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Gluon scattering in $\mathcal{N} = 4$ super-Yang-Mills theory from weak to strong coupling

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I describe some recent developments in the understanding of gluon scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory in the large-N_c limit. These amplitudes can be computed to high orders in the weak coupling expansion, and also now at strong coupling using the AdS/CFT correspondence. They hold the promise of being solvable to all orders in the gauge coupling, with the help of techniques based on integrability. They are intimately related to expectation values for polygonal Wilson loops composed of light-like segments.

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1. Introduction

In this talk I would like to describe some remarkable progress that has been made in the past few years in understanding the structure of gauge boson scattering amplitudes in a particular gauge theory, $\mathcal{N}=4$ super-Yang-Mills theory. While this theory differs in many details from the electroweak and QCD theories whose radiative corrections were the subject of this symposium, there are many common issues, particularly associated with infrared structure. Indeed, the understanding of infrared divergences in QCD acquired over the last few decades has proved extremely useful in unraveling some of the structure of $\mathcal{N}=4$ super-Yang-Mills theory.

$\mathcal{N}=4$ super-Yang-Mills theory is the most supersymmetric theory possible without gravity. In the free theory, starting from the helicity $+1$ massless gauge boson ("gluon") state, the four supercharges can be used to lower the helicity by $4\times\frac{1}{2}=2$ units, until the helicity -1 gluon state is reached. If one had more supercharges, one would need spin >1 states, and it is not known how to quantize such theories in a unitary way without including at least spin 2 gravitons. Along the way from the helicity $+1$ to the helicity -1 gluon state, one passes through the 4 massless (Majorana) spin $1/2$ gluinos, and 6 real (or 3 complex) massless spin 0 scalars. In this maximally supersymmetric Yang-Mills theory (MSYM), all the massless states are in the adjoint representation of the gauge group, which we will take to be $SU(N_c)$. The interactions are all uniquely specified by the choice of gauge group, and one dimensionless gauge coupling g. The theory is an exactly scale-invariant, conformal field theory; that is, the beta function vanishes identically for all values of the coupling [1].

Here we will consider the 't Hooft limit of MSYM, in which the number of colors $N_c\to\infty$, with the 't Hooft parameter $\lambda\equiv g^2N_c$ held fixed [2]. In this limit, only planar Feynman diagrams contribute. Also, the anti-de Sitter space / conformal field theory (AdS/CFT) duality [3] suggests that for $N_c\to\infty$ the weak-coupling perturbation series in λ might have some very special properties. The reason is that, according to AdS/CFT, the strongly-coupled (large λ) limit of the four-dimensional conformal gauge theory has an equivalent description in terms of a weakly-coupled string theory. The intuition is that the perturbative series should know about this simple strong-coupling limit, and organize itself accordingly [4].

Figure 1 sketches how events such as gluon scattering look in the AdS/CFT duality [3, 5]. Five-dimensional anti-de Sitter space, AdS_5, contains, besides the usual four-dimensional space-time $R^{1,3}$, an additional radial variable r, which corresponds to a resolution scale in the four-dimensional theory. Large values of r correspond to the ultraviolet (UV) region; small values to the infrared (IR). The figure shows a "big" glueball state in the IR, and a "small" glueball state in the UV. The arrows represent the motion of plane-wave single gluon states in $R^{1,3}$ for $gg\to gg$ scattering at 90°. We'll discuss the motion in r later. The radius of curvature of AdS_5 is proportional to $\lambda^{1/4}$. Large λ means that the space-time is only weakly curved, which makes it much simpler to study the string theory; higher excitations of the string can usually be neglected.

The AdS/CFT duality is a weak/strong duality. Quantities that can be computed at weak coupling in one picture have a strong-coupling description in the other picture. This property makes AdS/CFT both powerful and difficult to check explicitly — although there is certainly convincing evidence in its favor. There are a few quantities that are known (modulo a few assumptions) to all orders in λ; that is, for which one can interpolate all the way from weak to strong coupling.

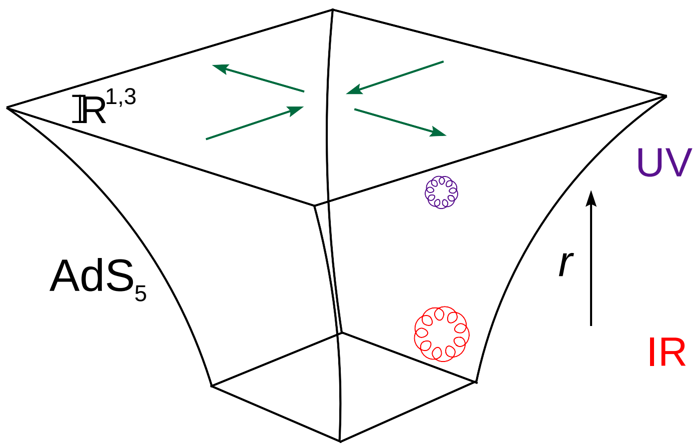

Figure 1: Cartoon of the AdS/CFT duality.

Notable among these is the cusp (or soft) anomalous dimension $\gamma_K(\lambda)$. The QCD version of this quantity crops up a lot in soft-gluon resummation. Beisert, Eden and Staudacher [6] have given an all-orders proposal for $\gamma_K(\lambda)$, based on integrability, plus a number of other properties. Their proposal is consistent with the first four loops in the weak-coupling expansion [7, 8], and also agrees [9, 10] with the first three terms in the strong-coupling expansion [11, 12, 13].

In this talk I would like to discuss the evidence for another proposal [14], namely that gluon-gluon scattering $gg \to gg$ in MSYM, for any scattering angle θ can be fully specified by just three functions of λ, independent of θ. One of these three functions is already "known", because it is just $\gamma_K(\lambda)$. This proposal has received some confirmation at strong coupling, through the work of Alday and Maldacena [5]. It was motivated by the structure of IR divergences in gauge theory.

2. Infrared divergences

In a conformal field theory, scale invariance implies that the interactions never shut off, so that a scattering process cannot really be defined. While strictly speaking this is true, we are able to get around it in practice by regulating the theory in the IR. We'll use dimensional regularization with $D = 4-2\varepsilon$ and $\varepsilon < 0$ (actually a version of it that preserves all the supersymmetry [15]). The regulator breaks the conformal invariance, but we can recover it by performing a Laurent expansion around $\varepsilon = 0$, up to and including the $\mathcal{O}(\varepsilon^0)$ terms.

