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Large-scale modifications of gravity and the graviton mass

P. Tinyakov*
Service de Physique Théorique, Université Libre de Bruxelles, CP225, blv. du Triomphe, B-1050 Bruxelles, Belgium
E-mail: petr.tiniakov@ulb.ac.be

Motivated by the puzzles of the dark matter and dark energy, we address the question whether the gravitational interaction can be consistently modified to include a non-zero graviton mass. After discussing generic difficulties of such a scenario in the Lorentz-invariant framework we present a model with spontaneously broken Lorentz symmetry which is theoretically consistent and experimentally viable. In this model the Newtonian potential is unmodified at the linear level, so the mass of the graviton is weakly constrained and can be as large as $(10^{14}\text{cm})^{-1}$. The model possesses cosmological solutions with extra contributions to the energy density which behave like the cosmological constant and the matter with the equation of state depending on the parameters of the model. The massive graviton can play the role of the dark matter, in which case the model is falsifiable by the gravitational wave experiments.

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1. Introduction

The question of whether the gravitational interaction is described by the Einstein theory of relativity at all scales is of both theoretical and practical interest. On theory side, the attempts to construct an alternative model, successful or not, serve to better understanding of the fundamental principles lying behind the theory of gravity. The requirement of general covariance fixes the form of the gravitational Lagrangian almost uniquely. There exist only a few modifications of gravity which do not involve higher-derivative terms, the most known being scalar-tensor models of the Brans-Dicke type [1]. A natural question then is whether the general covariance can be broken, say, spontaneously, in a manner similar to the Higgs mechanism in gauge theories. If that were the case one would expect, by analogy with the gauge theory, that the graviton gets a non-zero mass. More generally, the question is whether at all one can construct a consistent theory of gravity where the graviton has a non-zero mass. Whatever is the answer, it will certainly contribute to better understanding of the gravitational interaction.

On phenomenological side, the conventional theory of gravity is completely successful at scales of order and below the solar system size up to scales of order a fraction of a millimeter. At larger scales there is a hint of a problem: one needs to introduce the (otherwise undetected) dark matter in order to explain the rotation curves of the galaxies and galaxy clusters. At cosmological scales yet another form of matter — the one behaving like the cosmological constant — is also needed [2]. With these two additions the Einstein's theory apparently works quite well at all scales. However, it is disturbing that the new components are only needed to correct the gravitational interaction at very large scales. Moreover, at those scales the new components must play a dominant role in order to fit the observations.

Before accepting the existence of the new forms of matter it is natural to wonder whether the gravitational interaction itself can be modified at large distances so as to explain the existing observations without the need of the dark matter and the dark energy. Whether likely or not, this is a logical possibility.

Perhaps one of the first attempts to find an alternative to the dark matter was the model known as MOND (modified Newtonian dynamics) [3, 4]. In this model one postulates the existence of a critical acceleration at which the $1/r^2$ fall off of the Newtonian force changes to a slower dependence. There is an ongoing discussion in the literature whether this model is viable phenomenologically (see, e.g., Ref.[5] and references therein) and whether it can be generalized to a fully relativistic theory [6].

The idea to modify the gravitational interaction at large distances lies behind several recent attempts to find alternative models of gravity. One of the first such attempts was performed in Ref.[7] in the context of extra dimensions. The model developed there involved branes with a negative tension and was later shown to possess ghosts [8, 9]. Another attempt employing extra dimensions is the DGP model [10]. This model has interesting cosmological solutions [11, 12]. However it is still debated whether it is consistent theoretically [13, 14, 15]. Yet another approach is based on the actions which are singular in the low curvature limit, the so-called $f(R)$ gravities (see, e.g., [16]). These models are widely discussed now in the cosmological context [17, 18, 19], however their ability to pass the solar system tests is questioned [20, 21] (see, however, Ref.[22]). One should mention also the bi-gravity models which involve two metric tensors [23, 24, 25]. None of these models possesses massive gravitons.

In these lectures I will concentrate on the question of whether the graviton can be given a mass. This question has a long history dating back to late 30th when Fierz and Pauli [26] have constructed a Lorentz-invariant mass term for the spin 2 field and shown that the resulting theory has no ghosts. It was noticed much later [27, 28] that the model constructed by Fierz and Pauli has a problem which makes it phenomenologically unacceptable: it predicts the wrong value of the light bending by massive bodies, and moreover, this (wrong) prediction persists in the limit of vanishing graviton mass. This phenomenon of the absence of a smooth zero-mass limit is known as the van Dam-Veltman-Zakharov (vDVZ) discontinuity. The bending of light has been experimentally measured and shown to be in agreement with the Einstein's general relativity, thus apparently ruling out the Fiertz-Pauli model. The conclusion was that the graviton mass has to be strictly zero.

The status of the graviton mass seemed clear until it was noticed [29] that the discontinuity argument leading to the contradiction with the experiment has a loophole. The light bending in the massive case was calculated within the linear approximation, while the model is actually non-linear at the relevant scale. It was shown that the onset of the non-linear regime at non-zero graviton mass happens at much larger distances from the source than one would naively expect. What precisely is the corresponding distance scale may depend on the non-linear terms in the action [30], but this scale cannot be made short enough. Another manifestation of the same phenomenon is the strong coupling between longitudinal polarizations of massive gravitons which sets in at unacceptably low energies [31].

A new approach to the modification of gravity has been developed recently which involves the spontaneous breaking of the Lorentz symmetry. The first model of this type is the so-called ghost condensate model [32]. Although the graviton is massless in this model, we will show that it can be generalized in such a way that the graviton gets a mass, while the problems mentioned above do not arise. One gets, therefore, a consistent theory which is well-defined below a certain scale and in which the Lorentz symmetry is spontaneously broken, the graviton is massive and obvious pathologies are absent. We will argue below that this theory is a perfect theoretical laboratory for studying modifications of gravity and may even be interesting from the phenomenological point of view.

Before we proceed to the discussion of this low-energy effective theory, an important remark is in orded. The model which we will consider in these lectures is a low-energy effective theory which presumably arises from some fundamental theory in the low-energy regime. However, no such fundamental theory (the UV-completion) has been constructed so far. Moreover, neither uniqueness nor even existence of the UV-completion is guaranteed *a priori*. Finding such a UV-completion remains the major unsolved problem of models with the large-scale modifications of gravity. We will not discuss this problem in more detail here.

