& + a_s^2 C_F^2 \left( -\frac{313264177}{104781600} + 8\zeta_3 \right) \\
& + a_s^2 C_A C_F \left( \frac{1984474543}{209563200} - 4\zeta_3 \right) \\
& + a_s^3 C_F n_f^2 \frac{1355317}{255150} \\
& + a_s^3 C_F^2 n_f \left( \frac{355389053714327}{8307001422720} - \frac{30048196}{405405} \zeta_3 \right)
\end{aligned}$$

$$\begin{aligned}
& + a_s^3 C_F^3 \left( -\frac{9102853788548773397}{27676692240120000} + \frac{4240}{3} \zeta_5 - \frac{505404892}{571725} \zeta_3 \right) \\
& + a_s^3 C_A C_F n_f \left( -\frac{119231997944341913}{1213685272800000} + \frac{1849566364}{42567525} \zeta_3 \right) \\
& + a_s^3 C_A C_F^2 \left( -\frac{5696808539467201991}{290745049795200000} - 1480 \zeta_5 + \frac{1327717221659}{936485550} \zeta_3 \right) \\
& + a_s^3 C_A^2 C_F \left( \frac{278571247117091333}{882680198400000} + \frac{1160}{3} \zeta_5 - \frac{89146587169}{170270100} \zeta_3 \right), \tag{A.2}
\end{aligned}$$

$$C_{3,12}^{\text{ns}} = 1$$

$$\begin{aligned}
& + a_s C_F \frac{180008419}{10810800} \\
& + a_s^2 C_F n_f \left( -\frac{56084621695749173}{1403883240439680} \right) \\
& + a_s^2 C_F^2 \left( \frac{3728023695177860551381}{52698267138004488000} + \frac{88882}{1155} \zeta_3 \right) \\
& + a_s^2 C_A C_F \left( \frac{41265384392827325207}{175485405054960000} - \frac{1424581}{15015} \zeta_3 \right) \\
& + a_s^3 C_F n_f^2 \left( \frac{1855782116991095763359887}{28457064254522423520000} + \frac{3387392}{1216215} \zeta_3 \right) \\
& + a_s^3 C_F^2 n_f \left( -\frac{22132972736542162287370477597}{28250103787216805894400000} + \frac{1693696}{45045} \zeta_4 - \frac{496067497273}{12174312150} \zeta_3 \right) \\
& + a_s^3 C_F^3 \left( -\frac{1556243722103827358575546080538633}{17108404104057433733678112000000} - \frac{551372}{1287} \zeta_5 + \frac{25648239313}{1352701350} \zeta_4 \right. \\
& \quad \left. + \frac{67198327643555959}{49853808254250} \zeta_3 \right) \\
& + a_s^3 C_A C_F n_f \left( -\frac{383732229667234059139588387}{379427523393632313600000} - \frac{1693696}{45045} \zeta_4 + \frac{31504273327}{133783650} \zeta_3 \right) \\
& + a_s^3 C_A C_F^2 \left( \frac{55562394749867369342925789360923}{17091312791266167566112000000} - \frac{58298}{99} \zeta_5 - \frac{25648239313}{901800900} \zeta_4 \right. \\
& \quad \left. - \frac{518595205781183}{6499459446480} \zeta_3 \right) \\
& + a_s^3 C_A^2 C_F \left( \frac{145036477650625443670858543223}{45531302807235877632000000} + \frac{2899516}{3003} \zeta_5 + \frac{25648239313}{2705402700} \zeta_4 \right. \\
& \quad \left. - \frac{118955981330663}{48697248600} \zeta_3 \right). \tag{A.3}
\end{aligned}$$

## B. Appendix

Here I present the analytic expressions for the Mellin-space coefficient-function differences  $\delta c_{2,11}^{(3)}$ ,  $\delta c_{L,11}^{(3)}$  and  $\delta c_{3,12}^{(3)}$  given numerically in Eqs. (2.4). We use the notations and conventions as specified in Sec. 2. The moments of this quantity are given by