At one loop, there are two types of IR divergences: *soft*-gluon exchange, in which the virtual gluon energy $\omega \to 0$; and *collinear* regions, in which the gluon's transverse momentum (with respect to a massless external line) $k_{\mathrm{T}} \to 0$. The soft and collinear regions each produce a $1/\varepsilon$ pole, resulting in a $1/\varepsilon^2$ leading behavior for on-shell amplitudes at one loop. At L loops, the leading behavior is $1/\varepsilon^{2L}$, coming from multiple soft-gluon exchange that is arranged hierarchically, so that the outermost gluons are softer and more collinear than the innermost ones.

In fact, all the pole terms for L-loop amplitudes are predictable in planar gauge theory, thanks to decades of work on the soft/collinear factorization and exponentiation of amplitudes, and of quark and gluon form factors, in QCD [16, 17, 18, 19, 20, 21]. For both QCD and MSYM, in the planar limit the pole terms are given in terms of three quantities (in the notation of refs. [19, 21]):

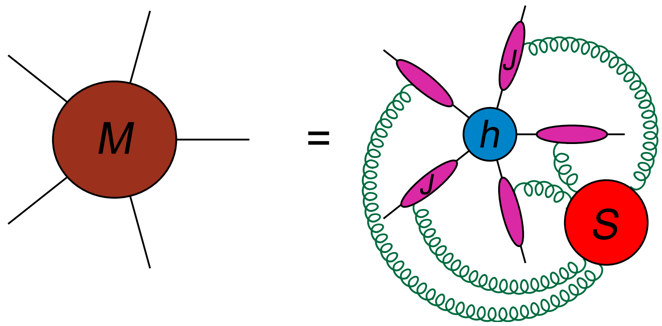

Figure 2: Factorization of soft and collinear singularities.

- the beta function $\beta(\lambda)$ (but of course this vanishes in MSYM),
- the cusp anomalous dimension $\gamma_K(\lambda)$,
- a "collinear" anomalous dimension $G_0(\lambda)$.

The cusp anomalous dimension gets its name because it appears [22, 23, 24] in the renormalization group equation for the expectation value of a Wilson line $W(\rho,g)$ for two semi-infinite straight lines, joined at a kink or cusp:

$$\left(\rho\frac{\partial}{\partial\rho}+\beta(g)\frac{\partial}{\partial g}\right)\ln W(\rho,g) = -2\gamma_K(\lambda)\,\ln\rho^2+\mathscr{O}(\rho^0), \tag{2.1}$$

where $\rho^2 \equiv n_1\cdot n_2/(\sqrt{n_1^2 n_2^2}) \to \infty$ as the two straight lines become light-like, $n_1^2, n_2^2 \to 0$. The cusp anomalous dimension also controls [18] the universal (flavor independent) large-spin limit of anomalous dimensions γ_j of leading-twist operators with spin j, such as the quark operators $O_j \equiv \bar{q}(\gamma_+\mathscr{D}^+)^j q$:

$$\gamma_j = \frac{1}{2}\gamma_K(\lambda)\,\ln j+\mathscr{O}(j^0)\,, \qquad j\to\infty. \tag{2.2}$$

Finally, through a Mellin transform of eq. (2.2), $\gamma_K(\lambda)$ appears in the large x limit of the DGLAP kernel for evolving the parton distributions,

$$P_{aa}(x) = \frac{1}{2}\frac{\gamma_K(\lambda)}{(1-x)_+}+\cdots\,, \qquad x\to 1. \tag{2.3}$$

Thus, in the study of QCD at colliders it is an important quantity for resumming the effects of soft gluon emission.

The general infrared structure of massless gauge amplitudes can be exposed [17, 20, 21] by factoring off soft singularities, which arise from long-distance gluon exchange, and collinear singularities, which are also at long distances, but only out along the axis of a hard parton. This space-time picture is shown in fig. 2. Defining $\mathscr{M}_n$ to be the full amplitude $\mathscr{A}_n$ divided by the tree amplitude $\mathscr{A}_n^{\rm tree}$, the factorization formula reads,

$$\mathscr{M}_n = S(\{k_i\},\mu,\varepsilon)\times\prod_{i=1}^{n} J_i(k_i,\mu,\varepsilon)\times h_n(\{k_i\},\mu)\,, \tag{2.4}$$

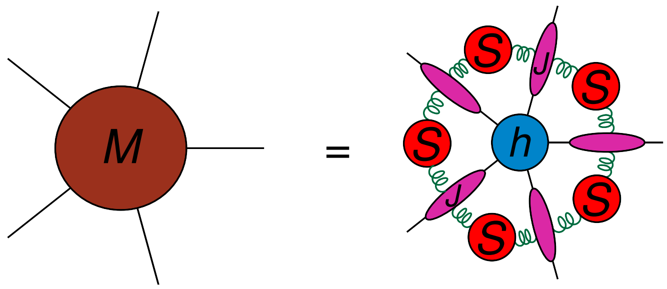

Figure 3: Soft-collinear factorization in the planar limit.

where μ is the factorization scale, and h_n is the hard remainder function, and is finite as $\varepsilon \to 0$. The soft function S only sees the classical color charge of the i^{th} particle. In general it is a complicated matrix acting on the possible color configurations for h_n, because soft gluons can attach to any pair of external partons. The jet function J_i is color-diagonal, but depends on the i^{th} spin. Terms that are color-diagonal and spin-independent can be moved arbitrarily between S and J_i.

In the large-N_c planar limit, the picture simplifies, to that shown in fig. 3. Here M represents the coefficient of a particular color structure, $\text{tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]$. Now soft gluons can only connect adjacent external partons; and indeed there is no mixing of different color structures at large N_c. Because of the color-triviality of the planar limit, one can absorb the entire soft function S into jet functions, or break up the right-hand side of fig. 3 into n wedges. Each wedge represents the square root of the Sudakov form factor, the amplitude $\mathcal{M}^{[1\to gg]}$ for a color-singlet state "1" to decay to a pair of partons, say gluons. Hence the planar version of eq. (2.4) is

$$\mathcal{M}_n = \prod_{i=1}^{n} \left[\mathcal{M}^{[1\to gg]} \left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \varepsilon \right) \right]^{1/2} \times h_n(\{k_i\}, \mu, \alpha_s) \,. \qquad (2.5)$$

The only dependence of the singular terms on the kinematics is through the momentum scale, $s_{i,i+1} = (k_i + k_{i+1})^2$, entering the i^{th} Sudakov form factor.