The outline of these lectures is as follows. We start in Sect. 2 by discussing the generic obstructions to massive gravity. In Sect. 3 we outline the ways to overcome the difficulties and present a class of models where these ideas are realized and the graviton is made massive. We investigate this class of models at the liner level in Sect. 4. Sec. 5 deals with some phenomenological consequences of the graviton mass in a concrete model. In particular, the cosmological solutions are considered. Finally, in Sect. 6 we summarize the results and outline the open questions.

2. Theoretical obstructions to massive gravity

Let us discuss in more detail the origin of the problems which arise when one tries to modify the general relativity in such a way that the graviton gets a mass. These problems are: the appearance of ghosts, the vDVZ discontinuity and the strong coupling at a low energy scale.

Instabilities and ghosts. There may occur several types of instabilities. In the simplest case of a single variable ϕ the quadratic action has the form

$$\int d^4x \left(\alpha\dot{\phi}^2 - \beta(\partial_i\phi)^2 - m^2\phi^2\right),$$

where α, β and m^2 are some real coefficients. We do not assume here the Lorentz invariance which would require $\alpha = \beta$. The equation of motion for the variable ϕ in the Fourier space reads

$$\alpha\ddot{\phi} + \beta k_i^2\phi + m^2\phi = 0. \tag{2.1}$$

One usually has $\alpha > 0$, $\beta > 0$, $m^2 > 0$ and the solution to eq. (2.1) is oscillatory. This is the "normal" case which corresponds to usual particles.

If $\alpha > 0$ but $\beta k^2 + m^2 < 0$ for some k, then the solutions to eq. (2.1) are exponentially decaying or growing, so the instability is present. Two cases should be distinguished. If $\beta > 0$ then for $k > \sqrt{|m^2|/\beta}$ the instability disappears. If this critical value of k is very low, there may be not enough time for the instability to develop. Thus, this kind of instability is not necessarily a pathology. On the contrary, if $\beta < 0$, then the instability persists at an arbitrary large k and is therefore arbitrarily rapid. Instabilities of this type are unacceptable.

Finally, if $\alpha < 0$ then the contribution of the kinetic term $\alpha\dot{\phi}^2$ into the energy is negative and unbounded from below. This is physically unacceptable unless the field ϕ is completely decoupled from the rest of the system, which is an unrealistic situation. Note that if $\alpha < 0$ and $\beta k^2 + m^2 < 0$ at the same time, the solutions to eq. (2.1) oscillate, so the pathology does not show up in the equations of motion. However, both the kinetic and the potential term have negative energy. The field of this type is referred to as ghost.

The requirement of the absence of ghosts and instabilities is routinely used in field theory. It allows, for instance, to fix uniquely the conventional gauge-invariant form of the Lagrangian of the massless vector field, $-1/4F_{\mu\nu}F^{\mu\nu}$. This latter observation may be used to see very easily that there is only one Lorentz-invariant graviton mass term which gives a theory free from instabilities and ghosts. Indeed, consider a quadratic Lagrangian for the metric perturbation $h_{\mu\nu}$ around the flat Minkowski space and add all possible Lorentz-invariant mass terms,

$$\int d^4x \left\{L^{(2)}(h_{\mu\nu}) + \alpha h_{\mu\nu}^2 + \beta(h_\mu^\mu)^2\right\}. \tag{2.2}$$

The first term in eq. (2.2) is just the standard kinetic term which comes from the Einstein action $\int\sqrt{g}R$ and describes massless gravitational waves. Its precise form is not important for the argument. The second and third terms are the only possible Lorentz-invariant combinations which are quadratic in $h_{\mu\nu}$ and do not contain derivatives.

Let us see why only at $\beta = -\alpha$ the action (2.2) describes a non-pathological theory. To this end consider a *particular* metric perturbation,

$$h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu. \tag{2.3}$$

This form of the metric perturbation corresponds to a "pure gauge", as eq.(2.3) is precisely the coordinate transformation at the linear level. Like in the case of the gauge theory, one may show that in the presence of the mass terms the longitudinal perturbations dominate at high momenta.

Inserting eq. (2.3) into the action (2.2) we see that only the mass terms (the second and third terms) contribute (recall that the Hilbert-Einstein part of the action is gauge-invariant). Thus, we find

$$\int d^4x \left\{2\alpha(\partial_\mu \xi_\nu)^2 + (2\alpha+4\beta)(\partial_\mu \xi^\mu)^2\right\}.$$

This is a general Lorentz-invariant action for the vector field ξ_μ. As is well known from the field theory, this action describes a consistent model only when it is proportional to $(\partial_\mu \xi_\nu - \partial_\nu \xi_\mu)^2$, which leads to the condition $\beta + \alpha = 0$. Setting $\alpha = -\beta = -\mu^2 < 0$ one arrives at the Fiertz-Pauli model of massive gravity.

vDVZ discontinuity. Let us now see that the Fierz-Pauli model predicts the bending of light by massive bodies which is different from GR even in the limit of zero mass. We need to compare the interactions of a given mass with a massive test particle and with the photon in GR and in the Fierz-Pauli model. In the language of quantum field theory these interactions are proportional to the amplitude of the one-graviton exchange which in turn is proportional to the graviton propagator. To be more precise, the quantity which determines the interaction is the contraction of the graviton propagator with the energy-momentum tensors of the source and the test particle. The graviton propagator in both massive and massless cases has the form

$$P_{\mu\nu\lambda\rho} = \frac{\sum_i e^i_{\mu\nu} e^i_{\lambda\rho}}{p^2 - m^2}, \tag{2.4}$$

where the sum runs over all "polarization tensors" $e^i_{\mu\nu}$. In the massive case there are 5 such tensors. In the rest frame of a (massive) graviton they have the following form:

$$\frac{\sqrt{2}}{3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{2.5}$$

while in the massless case there are only two polarization tensors which can be written as follows (in the frame where the vector $\vec{p}$ is parallel to the z axis),

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.6}$$

Note that two of these tensors are the same as in the massive case. The substitution of the polarization tensors into eq. (2.4) gives the following graviton propagators,

$$
\begin{aligned}
P^{m\neq 0}_{\mu\nu\lambda\rho} &= \frac{1}{p^2-m^2}\left\{\frac{1}{2}\eta_{\mu\lambda}\eta_{\nu\rho}+\frac{1}{2}\eta_{\mu\rho}\eta_{\nu\lambda}-\frac{1}{3}\eta_{\mu\nu}\eta_{\lambda\rho}\right.\\
&\qquad \left. +(p\text{-dependent terms})\right\},
\end{aligned}
$$

$$
\begin{aligned}
P^{m=0}_{\mu\nu\lambda\rho} &= \frac{1}{p^2}\left\{\frac{1}{2}\eta_{\mu\lambda}\eta_{\nu\rho}+\frac{1}{2}\eta_{\mu\rho}\eta_{\nu\lambda}-\frac{1}{2}\eta_{\mu\nu}\eta_{\lambda\rho}\right.\\
&\qquad \left. +(p\text{-dependent terms})\right\}.
\end{aligned}
$$

The terms containing p are of no interest since the propagator is contracted with the conserved energy-momentum tensor and these terms give no contribution.