$$\begin{aligned}
\delta c_{2,11}^{(3)} = & C_F C_{FA}^2 \left( \frac{72985545040605109471734789941}{12566701747718550432000000} - \frac{23832}{77} \zeta_5 + \frac{55689927719519927}{1622461215375} \zeta_3 \right. \\
& - \frac{1472}{3} \zeta_3^2 - \frac{15177966246339422387}{1537594013340000} \zeta_2 + \frac{8654312}{945} \zeta_2 \zeta_3 - \frac{31781239759}{1819125} \zeta_2^2 \\
& \left. - \frac{8992}{63} \zeta_2^3 \right) \\
& + C_F^2 C_{FA} \left( -\frac{417272486089283995425649952101}{69116859612452027376000000} + \frac{66993964}{10395} \zeta_5 \right. \\
& - \frac{436861012176654887}{12979689723000} \zeta_3 + \frac{1456}{3} \zeta_3^2 + \frac{1514141807453257771}{109828143810000} \zeta_2 \\
& \left. - \frac{77266484}{10395} \zeta_2 \zeta_3 + \frac{2125039226077}{180093375} \zeta_2^2 - \frac{56432}{315} \zeta_2^3 \right) \\
& + n_f C_F C_{FA} \left( \frac{188706965915502835794319}{369391585764801600000} - \frac{1792}{9} \zeta_5 + \frac{22120534934}{10405395} \zeta_3 \right. \\
& \left. - \frac{3063427279802429}{2995313013000} \zeta_2 + \frac{1408}{9} \zeta_2 \zeta_3 - \frac{9519404}{17325} \zeta_2^2 \right), \tag{B.1}
\end{aligned}$$

$$\begin{aligned}
\delta c_{L,11}^{(3)} = & C_F C_{FA}^2 \left( \frac{13469264008826648669111}{9251922834554400000} - \frac{8288}{3} \zeta_5 + \frac{10937259706}{4729725} \zeta_3 \right. \\
& \left. - \frac{12442705009487}{6483361500} \zeta_2 + 1312 \zeta_2 \zeta_3 - \frac{1413232}{4725} \zeta_2^2 \right) \\
& + C_F^2 C_{FA} \left( \frac{29154136441960450330061}{226672109446582800000} - \frac{368}{3} \zeta_5 - \frac{560828132092}{468242775} \zeta_3 \right. \\
& \left. + \frac{244988263837}{480249000} \zeta_2 + 48 \zeta_2 \zeta_3 + \frac{9705316}{51975} \zeta_2^2 \right) \\
& + n_f C_F C_{FA} \left( \frac{717577454838293479}{32708818101960000} + \frac{26725912}{405405} \zeta_3 - \frac{67470181}{1403325} \zeta_2 - \frac{368}{45} \zeta_2^2 \right), \tag{B.2}
\end{aligned}$$

$$\begin{aligned}
\delta c_{3,12}^{(3)} = & C_F C_{FA}^2 \left( -\frac{29715680590481634053309842446559253}{4665928392015663745548576000000} + \frac{2637896}{3003} \zeta_5 \right. \\
& - \frac{9720046002002881973}{274195945398375} \zeta_3 + \frac{1472}{3} \zeta_3^2 + \frac{151219518094861415132729}{14638407538334580000} \zeta_2 \\
& \left. - \frac{1290033016}{135135} \zeta_2 \zeta_3 + \frac{20250388084007}{1127251125} \zeta_2^2 + \frac{8992}{63} \zeta_2^3 \right)
\end{aligned}$$

$$\begin{aligned}
& + C_F^2 C_{FA} \left( \frac{165791456833171990576039064477409253}{25662606156086150600517168000000} - \frac{902648332}{135135} \zeta_5 \right. \\
& \quad + \frac{6932482134993002293}{199415233017000} \zeta_3 - \frac{1456}{3} \zeta_3^2 - \frac{3478984519432985388157}{241292431950570000} \zeta_2 \\
& \quad \left. + \frac{79428644}{10395} \zeta_2 \zeta_3 - \frac{369039205594393}{30435780375} \zeta_2^2 + \frac{56432}{315} \zeta_2^3 \right) \\
& + n_f C_F C_{FA} \left( -\frac{1297953990099826399951147609}{2434659941775807345600000} + \frac{1792}{9} \zeta_5 - \frac{876359298658}{405810405} \zeta_3 \right. \\
& \quad \left. + \frac{40974440556905057}{38939069169000} \zeta_2 - \frac{1408}{9} \zeta_2 \zeta_3 + \frac{13881228}{25025} \zeta_2^2 \right). \tag{B.3}
\end{aligned}$$

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