Factorization also implies that the Sudakov form factor obeys a differential equation in the momentum scale [16, 24, 18, 19],

$$\frac{\partial}{\ln Q^2} \ln \mathcal{M}^{[1\to gg]}(Q^2/\mu^2, \alpha_s, \varepsilon) = \frac{1}{2}\Big[K(\varepsilon, \alpha_s) + G(Q^2/\mu^2, \alpha_s, \varepsilon) \Big] \,. \qquad (2.6)$$

Here $K(\varepsilon, \alpha_s)$ is a pure counterterm, or series of $1/\varepsilon$ poles. By analogy with the D-dimensional β-function, $\beta(\varepsilon, \alpha_s)$, the single poles (related to γ_K) determine K completely. The function G is finite as $\varepsilon \to 0$, but contains all the Q^2 dependence; it will generate a single pole in $\ln \mathcal{M}^{[1\to gg]}$ upon integrating eq. (2.6) with respect to Q^2. The functions K and G obey renormalization group equations,

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} \right) K = - \left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} \right) G = -\gamma_K(\lambda). \qquad (2.7)$$

The collinear anomalous dimension $G_0(\lambda)$ arises as a constant of integration for the differential equation for G.

Solving the differential equations for K, G and the Sudakov form factor is particularly easy in a conformal theory because the four-dimensional coupling does not run. Doing this, and inserting the form-factor solution into eq. (2.5) for the n-point amplitude, we obtain [14],

$$\begin{aligned}\mathcal{M}_n(\varepsilon) &= 1+\sum_{L=1}^{\infty} a^L M_n^{(L)}(\varepsilon)\\ &= \exp\left[-\frac{1}{8}\sum_{l=1}^{\infty} a^l \left(\frac{\hat{\gamma}_K^{(l)}}{(l\,\varepsilon)^2}+\frac{2\,\hat{G}_0^{(l)}}{l\,\varepsilon}\right)\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{l\,\varepsilon}\right]\times h_n(\{k_i\}),\end{aligned} \tag{2.8}$$

where

$$a \equiv \frac{N_c\alpha_s}{2\pi}(4\pi e^{-\gamma})^{\varepsilon} = \frac{\lambda}{8\pi^2}(4\pi e^{-\gamma})^{\varepsilon} \tag{2.9}$$

is the loop expansion parameter in the 't Hooft limit, and $\hat{\gamma}_K^{(l)}$ and $\hat{G}_0^{(l)}$ are the l-loop coefficients of $\gamma_K(a)$ and $G_0(a)$.

The argument of the exponential in eq. (2.8) looks very much like the one-loop amplitude, but with ε replaced by $l\,\varepsilon$, denoted by $M_n^{(1)}(l\varepsilon)$. Thus we are motivated to rewrite eq. (2.8) as

$$\mathcal{M}_n(\varepsilon) = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\varepsilon) M_n^{(1)}(l\varepsilon) + h_n(\{k_i\}) + \mathcal{O}(\varepsilon)\right)\right], \tag{2.10}$$

where $f^{(l)}(\varepsilon) \equiv f_0^{(l)} + \varepsilon\, f_1^{(l)} + \varepsilon^2\, f_2^{(l)}$ collects three series of constants. Two of these are identified with the previous quantities as,

$$f_0^{(l)} = \frac{1}{4}\hat{\gamma}_K^{(l)}, \qquad f_1^{(l)} = \frac{l}{2}\hat{G}_0^{(l)}, \tag{2.11}$$

while the third quantity, $f_2^{(l)}$, is related to the consistency of eq. (2.10) under collinear limits [4].

3. A surprising relation

The surprise in planar MSYM is that in some cases the hard remainder function $h_n(\{k_i\})$ defined through eq. (2.10) is actually a constant, independent of the kinematics. This result, which has been tested perturbatively for $n=4$ through three loops [4, 14], and for $n=5$ at two loops [25], is a conjecture beyond that:

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\varepsilon) M_n^{(1)}(l\varepsilon) + C^{(l)} + \mathcal{O}(\varepsilon)\right)\right]. \tag{3.1}$$

The dependence of the finite part of the logarithm of the amplitude is predicted to all orders by eq. (3.1), in terms of the cusp anomalous dimension. The prediction for four-gluon scattering is

$$\mathcal{M}_4^{\text{finite}} = \exp\left[\frac{1}{8}\gamma_K(a)\ln^2\left(\frac{s}{t}\right) + \text{const.}\right], \tag{3.2}$$

where $s = s_{12}$, $t = s_{23}$. As we shall discuss in section 9, this formula was confirmed at strong coupling by Alday and Maldacena [5] using the AdS/CFT correspondence [3]. In contrast, even at two loops there does not appear to be any comparably simple formula for the finite parts of four-gluon scattering amplitudes in QCD, or for the subleading-in-N_c terms in MSYM [4]. Instead of a constant, as in eq. (3.1), one finds that $h_n^{(2)}$ in eq. (2.10) is given by a complicated combination of polylogarithms involving the dimensionless ratio t/s. On the other hand, eq. (3.1) is reminiscent of the observation [26] that finite terms can also exponentiate in QCD, in *e.g.* the Drell-Yan cross section near partonic threshold.

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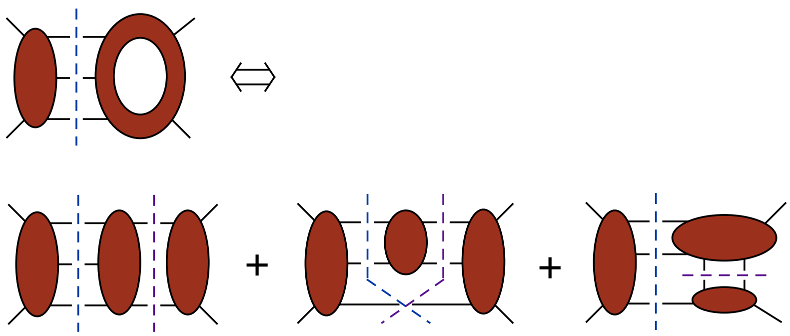

Figure 4: Example of generalized unitarity at three loops.

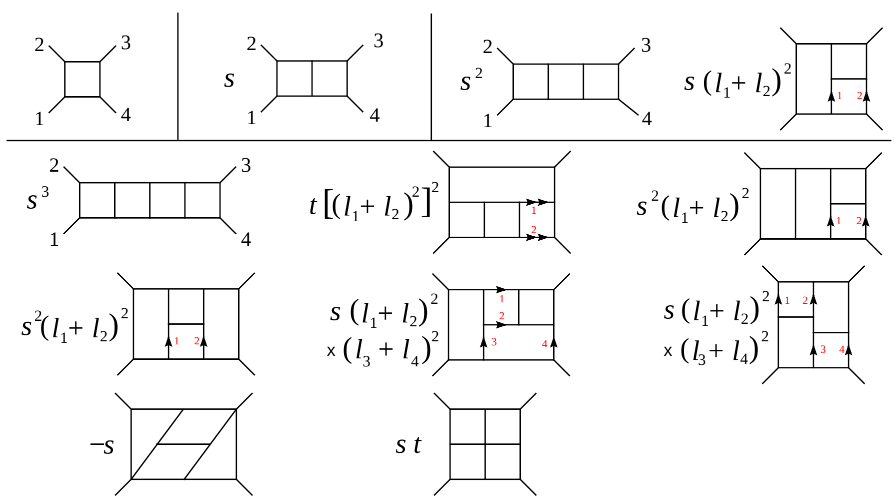

Figure 5: Integrals contributing to four-gluon scattering in planar MSYM, from one to four loops.