The crucial difference between the two cases is the coefficient in front of the term $\eta_{\mu\nu}\eta_{\lambda\rho}$ which couples to the trace of the energy-momentum tensor. This difference does not vanish in the zero mass limit. Clearly, it is due to a different number of graviton polarizations in the massive and massless cases, as can be seen by comparing eqs. (2.5) and (2.6).

Normalizing the interaction with the test massive particle to the observed value, one can predict the bending of light in both cases. It turns out that the predictions are different. To see this let us denote the gravitational interaction constant as G and $\tilde{G}$ in the massless and massive cases, respectively. In the two cases the interaction between the non-relativistic masses is proportional to the following combinations,

$$
\begin{aligned}
\text{massless case:}\quad G\,T_{\mu\nu}\,P^{m=0}_{\mu\nu\lambda\rho}\,T'_{\lambda\rho} &= \frac{1}{2}G\,T_{00}T'_{00}\frac{1}{p^2},\\
\text{massive case:}\quad \tilde{G}\,T_{\mu\nu}\,P^{m\neq 0}_{\mu\nu\lambda\rho}\,T'_{\lambda\rho} &= \frac{2}{3}\tilde{G}\,T_{00}T'_{00}\frac{1}{p^2-m^2}.
\end{aligned}
$$

This implies in the small mass limit that

$$
\tilde{G} = \frac{3}{4}G. \tag{2.7}
$$

In the case of the light bending by a non-relativistic mass the third term in the propagator does not contribute because of the vanishing trace of the electromagnetic energy-momentum tensor, ${T'}^{\mu}_{\mu}=0$, and the result is the same in both cases,

$$
\begin{aligned}
\text{massless case:}\quad & GT_{00}T'_{00}\frac{1}{p^2},\\
\text{massive case:}\quad & \tilde{G}T_{00}T'_{00}\frac{1}{p^2-m^2}.
\end{aligned}
$$

In view of eq. (2.7) the light bending predicted in the massive theory in the limit of the vanishing graviton mass is $3/4$ of that predicted in general relativity. Thus, there exists a discontinuity in the limit when the graviton mass goes to zero.

The low-scale strong coupling. The calculation of the gravitational potential and of the light bending outlined above were performed in the linearized approximation where the self-interaction of the gravitational field is neglected. In general relativity this is a good approximation at distances which are much larger than the Schwarzschild radius of the mass producing the gravitational field. One may wonder whether this approximation is correct in the Feirz-Pauli massive gravity as well. It was first noted by Vainstein [29] who considered the spherically-symmetric solutions in the Fierz-Pauli model that for these solutions the weak-field approximation actually breaks down much further from the source than the gravitational radius. Vainstein argued that the validity of the linear approximation is controlled by the parameter

$$\varepsilon = \frac{R_g}{m_g^4 r^5}, \tag{2.8}$$

where m_g is the graviton mass and $R_g = 2M/M_{\rm Pl}$ is the Schwarzschild radius corresponding to the mass M. Eq. (2.8) is remarkable in that the graviton mass enters in the denominator, so that the expansion parameter always becomes large when the mass of the graviton goes to zero.

Let us see what are the numbers. Assuming the solar system planets feel the gravitational field of the Sun in the linear regime implies that the graviton mass should be smaller than the inverse radius of the Pluto orbit,

$$m_g < (40AU)^{-1} \sim 3\times 10^{-20} \text{ eV}. \tag{2.9}$$

Then the parameter ε at the Mercury orbit $r_M \sim 0.3AU$ equals

$$\varepsilon \sim 15,$$

so that the motion of Mercury should be strongly affected by non-linearities. However, even the estimate (2.9) is way too optimistic. From observations of the star motion in galaxies the graviton mass should be smaller than at least kpc^{-1}. The parameter ε at the Earth orbit around the Sun would then be $\varepsilon \sim 10^{25}$, so that the gravitational interaction would be deeply in the non-linear regime. Thus, the problem of discontinuity is replaced by the strong coupling problem.

By itself, the strong coupling does not mean that the theory is inconsistent with observations. Indeed, the arguments of Ref.[29] were reconsidered in Ref. [30] where it was argued, within the DGP model of modified gravity, that particular non-linear effects may make the transition to the zero graviton mass continuous and weaken the experimental limits on the mass of the graviton.

From the point of view of the quantum field theory, the onset of the non-linear regime shows up as the strong coupling at high energies. When graviton is given a mass, the contribution of the longitudinal polarizations of the graviton to the graviton-graviton cross section grows with energy, as has been checked by the direct calculations [31]. This leading to the strong coupling at some energy scale. These two strong coupling phenomena — classical and quantum — are interrelated and are manifestations of the same problem [33].

3. Constructing the Lorentz-breaking massive gravity models

Our goal now is to construct a model where the graviton is massive and which is compatible with observations despite the problems outlined above. To be mode precise, we will require that

PoS(cargese)011

the flat Minkowski space is a solution to the equations of motion, that there are no instabilities and ghosts in perturbation theory around the flat space, there is no discontinuity in physical quantities when the graviton mass goes to zero, and that the theory is weakly coupled below some sufficiently high scale Λ. We will see that all these requirements can be satisfied if one allows the Lorentz invariance to be spontaneously broken.

To see whether such models exist at all one may consider a generic graviton mass term which preserves rotational invariance but not necessarily the Lorentz invariance. Following Ref.[34], this term may be written as

$$\frac{1}{2}M_{\rm Pl}^2\left\{m_0^2h_{00}^2+2m_1^2h_{0i}^2-m_2^2h_{ij}^2+m_3^2h_{ii}^2-2m_4^2h_{00}h_{ii}\right\}, \tag{3.1}$$

where m_i are 5 mass parameters which are, in general, different. The graviton mass (the mass of the tensor perturbations) is given by m_2. Clearly, the Lorentz invariance requires that only two of the mass parameters be independent,

$$m_1^2=m_2^2=-\alpha^2,\quad m_3^2=m_4^2=-\beta^2,\quad m_0^2=\alpha^2+\beta^2,$$

where α and β are arbitrary. The Fierz-Pauli model corresponds to the case $\alpha^2=-\beta^2$. It has been shown in Ref.[34] that the choice $m_0=0$ leads to a model free of ghosts, vDVZ discontinuity and low strong coupling scale. Thus, all the above requirements may, in principle, be satisfied at the same time. We will discuss other possible values of the parameters shortly.