4. Evidence

The evidence in favor of eq. (3.1) was collected from explicit computations of the multi-loop scattering amplitudes. The amplitudes were constructed by evaluating (generalized) unitarity cuts [27, 28, 29, 30, 31, 32] and matching them to compact representations in terms of a relatively small number of multi-loop integrals, which turn out to have rather interesting properties. Ordinary unitarity relates discontinuities (cuts) in a given channel to products of lower-loop amplitudes, summed over the possible intermediate states in that channel. Generalized unitarity allows the lower-loop amplitudes to be further sliced, all the way down to tree amplitudes. Figure 4 shows an ordinary three-particle cut for the four-gluon amplitude. The information in this cut can be extracted more easily by further cutting the one-loop five-point amplitude on the right-hand side of the cut, decomposing it into the product of a four-point tree and a five-point tree; as illustrated, there are three inequivalent ways to do this. If one finds a representation of the amplitude that reproduces all the generalized cuts (in D dimensions), then that representation is correct.

Figure 5 shows the integrals that enter the four-gluon scattering amplitude in planar MSYM, from one to four loops [33, 7], along with their numerator factors. An overall factor of st is omitted from the rescaled amplitude $\mathcal{M}_4(s,t)$, and only one permutation of each integral is shown. At one

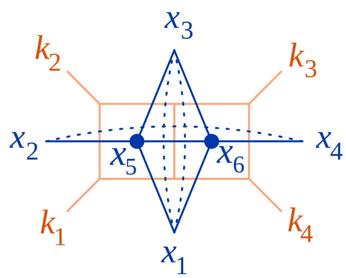

Figure 6: The two-loop planar double box integral (in orange) and associated dual graph (in blue).

and two loops, only scalar integrals appear; that is, the numerator factors in the integrand depend only on the external momentum invariants. At three loops, there are two integrals, the scalar triple ladder integral and the "tennis-court" integral shown at the top right of fig. 5. The latter integral marks the first appearance of a loop-momentum factor in the numerator, of the form $(l_i + l_j)^2$, as dictated by the "rung rule" [33]. The rung-rule correctly describes all integral topologies that can be reduced to trees by a sequence of two-particle cuts. At four loops, the last two integrals in fig. 5 have no two-particle cuts, and are somewhat more work to determine. At five loops (not shown) there are a total of 34 distinct integrals [34]. Still, it is remarkable that so few integrals are required to describe the amplitude.

5. Pseudo-conformal integrals

In fact, the integrals that appear in the four-point amplitude through five loops are all *pseudo-conformal*. To describe what this means [35], first consider taking all the external legs off shell, $k_i^2 \neq 0$, in order to be able to perform the integral without dimensional regularization, in $D = 4$. Next define dual momentum or sector variables x_i, such that the original momentum variables k_i are differences of the x_i, with $k_i^\mu = x_{i+1}^\mu - x_i^\mu$. Similarly define an x_i associated with each loop, such that $x_{ij} \equiv x_i - x_j$ is equal to the momentum flowing through the propagator that separates x_i from x_j. Figure 6 illustrates the dual diagram (in blue) associated with the planar double box integral (in orange) which appears in the two-loop MSYM amplitude. The dual propagators (denominator factors) are shown as solid blue lines, while dashed blue lines correspond to numerator factors in the integrand. The integral is given by

$$I^{(2)}(\{k_i\}) = s^2 t \int \frac{d^4p\, d^4q}{p^2(p-k_1)^2(p-k_1-k_2)^2 q^2 (q-k_4)^2 (q-k_3-k_4)^2 (p+q)^2} \tag{5.1}$$

$$= (x_{13}^2)^2 x_{24}^2 \int \frac{d^4x_5\, d^4x_6}{x_{15}^2 x_{25}^2 x_{35}^2 x_{36}^2 x_{46}^2 x_{16}^2 x_{56}^2}\,, \tag{5.2}$$

using $s = (k_1 + k_2)^2 = x_{13}^2$, $p^2 = x_{15}^2$, and so forth.

Under an inversion, $x_i^\mu \to x_i^\mu / x_i^2$, we have

$$x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}\,, \qquad d^4x_5 \to \frac{d^4x_5}{(x_5^2)^4}\,, \qquad d^4x_6 \to \frac{d^4x_6}{(x_6^2)^4}\,, \tag{5.3}$$

and it is easy to see that eq. (5.2) is left invariant. In general, an integral is invariant under inversion if there is a net of zero (four) lines emerging from each external (internal) x_i vertex, where "net" means solid lines minus dashed lines. Every integral is automatically invariant under translations of the dual variables, $x_i \to x_i + c$, and under Lorentz transformations. Because these transformations, together with inversions, generate the conformal group, invariance under inversion suffices to guarantee dual conformal invariance for the integral. Now we can define a pseudo-conformal integral to be one which is finite in $D = 4$, after all the k_i^2 are taken off-shell, is dual conformal invariant, and possesses a smooth $k_i^2 \to 0$ limit. The last condition ensures that the integral does not become infinite or vanish as we return to the on-shell limit.

Dual conformal symmetry arose in the context of multi-loop ladder integrals [36], and in two dimensions in the theory of (planar) Reggeon interactions [37]. Its relevance for the structure of MSYM amplitudes was first pointed out by Drummond, Henn, Smirnov and Sokatchev [35], based on the structure of the amplitudes through three loops, and the rung-rule contributions at four loops. The four- and five-loop four-gluon amplitudes can be organized as well, according to the two principles:

- Only pseudo-conformal integrals appear.
- The pseudo-conformal integrals appear only with weight ± 1.

Originally it appeared that two integrals at four loops [7] and 25 integrals at five loops [34] were pseudo-conformal but did *not* appear in the amplitude. However, it was later pointed out that those integrals were not actually finite in $D = 4$ [38]. Recently, some intuition into the signs ± 1 has been given by considering the singularity structure of the various integrals more carefully [39].

6. Evaluating integrals

Once the structure of the amplitude is known in terms of basic integrals, the next task is to evaluate those integrals, analytically if possible, otherwise numerically. For example, to test eq. (3.1) at three loops, we first expand it out to third order, obtaining the iterative relation,

$$M_n^{(3)}(\varepsilon) = -\frac{1}{3}\Big[M_n^{(1)}(\varepsilon)\Big]^3 + M_n^{(1)}(\varepsilon) M_n^{(2)}(\varepsilon) + f^{(3)}(\varepsilon) M_n^{(1)}(3\varepsilon) + C^{(3)} + \mathscr{O}(\varepsilon). \qquad (6.1)$$

To test this relation at order ε^0 for $n = 4$ [14], we need the following integrals:

- The one-loop box integral through ε^4 — because it has $1/\varepsilon^2$ poles, and appears cubed in eq. (6.1).
- The planar double box integral [40] in fig. 6 through ε^2 — because $M_4^{(2)}(\varepsilon)$ appears in eq. (6.1) multiplied by $M_4^{(1)}(\varepsilon)$.
- The triple ladder [41] and tennis-court [14] integrals through ε^0.

Mellin-Barnes techniques (see *e.g.* ref. [42]) are very useful in this regard. Inserting the results into eq. (6.1), and using identities among weight 6 harmonic polylogarithms [43], the relation (6.1) was verified, and three of the four constants at three loops could be extracted:

$$f_0^{(3)} = \frac{11}{5}(\zeta_2)^2, \qquad f_1^{(3)} = 6\zeta_5 + 5\zeta_2\zeta_3, \qquad f_2^{(3)} = c_1\,\zeta_6 + c_2\,(\zeta_3)^2, \qquad (6.2)$$

$$C^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)(\zeta_3)^2\,. \qquad (6.3)$$

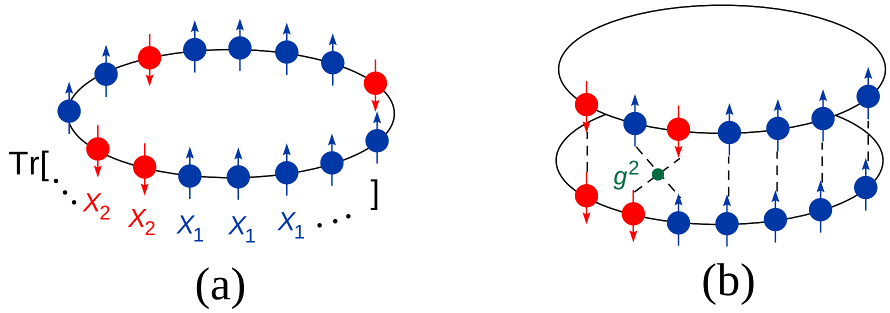

Figure 7: (a) Mapping a single-trace operator to a spin chain. (b) One-loop contribution to the anomalous dimension matrix at large N_c.

The first two of these constants control infrared divergences. The value of $f_0^{(3)} = \hat{\gamma}_K^{(3)}/4$ confirms a result for the three-loop cusp anomalous dimension in planar MSYM, which was first obtained [44] by applying the principle of "maximal transcendentality" to the corresponding result in QCD [45]. The value of $f_1^{(3)} = (3/2)\hat{G}_0^{(3)}$ gives the three-loop collinear anomalous dimension, which was found to agree (applying the same principle) with the QCD result [46]. The constants $f_2^{(3)}$ and $C^{(3)}$ are inseparable using only the four-gluon amplitude; either the five-gluon amplitude or a collinear analysis would be required to separate them. The numbers c_1 and c_2 are expected to be rational.

A similar analysis can be performed at four loops [7, 8, 47], except that the integrals become less tractable analytically. Fortunately, there are methods available for automating the construction of Mellin-Barnes representations [48], the extraction of $1/\varepsilon$ poles, and the setting up of numerical integration over multiple contours for the Mellin inversion [49, 50]. Before describing the four-loop results, let us turn to some very interesting developments that have taken place, based on integrability.

7. Integrability and anomalous dimensions

In large-N_c gauge theory, a preferred role is played by local "single-trace operators". In the case of MSYM, one subsector of such operators is provided by products of the 3 complex scalar fields, X_i, $i = 1,2,3$. The operator $\mathrm{Tr}[X_1^n]$ is a so-called BPS operator, and is unrenormalized to all orders in λ. A set of operators with more interesting renormalization properties are close to BPS [51], and contain X_2 fields as well as X_1, for example,

$$\mathrm{Tr}[\ldots X_2 X_2 X_1 X_1 X_1 \ldots]\,. \tag{7.1}$$

As shown in fig. 7(a), this set of operators can be mapped to a one-dimensional, periodic spin chain, in which X_1 (X_2) is mapped to spin up (spin down), corresponding to a finite-dimensional (spin 1/2) representation of $SU(2)$ spin symmetry.

The anomalous dimensions of the set of operators (7.1) are found by diagonalizing the dilatation operator, which can be mapped to a Hamiltonian for the spin chain. In the large-N_c limit, this Hamiltonian is local, because non-local interactions correspond to non-planar diagrams. For example, as shown in fig. 7(b), a one-loop contribution from a four-scalar interaction can only affect color-adjacent X_i fields (spins). (The range of the interactions does increase with the number of loops.) Minahan and Zarembo [52] showed that the one-loop Hamiltonian was *integrable*; that is, the system possesses

- infinitely many conserved charges,
- a spectrum of quasi-particles (spin waves, or magnons),
- magnon scattering via a $2 \to 2$ S matrix obeying the Yang-Baxter equation,
- solutions for the anomalous dimensions (energies) via a Bethe ansatz.

Integrable structures in QCD had been identified previously [53, 54, 55]. In planar MSYM, however, the integrability appears to persist to all orders in λ; indeed, it is known to be present at strong coupling, from the form of the classical sigma model on target space $\mathrm{AdS}_5 \times S^5$ [56].

There is a rich literature of extensions of the one-loop results of ref. [52] to higher loops, even all loop orders, and to more general sectors of planar MSYM, which I can only touch on here [6, 57, 58, 59, 60]. The sector most relevant to gluon scattering amplitudes is not the spin 1/2 $SU(2)$ sector (7.1), but that in which the X_2 fields are replaced by covariant derivatives $\mathscr{D}^+$ acting in the $+$ (light-cone) direction,

$$\mathrm{Tr}[\ldots \mathscr{D}^+ \mathscr{D}^+ X_1 X_1 X_1 \ldots] \,. \tag{7.2}$$

These derivatives act as an infinite-dimensional representation of the noncompact version of $SU(2)$, namely $SL(2)$. Within this sector, the cusp anomalous dimension can be found by taking the limit of a small number of fields (spin chain length) L, and a large number of derivatives j, to get the operator

$$O_j = \mathrm{Tr}\big[X_1 (\mathscr{D}^+)^j X_1\big] , \quad j \to \infty. \tag{7.3}$$

By the universality of the cusp anomalous dimension, it does not matter which leading-twist large j operator is used; they all have the behavior (2.2) at large j.