The mass term (3.1) can be considered as the quadratic part of a more general action depending on the metric components,

$$S=\int d^4x\sqrt{g}\left\{M_{\rm Pl}^2R+\Lambda^4F(g_{\mu\nu})+\text{matter}\right\}, \tag{3.2}$$

where the function $F(g_{\mu\nu})=F(g_{00},g_{0i},g_{ij})$ is assumed to preserve rotations but not necessarily the Lorentz invariance. The scale Λ which will eventually play the role of the cutoff of the model is related to the graviton masses as

$$m_i^2\sim\frac{\Lambda^4}{M_{\rm Pl}^2}.$$

We will assume in what follows that there are no other scales in the function F. Our purpose now is to investigate the models with the action (3.2).

The analysis is greatly simplified by the so-called Stückelberg's trick which consists in restoring the gauge invariance by introducing auxiliary scalar fields, a sort of "inverse" Higgs mechanism. To illustrate how this works consider an example of the (Lorentz-invariant) massive electrodynamics with the action

$$S=\int d^4x\left\{-\frac{1}{4}F_{\mu\nu}^2+m^2A_\mu^2\right\}.$$

There are three degrees of freedom: two transverse polarizations of the photon, and the longitudinal polarization. The gauge invariance is broken by the photon mass term. Let us now add a scalar field ϕ in such a way as to restore the gauge invariance,

$$S=\int d^4x\left\{-\frac{1}{4}F_{\mu\nu}^2+m^2(A_\mu-\partial_\mu\phi)^2\right\}.$$

The new action is explicitly gauge-invariant under the transformations

$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu \phi, \\ \phi &\to \phi + \alpha, \end{aligned}$$

and has the same three degrees of freedom as the original action. Moreover, one may use the gauge transformation to set $\phi = 0$. Then the original action is recovered. On the other hand, one may concentrate on the "Goldstone" part of the action $m^2(\partial_\mu \phi)^2$ and recover the action for the phase part of the Higgs field in the standard $U(1)$ Higgs mechanism. This is this freedom of the gauge choice which simplifies the analysis.

This trick can be generalized [33, 35] to the case of the action (3.2). The symmetries of general relativity are four coordinate transformations

$$x^\mu \to x^\mu + \xi^\mu(x).$$

Thus one has to introduce 4 scalar "Goldstone" fields ϕ^0 and ϕ^i. We stress that these fields are scalars and thus must transform as scalars under the coordinate transformations. For this reason it is not straightforward to introduce these fields into the action (3.2) in such a way that they restore the general covariance. The job is done by the following combinations [35],

$$\begin{aligned} X &= \frac{1}{\Lambda^4} g^{\mu\nu} \partial_\mu \phi^0 \partial_\mu \phi^0, \\ V^i &= \frac{1}{\Lambda^4} g^{\mu\nu} \partial_\mu \phi^i \partial_\mu \phi^0, \\ Y^{ij} &= \frac{1}{\Lambda^4} g^{\mu\nu} \partial_\mu \phi^i \partial_\mu \phi^j. \end{aligned} \tag{3.3}$$

The factor $1/\Lambda^4$ is introduced to make these combinations dimensionless. Thus, the general action for the massive gravity becomes

$$S = \int d^4x \sqrt{g} \left\{ M_{\rm Pl}^2 R + \Lambda^4 F(X, V^i, Y^{ij}) + L_{\rm matter} \right\}. \tag{3.4}$$

Several remarks are in order. First, the Goldstone fields only enter the action through the derivatives, as they should. Second, they only couple to the metric and not to the matter fields directly, so that they do not introduce extra interactions except the modification of the gravity law. Finally, as in our toy example, one can choose the gauge (the reference frame) in such a way that the action (3.4) reduces to the action (3.2), as we will now show.

Before that we need to discuss one important point — the vacuum solutions in the model (3.4). More precisely, we have to determine under which conditions the flat space is the vacuum solution. To this end let us assume that there is no ordinary matter. Then the Einstein equations derived from the action (3.4) are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = M_{\rm Pl}^2 T^G_{\mu\nu},$$

where $T^G_{\mu\nu}$ is the energy-momentum tensor of the Goldstone fields. Since the left hand side of this equation is zero in the flat space, the right hand side should also be zero. Thus, we have to determine conditions under which the Goldstone energy-momentum tensor vanishes in the Minkowski background.

As usual, the calculation of the energy-momentum tensor is done by the variation of the Goldstone action with respect to the metric. Denote

$$\delta F \equiv F_X \delta X + F_i \delta V^i + F_{ij} \delta Y^{ij},$$

where

$$\begin{aligned}
\delta X &= \frac{1}{\Lambda^4} \partial_\mu \phi^0 \partial_\nu \phi^0 \delta g^{\mu\nu}, \\
\delta V^i &= \frac{1}{\Lambda^4} \partial_\mu \phi^0 \partial_\nu \phi^i \delta g^{\mu\nu}, \\
\delta Y^{ij} &= \frac{1}{\Lambda^4} \partial_\mu \phi^i \partial_\nu \phi^j \delta g^{\mu\nu}.
\end{aligned}$$

Making use of the definition $\delta S = 1/2 \int T_{\mu\nu} \delta g^{\mu\nu}$, one finds

$$\begin{aligned}
T^G_{\mu\nu} = &\; 2F_X \partial_\mu \phi^0 \partial_\nu \phi^0 + F_i(\partial_\mu \phi^0 \partial_\nu \phi^i + \partial_\mu \phi^i \partial_\nu \phi^0) + \\
&\; 2F_{ij} \partial_\mu \phi^i \partial_\nu \phi^j - g_{\mu\nu} \Lambda^4 F.
\end{aligned}$$

The requirement $T^G_{\mu\nu} = 0$ should be considered as the set of equations for the Goldstone fields. For an arbitrary metric these equations are impossible to satisfy because there are 10 equations (which are all in general independent) for only 4 unknowns. However, for the flat space the solution is easy to guess. Consider the Goldstone fields of the following form,

$$\begin{aligned}
\phi^0 &= a\Lambda^2 t, \\
\phi^i &= b\Lambda^2 x^i,
\end{aligned}$$

where a and b are two unknown constants. For this ansatz the equations $T^G_{\mu\nu} = 0$ reduce to the following two *algebraic* equations[1],

$$\begin{aligned}
&2a^2 F_X(a^2, b^2) - F(a^2, b^2) = 0, \\
&2b^2 F_Y(a^2, b^2) - F(a^2, b^2) = 0,
\end{aligned} \tag{3.5}$$

where we have used the notation $F_{ij} = F_Y \delta_{ij}$. Since these are two equations for two variables, in general they have a solution. Without loss of generality we will assume in what follows that this solution is such that $a = b = 1$. Thus, the vacuum in our model is

$$\begin{aligned}
\phi^0 &= \Lambda^2 t, \\
\phi^i &= \Lambda^2 x^i.
\end{aligned} \tag{3.6}$$

Note that the fields themselves do not enter the action, so there is nothing wrong with them growing at infinity. However, since the vacuum values of these fields are space-time dependent, they break the Lorentz symmetry. The rotational symmetry remains unbroken if the action preserves the global rotations of the fields ϕ^i with respect to the index i. Indeed, in this case the space rotations of the

[1] For symmetry reasons T_{0i} vanishes identically, while the T_{ij} is proportional to δ_{ij}. Thus, only two equations are independent.

vacuum manifold (3.6) can be compensated by the global rotations of the fields, so that of the two rotation groups the diagonal part remains unbroken.

We can now see that the action (3.4) is equivalent to the original action (3.2). Indeed, for an arbitrary metric we can choose the gauge in which the Goldstone fields equal to their vacuum values (3.6) (the "unitary" gauge). Then we have

$$X = g^{00}, \quad V^i = g^{0i}, \quad Y^{ij} = g^{ij}, \tag{3.7}$$

so that the function F becomes a function of the metric components as in eq. (3.2).

4. Linear perturbations

Let us discuss the behavior of the linear perturbations in the flat background. We have to consider both perturbations of the metric $h_{\mu\nu}$ and perturbations of the Goldstone fields π_μ,

$$\begin{aligned} \phi^\mu &= \Lambda^2 x^\mu + \pi^\mu, \\ g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu}. \end{aligned}$$

The purpose is to show that there are neither ghosts nor instabilities present for some values of the mass parameters.

First, let us clarify the relation between the masses entering eq. (3.1) and the function F in the action (3.4). The mass term (3.1) is recovered from eq. (3.4) in the unitary gauge where the perturbation of the Goldstone fields are zero. To calculate the mass parameters one has to expand $\sqrt{g}F$ up to the second order in metric perturbations. The zeroth order term is an irrelevant constant. The linear terms must vanish; this is the condition that our background is a solution to the Einstein equations. If we start with an arbitrary function F, the vanishing of the linear terms will be equivalent to the conditions (3.5) which ensure that the energy-momentum tensor of the Goldstone fields is zero. Finally, the quadratic terms should be identified with the mass parameters. The overall mass scale is already clear: assuming the function F does not contain other scales apart from Λ, the masses are of the order $m_i^2 \sim \Lambda^4/M_{\rm Pl}^2$. Carrying out the expansion one finds, for instance

$$\begin{aligned} m_0^2 &= \frac{\Lambda^4}{M_{\rm Pl}^2}\left(\frac{1}{2}F_X + F_{XX}\right), \\ m_1^2 &= 2\frac{\Lambda^4}{M_{\rm Pl}^2}(F_Y + F_{VV}), \\ &\dots \end{aligned}$$

where the subscript on F denotes the derivative with respect to the corresponding variable. An important observation which will be useful in what follows is that if one takes the function F which depends only on the two arguments X and

$$W^{ij} = Y^{ij} - V^i V^j / X, \tag{4.1}$$

then one has $m_1^2 = 0$. This follows from the fact that both X and W^{ij} are invariant under the following symmetry:

$$\begin{aligned} &\phi^0 \to \phi^0 \\ &\phi^i \to \phi^i + \xi^i(\phi^0), \end{aligned} \tag{4.2}$$

where x^i are arbitrary functions, while the term proportional to m_1^2 breaks this symmetry. As we will discuss shortly, imposing this symmetry leads to a particularly interesting class of models.

The next question to discuss is the absence of instabilities and ghosts. As we have already discussed, the problem comes from the longitudinal polarizations of the graviton, i.e., from the purely Goldstone sector. This part of the quadratic action may be obtained by substituting

$$h_{\mu\nu} = \partial_\mu \pi_\nu + \partial_\nu \pi_\mu \tag{4.3}$$

into the full quadratic action for perturbations. Since in general relativity the perturbation (4.3) is a pure gauge, the Einstein part of the action does not contribute, and the only contribution comes from the mass term which takes the form

$$\frac{1}{2}M_{\rm Pl}^2\Big\{2m_0^2(\partial_0\pi_0)^2 + m_1^2(\partial_0\pi_i)^2 + m_1^2(\partial_i\pi_0)^2 + (4m_4^2 - 2m_1^2)\pi_0\partial_0\partial_i\pi_i$$
$$-m_2^2(\partial_i\pi_j)^2 - (m_2^2 - 2m_3^2)(\partial_i\pi_i)^2\Big\}.$$

We need to determine under which constraints on the masses this Lagrangian defines a consistent model.

It is convenient to use the internal $O(3)$-symmetry and separate the vector and scalar representations. The field π_0 is a scalar under $O(3)$, while the vector π_i can be decomposed in the transverse and longitudinal parts,

$$\pi_i = \pi_i^T + \pi_i^L,$$

where π_i^T is transverse,

$$\partial_i\pi_i^T = 0,$$

and π_i^L is a divergence of a scalar, $\pi_i^L = \partial_i\pi^L/\sqrt{-\partial_j^2}$. The vector and scalar sectors separate. The Lagrangian of the vector sector reads

$$L^{(\rm vec)} = \frac{1}{2}M_{\rm Pl}^2\left\{m_1^2(\partial_0\pi_i^T)^2 - m_2^2(\partial_i\pi_j^T)^2\right\}.$$

For the absence of pathologies it is sufficient to require $m_1^2, m_2^2 > 0$.

In the scalar sector the analysis proceeds in a similar way but is more involved as one has to deal with the coupled system of equations. Without going into details of the calculations which can be found in Ref. [35], let us summarize the results.

- At general values of the mass parameters there are 6 propagating degrees of freedom (two tensor, two vector and two scalar modes). One of them is necessarily either ghost or unstable. The consistent model arises only in special cases.