8. An all-orders proposal

In brief, and omitting many subtleties, the Bethe-ansatz solution consists of taking the eigenstates of the Hamiltonian to be multi-magnon states, with phase-shifts induced by repeated $2 \to 2$ scatterings. The periodicity of the wave function on the closed chain leads to the Bethe condition, which depends on the chain length L. In the limit $L \to \infty$, the Bethe condition becomes an integral equation, which depends on the form of the $2 \to 2$ magnon S matrix [60]. This S matrix is *almost* fixed by the symmetries, but an overall phase, the *dressing factor*, is not so easily deduced. Finally, there is a potential *wrapping problem* in extrapolating to the cusp anomalous dimension: The Bethe ansatz is only rigorously valid when the interaction range (the number of loops) is smaller than the chain periodicity L. However, even though the cusp anomalous dimension has $L = 2$, it has been argued that its universality leads it to appear within large-L sectors, and renders it immune to the wrapping problem [55, 60, 61].

Eden and Staudacher [60] derived an integral equation for the all-orders behavior of the cusp anomalous dimension from an all-loop Bethe ansatz [58], by assuming that the dressing factor did

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not play a role perturbatively. This equation agreed with the known one-, two-, and three-loop coefficients of $\gamma_K(\lambda)$, and made the four-loop prediction,

$$f_0^{(4)}\Big|_{\rm ES} = \frac{1}{4}\hat{\gamma}_K^{(4)}\Big|_{\rm ES} = -\frac{73}{2520}\pi^6 + (\zeta_3)^2 = -26.4048255\ldots, \tag{8.1}$$

motivating the computation of the four-loop four-gluon scattering amplitude, and the numerical extraction of $f_0^{(4)}$ from it. The result found [7],

$$f_0^{(4)} = -29.335 \pm 0.052, \tag{8.2}$$

and later with much improved precision [8],

$$f_0^{(4)} = -29.29473 \pm 0.00005, \tag{8.3}$$

was consistent, not with eq. (8.1), but with a version in which the sign of the $(\zeta_3)^2$ term was flipped,

$$f_0^{(4)}\Big|_{\rm BES} = \frac{1}{4}\hat{\gamma}_K^{(4)}\Big|_{\rm BES} = -\frac{73}{2520}\pi^6 - (\zeta_3)^2 = -29.2947071202\ldots. \tag{8.4}$$

Remarkably, the latter value was predicted, simultaneously with ref. [7], by Beisert, Eden and Staudacher (BES) [6], based on a modified integral equation taking into account a new proposal for the dressing factor, with nontrivial effects beginning at four loops. The proposed dressing factor was deduced by using its properties at strong-coupling, where it had been known to be nontrivial [62]. Perhaps even more remarkably, the only effect of including the dressing-factor term on the weak-coupling expansion of the integral equation, is to make the substitution $\zeta_{2k+1} \to i\,\zeta_{2k+1}$, which affects only the signs of the odd-zeta terms in the perturbative expansion. At five loops, this sign-flip is

$$f_0^{(5)}\Big|_{\rm ES} = (887/56700)\pi^8 - 2\,\zeta_2\,(\zeta_3)^2 - 10\,\zeta_3\,\zeta_5 = 131.21\ldots \tag{8.5}$$

$$\to\ f_0^{(5)}\Big|_{\rm BES} = (887/56700)\pi^8 + 2\,\zeta_2\,(\zeta_3)^2 + 10\,\zeta_3\,\zeta_5 = 165.65\ldots, \tag{8.6}$$

which also agrees with interpolation-based estimates [7].

The BES integral equation was solved numerically [9], and later expanded analytically to all orders in the strong-coupling $(1/\sqrt{\lambda})$ expansion [10]. Its strong-coupling behavior is consistent with the known first three terms in this expansion [11, 12, 13]. This concordance, plus the agreement with the first four loops at weak coupling, strongly suggests that the BES equation is an exact solution for the cusp anomalous dimension, valid for arbitrary λ.

The next quantity appearing in the planar MSYM gluon scattering amplitudes, $G_0(\lambda)$, which controls single poles in the argument of the exponential in eq. (2.10), is not quite as well known. The first four loop coefficients are known, the fourth numerically [47],

$$G_0(\lambda) = -\zeta_3\left(\frac{\lambda}{8\pi^2}\right)^2 + \frac{2}{3}(6\zeta_5 + 5\zeta_2\zeta_3)\left(\frac{\lambda}{8\pi^2}\right)^3 - (77.56 \pm 0.02)\left(\frac{\lambda}{8\pi^2}\right)^4 + \cdots, \tag{8.7}$$

and one coefficient is now known in the strong-coupling expansion [5]. A Padé approximant incorporating this data has been constructed [47]. Clearly, it would be of great interest if an integral equation could be found governing $G_0(\lambda)$ for all values of the coupling. Finding a cleaner operator interpretation for this quantity may be quite useful in this respect.

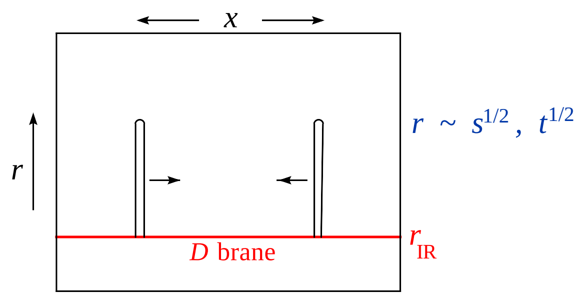

Figure 8: Gluon scattering in anti-de Sitter space. Four-dimensional space-time has coordinates x. Hard-scattering kinematics force the strings to stretch a long distance in the radial direction r, from their infrared "anchor", a D brane located at $r_{\rm IR}$.

9. Gluon scattering at strong coupling

Now let us return to the picture of gluon scattering at strong coupling developed by Alday and Maldacena [5]. Figure 8 is another view of the AdS space sketched in fig. 1, showing also a pair of incoming open string states prior to a hard scattering. The ends of the open strings are anchored on a D brane, which serves as an infrared regulator and is located at a small value of the AdS radial variable, $r_{\rm IR}$. The short-distance (UV) nature of the hard scattering forces part of the string to penetrate to large values of $r \sim \sqrt{s}, \sqrt{t}$. Gluons correspond to this part of the string, and the rest of the string can be thought of as the color string a gluon has to drag along with it, which is particularly important at strong coupling. Because the string has to stretch a long way, the scattering is semi-classical [5].