- In the case $m_0^2 = 0$, as was found in Ref. [34], one of the scalar modes does not propagate. Five other modes, 2 tensor, 2 vector and 1 scalar, form five polarizations of the massive graviton. Note, however, that the masses of these modes are, in general, different. This is the manifestation of the Lorentz symmetry breaking. There are no pathologies in this model provided the masses $m_1^2 \dots m_4^2$ satisfy certain inequalities [34].

- In the case $m_2^2 = m_3^2$ one of the scalars does not propagate. As in the previous case, the remaining 5 modes can be viewed as 5 polarizations of the massive graviton.

- Finally, in the case $m_1^2 = 0$ none of the scalar and vector modes are dynamical, so the only propagating degrees of freedom are the two tensor modes. These modes are massive and have the mass m_2. No ghosts or instabilities are present provided the masses satisfy certain inequalities.

There is one more important issue which has to be discussed in the context of linear perturbations. The action (3.4) is no more than the low-energy effective action. One should expect the appearance of higher terms suppresses by the powers of the energy divided by the cutoff scale Λ. These terms may contain, in particular, higher derivatives of the fields ϕ_0 and ϕ_i. Usually these terms can be neglected at low energies. However, the absence of instabilities requires the fine-tuning relations as was explained above. The violation of these fine-tuning relations may result in instabilities even if this violation is tiny. For instance, if a dispersion relation $\omega^2 = 0$ which corresponds to a non-propagating mode acquires a correction and changes to $\omega^2 = -\alpha k^4$, this may lead to a rapid instability at sufficiently high momentum even if the coefficient α is small. So, one has to make sure that the fine-tuning relations needed for the stability of the model can be protected by symmetries. This is probably not the case for the phase $m_0^2 = 0$ [35]. On the contrary, the phase $m_1^2 = 0$ can be protected against higher-order corrections by the symmetry

$$x^i \to x^i + \xi^i(t),$$

which is a part of the group of coordinate transformations. In terms of the Goldstone fields this is precisely the symmetry (4.2).

This is this last case that we will consider in more detail in the remaining part of these lectures. We will see that it has a number of other attractive features apart from being stable against higher-order corrections.

5. Some phenomenological implications

5.1 Newton's potential

Consider, from the phenomenological point of view, a particular class of models with the function F of the form

$$F = F(X, W^{ij}).$$

The first question which we have to address is whether — and how — the Newton's law is modified in these models. Thus, we have to calculate the linear response of the system to a point-like source of the gravitational field.

It is convenient to work in the "unitary gauge" where the Goldstone fields are set to their vacuum values (3.6). In this gauge the remaining perturbations are the perturbations of the metric $\delta g_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}. \tag{5.1}$$

In the notations of Ref. [36] they are parameterized as follows,

$$\begin{aligned}\delta g_{00} &= 2\varphi;\\ \delta g_{0i} &= S_i - \partial_i B;\\ \delta g_{ij} &= -h_{ij} - \partial_i F_j - \partial_j F_i + 2(\psi\delta_{ij} - \partial_i\partial_j E),\end{aligned}$$

where h_{ij} are the transverse and traceless tensor perturbations, S_i and F_i are the transverse vector perturbations, while φ, ψ, B and E are the scalar perturbations.

The quadratic Lagrangian for perturbations consists of the Einstein-Hilbert term, the mass term and the source term,

$$L = L_{EH} + L_m + L_s. \tag{5.2}$$

Explicitly, the Einstein-Hilbert part reads

$$\begin{aligned}L_{EH} = M_{Pl}^2\Big\{&-\frac{1}{4}h_{ij}(\partial_0^2 - \partial_i^2)h_{ij} - \frac{1}{2}(S_i + \partial_0 F_i)\partial_j^2(S_i + \partial_0 F_i)\\ &+4(\varphi + \partial_0 B - \partial_0^2 E)\partial_i^2\psi + 6\psi\partial_0^2\psi - 2\psi\partial_i^2\psi\Big\},\end{aligned} \tag{5.3}$$

while for the mass term one finds

$$\begin{aligned}&M_{Pl}^2\Big\{-\frac{1}{4}m_2^2 h_{ij}^2 - \frac{1}{2}m_2^2(\partial_i F_j)^2 + m_0^2\varphi^2 + \left(m_3^2 - m_2^2\right)\left(\partial_i^2 E\right)^2 -\\ &-2(3m_3^2 - m_2^2)\psi\partial_i^2 E + 3\left(3m_3^2 - m_2^2\right)\psi^2 + 2m_4^2\varphi\partial_i^2 E - 6m_4^2\varphi\psi\Big\}.\end{aligned} \tag{5.4}$$

Note that the tensor, vector and scalar sectors do not mix, so they can be considered separately.

We also add the external source $T_{\mu\nu}$ which is assumed to be conserved, $\partial^\mu T_{\mu\nu} = 0$. The corresponding contribution to the Lagrangian can be written as

$$L_s = -T_{00}\left(\varphi + \partial_0 B - \partial_0^2 E\right) - T_{ii}\psi + (S_i + \partial_0 F_i)T_{0i} + \frac{1}{2}h_{ij}T_{ij}.$$

All the combinations of metric perturbations which enter this equation are gauge-invariant. The one multiplying T_{00},

$$\Phi \equiv \varphi + \partial_0 B - \partial_0^2 E,$$

plays the role of the Newtonian potential in the non-relativistic limit.

Tensor sector. In the tensor sector there is a single equation of the form

$$(-\partial_0^2 + \partial_i^2 - m_2^2)h_{ij} = 0. \tag{5.5}$$

This equation describes propagation of a massive gravitational wave. Note that this wave has only two polarizations. This is, of course, only possible because of the violation of the Lorentz invariance.

Vector sector. The field equations in the vector sector are

$$-\partial_j^2(S_i+\partial_0 F_i) = -T_{0i}, \tag{5.6}$$

$$\partial_0\partial_j^2(S_i+\partial_0 F_i) + m_2^2\partial_j^2 F_i = \partial_0 T_{0i}. \tag{5.7}$$

Taking the time derivative of eq. (5.6) and adding it to eq. (5.7) gives

$$F_i = 0,$$

provided that $m_2^2 \neq 0$. Thus, the vector sector of our model behaves in the same way as in the Einstein theory in the gauge $F_i = 0$. There are no propagating vector perturbations.

Scalar sector. The field equations for the scalar perturbations are

$$2\partial_i^2\psi + m_0^2\varphi + m_4^2\partial_i^2 E - 3m_4^2\psi = \frac{T_{00}}{2M_{Pl}^2}, \tag{5.8}$$

$$\begin{aligned}&2\partial_i^2\Phi - 2\partial_i^2\psi + 6\partial_0^2\psi - \left(3m_3^2 - m_2^2\right)\partial_i^2 E\\ &+3\left(3m_3^2-m_2^2\right)\psi - 3m_4^2\varphi = \frac{T_{ii}}{2M_{Pl}^2},\end{aligned} \tag{5.9}$$

$$\begin{aligned}&-2\partial_i^2\partial_0^2\psi + \left(m_3^2-m_2^2\right)\partial_i^4 E\\ &-\left(3m_3^2-m_2^2\right)\partial_i^2\psi + m_4^2\partial_i^2\varphi = -\frac{\partial_0^2 T_{00}}{2M_{Pl}^2},\end{aligned} \tag{5.10}$$

$$2\partial_i^2\partial_0\psi = \frac{\partial_0 T_{00}}{2M_{Pl}^2}. \tag{5.11}$$

Eq. (5.11) implies

$$\psi = \frac{1}{\partial_i^2}\frac{T_{00}}{4M_{Pl}^2} + \psi_0(x^i), \tag{5.12}$$

where $\psi_0(x^i)$ is some time-independent function. From Eqs. (5.8) and (5.10) one finds

$$\varphi = \frac{1}{\Delta}\left\{2m_2^2m_4^2\psi + 2(m_3^2-m_2^2)\partial_i^2\psi_0\right\}, \tag{5.13}$$

$$\partial_i^2 E = \frac{1}{\Delta}\left\{(3\Delta - 2m_0^2m_2^2)\psi - 2m_4^2\partial_i^2\psi_0\right\}, \tag{5.14}$$

where

$$\Delta = m_4^4 - m_0^2(m_3^2-m_2^2).$$

Finally, substituting eqs. (5.12), (5.13) and (5.14) into eq. (5.9) one finds the gauge-invariant potential Φ,

$$\begin{aligned}\Phi = {}& \frac{1}{\partial_i^2}\frac{T_{00}+T_{ii}}{4M_{Pl}^2} - 3\frac{\partial_0^2}{\partial_i^4}\frac{T_{00}}{4M_{Pl}^2} + \frac{m_2^2}{\Delta}(3\Delta - 2m_0^2m_2^2)\frac{1}{\partial_i^4}\frac{T_{00}}{4M_{Pl}^2}\\ &+\frac{m_2^2}{\Delta}(3\Delta-2m_0^2m_2^2)\frac{1}{\partial_i^2}\psi_0 + \left(1-\frac{2m_2^2m_4^2}{\Delta}\right)\psi_0.\end{aligned} \tag{5.15}$$

The first two terms on the r.h.s. of eq. (5.15) are the standard contributions of the Einstein theory, the first becoming the Newtonian potential in the nonrelativistic limit. The third term on the r.h.s.

is the new contribution specific to massive gravity. We will discuss this term in more detail shortly. Finally, the terms on the second line of eq. (5.15) are also new; they are all proportional to the time-independent integration constant $\psi_0(x^i)$. The value of this constant is determined by the initial conditions. Since there are no absolutely static objects in the Universe, the value of ψ_0 is not related to the gravitational fields created by massive bodies. Thus, these terms are irrelevant for us here.

When $\psi_0 = 0$, the gauge-invariant potentials Φ and ψ in differ from their analogs in the Einstein theory Φ_E and ψ_E by the mass-dependent third term on the r.h.s of eq. (5.15),

$$\begin{aligned} \psi &= \psi_E, \\ \Phi &= \Phi_E + \left(3 - \frac{2m_0^2 m_2^2}{\Delta}\right) \frac{m_2^2}{\partial_i^4} \frac{T_{00}}{4M_{Pl}^2}. \end{aligned} \tag{5.16}$$

This term vanishes if all masses uniformly go to zero, which implies the absence of the vDVZ discontinuity as expected. In the coordinate space this term leads to the extra contribution to the potential which has the "confining" form, so that the whole potential becomes

$$\Phi = G_N M \left(-\frac{1}{r} + \mu^2 r\right), \tag{5.17}$$

where

$$\mu^2 = -\frac{1}{2} m_2^2 \left(3 - \frac{2m_0^2 m_2^2}{\Delta}\right). \tag{5.18}$$

The growth of the second term indicates the breakdown of perturbation theory at distances $r \gtrsim 1/(G_N M \mu^2)$.

An interesting situation arises when there is no modification of the Newtonian potential. This may happen if

$$3\Delta - 2m_0^2 m_2^2 = 0 \tag{5.19}$$

and $\Delta \neq 0$. In this case the static interaction between two massive bodies is described by the standard Newtonian force proportional to $1/r^2$, so the deviations from the standard gravity would not be possible to detect in the Cavendish-type experiments. Note that eq. (5.19) does not require that the mass of the graviton m_2 is zero. Thus, in the case when the condition (5.19) is satisfied the non-zero graviton mass coexists with the long-range force. This is yet another manifestation of the violation of the Lorentz invariance in this model.

Since the Lorentz symmetry is broken in our model, one should expect the preferred frame effects. These effects, related to the motion of the gravitational field source with respect to the preferred frame, are proportional to the square of the velocity of this motion v^2. However, when obtaining eq. (5.16) we did not neglect the motion of the source, so in the linear approximation such effects are absent; they only arise at the non-linear level and thus contain the additional suppression by the measure of linearity, i.e., by the gravitational potential which is in turn of order v^2 in realistic systems. Thus, the preferred-frame effects are suppressed by the factor of the order of v^4 and can be neglected.

The condition (5.19) can be ensured by imposing the following dilatation symmetry,

$$\begin{aligned} t &\to \lambda t, \\ x^i &\to \lambda^\gamma x^i, \end{aligned} \tag{5.20}$$

where λ is the transformation parameter and γ is a constant. Requiring the symmetry (5.19) is equivalent to a taking a particular dependence of the function F on its arguments,

$$F = F(Z^{ij}) \tag{5.21}$$

where

$$Z^{ij} \equiv X^{\gamma} W^{ij}.$$

We will see in the next section that the model obtained in this way has a number of phenomenologically interesting features.