This regime is similar to very high-energy, fixed-angle scattering in string theory in flat space-time, which was studied long ago [63]. Evaluated on the classical solution, for the case of color-ordered scattering with gluons 1 and 3 incoming, 2 and 4 outgoing, the string world-sheet action is imaginary. The Euclidean action, or area, is real, and is logarithmically divergent, leading to a large exponential suppression [5],

$$\mathcal{M}_4 \sim \exp[iS_{\rm cl}] \sim \exp[-S^E_{\rm cl}] \sim \exp[-\sqrt{\lambda}\ln^2(r/r_{\rm IR})]\,, \tag{9.1}$$

where $r \sim \sqrt{s}, \sqrt{t}$. The coupling-constant dependence in eq. (9.1) originates from the formula for the radius of curvature of AdS, $R^2_{\rm AdS} = \sqrt{\lambda}$, which enters the world-sheet action. From the string point of view, the suppression can be attributed to a tunnelling suppression factor. From the point of view of a four-dimensional collider physicist, it is a typical Sudakov suppression factor [16]: The probability for a pair of gluons to make it all the way into and back out of the scattering without radiating at all is exponentially small — especially at strong coupling, $\lambda \to \infty$ — with a double log in the exponential.

To make contact with the perturbative results, Alday and Maldacena constructed a dimensionally-regularized version of $\mathrm{AdS}_5 \times S^5$, instead of using the D brane location $r_{\rm IR}$ as a regulator. They also introduced T-dual variables y^μ in place of the usual four-dimensional coordinates. The T-duality transformation is a kind of Fourier transform, so the y^μ are like momentum variables. Indeed, the asymptotic boundary value for the world-sheet, which resides at $r = 0$ in the

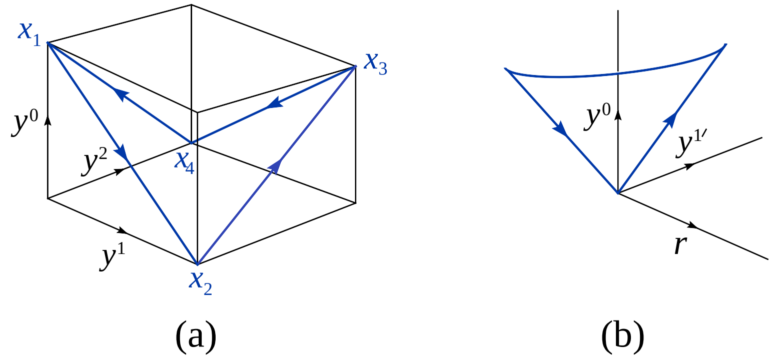

Figure 9: (a) Boundary condition at $r=0$ for $gg \to gg$ scattering at 90° in the u-channel. (b) The cusp solution, showing r as a function of y^0 and $y^{1\prime}$.

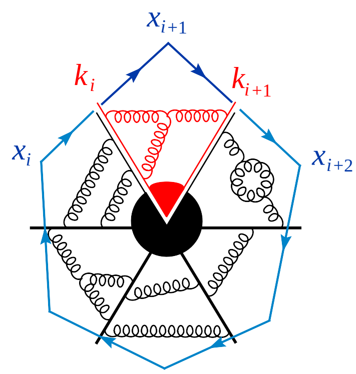

Figure 10: Planar Feynman diagrams, ringed by the strong-coupling boundary condition in dual momentum variables. Each Sudakov wedge has a single cusp associated to it.

dimensionally-regularized setup, is a polygon constructed from light-like segments in y^μ, with the corners y_i^μ satisfying

$$y_{i+1}^\mu - y_i^\mu = k_i^\mu \,, \tag{9.2}$$

where k_i^μ is the momentum of the i^{th} gluon. From eq. (9.2), we see that the y_i coincide precisely with the dual variables x_i introduced in section 5 to discuss dual conformal invariance! Figure 9(a) shows the light-like quadrilateral boundary satisfying eq. (9.2) for the case of $2 \to 2$ gluon scattering at 90° in the 1-2 plane, with k_1 and k_3 incoming. The vertical direction is the (dualized) time direction.

Near each corner of the polygonal boundary, the solution must look like a cusp solution, previously constructed by Kruczenski [64], in which r behaves like $r = \sqrt{(2+\varepsilon)[(y^0)^2-(y^{1'})^2]} = \sqrt{(2+\varepsilon)y^+y^-}$ for some spatial coordinate $y^{1'}$, and light-cone coordinates $y^\pm$. This hyperboloid is shown in fig. 9(b). The classical action (area) for this solution has a divergence regulated by ε,

$$iS_{\text{cl}} = -S_{\text{cl}}^E \;\to\; -R_{\text{AdS}}^2 \int_0 \frac{dy^+dy^-}{(y^+y^-)^{1+\varepsilon/2}} \;\sim\; -\frac{1}{\varepsilon^2}\frac{\sqrt{\lambda}}{2\pi} \;\sim\; -\frac{1}{\varepsilon^2}\frac{\gamma_K(\lambda)}{2} \,. \tag{9.3}$$

The coefficient of the leading divergence is just the strong-coupling limit of the cusp anomalous dimension [11],

$$\gamma_K(\lambda) \sim \frac{\sqrt{\lambda}}{\pi}, \qquad \text{as } \lambda \to \infty. \tag{9.4}$$

Figure 10 illustrates the situation heuristically. The singular part of the planar amplitude can be broken up into Sudakov wedges, as in fig. 3 and eq. (2.5). The overlap of soft and collinear divergences corresponds to regions between two hard lines, *e.g.* k_i and k_{i+1}. Thus each wedge is associated with a single divergent cusp [65], of the form shown in fig. 9(b).

The full classical solution, for arbitrary scattering angle, was found by Alday and Maldacena [5]. Its action gives a strong-coupling amplitude of the form,

$$\begin{aligned}
\mathcal{M}_4 &= \exp[-S_{\rm cl}^E], \\
-S_{\rm cl}^E &= -\frac{1}{\varepsilon^2}\frac{\sqrt{\lambda}}{\pi}\left[\left(\frac{\mu_{\rm IR}^2}{-s}\right)^{\varepsilon/2} + \left(\frac{\mu_{\rm IR}^2}{-t}\right)^{\varepsilon/2}\right] - \frac{1}{\varepsilon}\frac{\sqrt{\lambda}}{2\pi}(1-\ln 2)\left[\left(\frac{\mu_{\rm IR}^2}{-s}\right)^{\varepsilon/2} + \left(\frac{\mu_{\rm IR}^2}{-t}\right)^{\varepsilon/2}\right] \\
&\quad + \frac{\lambda}{8\pi}\left[\ln^2\left(\frac{s}{t}\right) + \tilde{C}\right] + \mathcal{O}(\varepsilon),
\end{aligned} \tag{9.5}$$

where $\mu_{IR}^2 = 4\pi e^{-\gamma}\mu^2$. This expression can be compared with the strong-coupling extrapolation of the ansatz (3.1) [5]. The $1/\varepsilon^2$ poles agree, using the strong-coupling value for $\gamma_K(\lambda)$ from eq. (9.4). The $1/\varepsilon$ poles give the strong-coupling limit of the collinear anomalous dimension $G_0(\lambda)$,

$$G_0(\lambda) \sim \sqrt{\lambda}\,\frac{(1-\ln 2)}{2\pi}, \qquad \text{as } \lambda \to \infty. \tag{9.6}$$

The finite part of $\mathcal{M}_4$ has a dependence on s and t which is precisely as predicted by eq. (3.2).

10. Dual variables and Wilson loops at weak coupling

The dual momentum variables x_i^μ play a prominent role in the strong-coupling computation of Alday and Maldacena, which is essentially the same as computing a Wilson loop vacuum expectation value at strong coupling. Inspired by this connection, there has been a sequence of recent Wilson-loop computations for loops corresponding to the dual-momentum boundary conditions for an n-point amplitude, namely polygons composed of n-light-like segments, with corners obeying eq. (9.2).

The first of these computations was by Drummond, Korchemsky and Sokatchev [38], for the one-loop expectation value of a quadrilateral ($n=4$) Wilson loop. Up to constants of the kinematics, attributable to a different regulator (in the UV) than the one used for the amplitudes (in the IR), the expectation value agreed, surprisingly, with the one-loop four-gluon amplitude, normalized by the tree amplitude, *i.e.* eq. (3.2). Next, Brandhuber, Heslop and Travaglini [66] showed that the same statement is actually true for the n-gon Wilson loop for any n, compared with the normalized one-loop amplitude [28] for the so-called maximally-helicity-violating (MHV) configuration of gluon helicities (two negative and $(n-2)$ positive). The Wilson-loop computation knows nothing about the polarizations of the external gluons. It is manifestly symmetric under cyclic permutations and reflections of the polygon. For $n=4$ and 5, a Ward identity for $\mathcal{N}=4$ supersymmetry shows that all helicity configurations in MSYM are equivalent, and that the normalized amplitudes have the same manifest symmetries as the polygonal Wilson loop [67]. However, beyond $n=5$ there are non-MHV configurations which do not have these symmetries. How does the Wilson loop know it is "supposed to" match the MHV amplitude alone?

Drummond, Henn, Korchemsky and Sokatchev (DHKS) then repeated the Wilson-loop computation in MSYM at two loops, first for the $n = 4$ case [68] and then[1] for the $n = 5$ case [69]. Again the results matched the full two-loop MSYM scattering amplitudes [4, 25], up to constants of the kinematics. Furthermore, DHKS first proposed [68] and then proved [69] an anomalous dual conformal Ward identity for Wilson loops, in which the anomaly arises from UV divergences proportional to $\gamma_K(\lambda)$. The solution to the Ward identity is unique for $n = 4$ and 5. Beyond $n = 5$, there are multiple solutions, due to the existence of nontrivial conformally-invariant cross ratios. For example, for $n = 6$ the quantity $u_1 \equiv x_{13}^2 x_{46}^2/(x_{14}^2 x_{36}^2) = s_{12}s_{45}/(s_{123}s_{345})$ is invariant under the inversion (5.3), and there are two other such cross ratios. (The appearance of $x_{i,i+1}^2 = k_i^2$ in a cross ratio is forbidden by the on-shell constraint $k_i^2 = 0$.)

DHKS also showed that the amplitude ansatz (3.1) obeys the anomalous dual conformal Ward identity. Given that the ansatz was known to be correct for $n = 4$ and 5 [4, 25], and the uniqueness of the Ward identity solution for these cases, this result could explain why the amplitude should match the Wilson loop in these cases. However, it was not clear what should happen for larger n. Indeed, Alday and Maldacena [70] gave an argument, based on approximating a Euclidean rectangular loop by a zig-zag configuration composed of many light-like segments, that the ansatz (3.1) should fail at strong coupling for sufficiently large n. DHKS [71] found that the hexagonal Wilson loop could *not* be described at two loops by the ansatz (3.1). This result left open the question, however, of whether the ansatz failed to describe MHV amplitudes beyond $n = 5$, or whether the relation between amplitudes and Wilson loops failed beyond two loops (or both).

The high-energy limits of the ansatz (3.1) have been examined for consistency with expected Regge behavior. For $n = 4$ and 5, the ansatz appears to have consistent behavior in all such limits [38, 72, 73, 74] However, there appears to be a difficulty with the ansatz for the six-gluon amplitude starting at two loops [74]. Very recently, a computation of the "parity even" part of the six-gluon MHV amplitude [75] has revealed directly that the ansatz (3.1) does fail for $n = 6$. However, a numerical comparison [75, 76] with the corresponding hexagonal Wilson loop [71] shows that the MHV-amplitude-Wilson-loop equivalence is still intact at two loops and $n = 6$. This result means that the scattering amplitude also obeys the dual conformal Ward identity. On the other hand, the solution to the Ward identity is not unique for $n = 6$. Hence some other principle, as yet unidentified, is needed to explain why MHV amplitudes are equivalent to Wilson loops in MSYM.

11. Conclusions

We have seen that gluon scattering amplitudes in planar $\mathcal{N} = 4$ super-Yang-Mills theory have some remarkable properties. It appears that the exact forms of the four-gluon and five-gluon amplitudes are given by the ansatz (3.1), which depends only on four different functions of the large-N_c coupling parameter λ: f_0, f_1, f_2 and C. Because an exact solution for one of the four functions — f_0, the cusp anomalous dimension — seems to be in hand [6], perhaps one can say that these cases are "1/4 solved". The fixed dependence of the ansatz (3.1) on the scattering angle(s) is apparently

[1] Most of the results reported from this point on appeared after this talk was presented, but before the write-up was completed. I include them here because of their close connection with the contents of the talk.

related to the uniqueness of solutions to a dual conformal Ward identity for $n = 4$ and 5 [68, 69], and an equivalence between (MHV) amplitudes and Wilson lines [5, 38, 66, 68, 69]. Although the ansatz (3.1) fails for the MHV six-gluon amplitude [75] at two loops, the equivalence remains valid [75, 76].

There are still many open questions. Are there simple(r) AdS/operator interpretations of the other three functions? Can one find integral equations for them, based on integrability? What is the precise relation between integrability and dual conformal invariance? Do non-MHV amplitudes obey any simple patterns, or bear any relation to Wilson loop expectation values? From the structure of the one-loop amplitudes, *e.g.* for six gluons [29], any such relations must be considerably more intricate. What happens in other conformal theories? Finally, we can hope that some of these advances may eventually help to shed light on scattering amplitudes in other gauge theories, particularly QCD, whose understanding — as exemplified by the other talks at this symposium — is vital to the search for new physics at the Large Hadron Collider.

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