5.2 Cosmological evolution

The symmetry (5.19) may seem artificial, but there is one more reason to consider models possessing the symmetry. To understand this reason we have to discuss the cosmological solutions in massive gravity. The action of our model is a full non-linear action of the low-energy effective theory. Thus, we can study the non-linear gravitational fields and, in particular, the cosmology, provided the relevant energy scale is below the cutoff scale Λ.

The homogeneous and isotropic ansatz in the spatially-flat case reads

$$\begin{aligned} ds^2 &= dt^2 - a^2(t) dx_i^2, \\ \phi^0 &= \phi(t), \\ \phi^i &= \Lambda^2 x^i. \end{aligned} \tag{5.22}$$

For this ansatz the variable W^{ij} takes the form

$$W^{ij} = -\frac{1}{a^2}\delta_{ij},$$

so the function F becomes a function of X and the scale factor a which one can consider as two independent variables. The equations which determine the cosmological evolution are the Friedmann equation

$$\left(\frac{\dot a}{a}\right)^2 = \frac{1}{6M_{Pl}^2}\left\{\rho_m + 2\Lambda^4 X F_X - \Lambda^4 F\right\} \equiv \frac{1}{6M_{Pl}^2}\left\{\rho_m + \rho_1 + \rho_2\right\}, \tag{5.23}$$

where ρ_m is the energy density of the ordinary matter not including the Goldstone fields, and the field equation for ϕ^0,

$$\partial_t\left(a^3\sqrt{X} F_X\right) = 0. \tag{5.24}$$

It is straightforward to solve this system of equations for any given function $F(X,a)$. After the integration, Eq. (5.24) gives an algebraic relation between X and the scale factor a. The dependence $X(a)$ as found from Eq. (5.24) determines the behavior of the Goldstone energy density $\rho_1 + \rho_2$ as a function of a. This makes Eq. (5.23) a closed equation for the scale factor $a(t)$.

Rather than solving eqs. (5.23) and (5.24) (see Ref.[37] for details) let us discuss general properties of the solutions. We are interested in solutions such that $a \to \infty$ at late times. Let us assume that the function $X(a)$ which results in the solution of eq. (5.24) asymptotes to some power of a. Then there exists a constant γ such that the combination X^{γ}/a^2 goes to a non-zero value as $a \to \infty$. Then eq. (5.24) implies

$$\rho_1 = \text{const}\frac{1}{a^{3-1/\gamma}}, \tag{5.25}$$

i.e., one of the contributions of the Goldstone fields to the energy density behaves like a matter with the equation of state $w = -1/(3\gamma)$.

By construction, we have $Z^{ij} = X^{\gamma} W^{ij} \to Z_0 \delta_{ij}$, where Z_0 is some constant. If we assume that the function F is regular in the limit $a \to \infty$, then it also has to go to some constant $F \to F_0 = F(Z_0)$. Thus at late times of the evolution the function F depends on the combination $Z^{ij} = X^{\gamma} W^{ij}$. In other words, this point is an attractor of the cosmological evolution. This is another reason to consider the actions which depend on the Goldstone fields in the combination $Z^{ij} = X^{\gamma} W^{ij}$. Note that the second contribution to the energy density behaves at late times as a cosmological constant.

5.3 Experimental signatures

Let us first discuss the experimental constraints on the graviton mass. We will concentrate on the case of the functions F satisfying the constraint (5.21). In this case the Newtonian potential remains unchanged, so the existing constraints from the Cavendish-type experiments and solar system tests [38, 39] do not apply. On the contrary, the presence of the mass affects emission of the gravitational waves. Precision observations of the slow down of the orbital motion in binary pulsar systems [40] imply that the mass of the gravitational waves cannot be larger than the frequency of the waves emitted by these systems. The latter is determined by the period of the orbital motion which is of order 10 hours, implying the following limit on the graviton mass,

$$m_2 \lesssim 2 \times 10^{-4}\text{Hz} \sim 10^{-19}\text{eV} \sim (10^{14}\text{cm})^{-1}.$$

This is a very large mass in cosmological standards. It would certainly be ruled out if it implied the Yukawa-type deviations from the Newtonian potential.

The non-zero mass of the graviton leads to interesting consequences for the primordial gravitational waves as well. The massive gravitons can be produced during the cosmological expansion. In the expanding Universe eq. (5.5) is modified in the following way,

$$(-\partial_0^2 - 3H\partial_0 + \partial_k^2/a^2 - m_2^2)h_{ij} = 0, \tag{5.26}$$

where $H = \dot{a}/a$ is the Hubble constant. This equation is identical to the one which governs the behavior of a massive scalar field such as axion. Thus, the massive gravitons will be efficiently produced during inflation (cf. Ref. [41, 42]). One may estimate the amount of the gravitational waves produced. Assuming the inflationary scenario in which the Hubble parameter is constant (e.g., as in the hybrid models of inflation [43]), the spectrum for the massive gravitons is that for the minimally coupled massive scalar field in the de Sitter space [44, 45, 46, 47],

$$\langle h_{ij}^2 \rangle \simeq \frac{1}{4\pi^2} \left(\frac{H_i}{M_{Pl}} \right)^2 \int \frac{dk}{k} \left(\frac{k}{H_i} \right)^{\frac{2m_2^2}{3H^2}} . \tag{5.27}$$

Superhorizon metric fluctuations remain frozen until the Hubble factor becomes smaller than the graviton mass, when they start to oscillate with the amplitude decreasing as $a^{-3/2}$. The energy density in massive gravitons at the beginning of oscillations is of order

$$\rho_o \sim M_{Pl}^2 m_2^2 \langle h_{ij}^2 \rangle \simeq \frac{3H_i^4}{8\pi^2}, \tag{5.28}$$

where we integrated in Eq. (5.27) over the modes longer than the horizon. Today the fraction of the energy density in the massive gravitational waves is, therefore,

$$\Omega_g = \frac{\rho_o}{z_o^3 \rho_c} = \frac{\rho_o}{z_e^3 \rho_c} \left(\frac{H_e}{H_o} \right)^{3/2}, \tag{5.29}$$

where z_o is the redshift at the start of oscillations, $H_o \sim m_2$ is the Hubble parameter at that time, $H_e \approx 0.4 \cdot 10^{-12}$ s^{-1} is the Hubble parameter at the matter/radiation equality, and $z_e \approx 3200$ is the corresponding redshift. Combining all the factors together one gets

$$\Omega_g \sim \cdot 10^4 (m_2 \cdot 10^{14} \mathrm{cm})^{1/2} \left(\frac{H_i}{\Lambda} \right)^4 . \tag{5.30}$$

This estimate assumes that the number of e-foldings during inflation is large, $\ln N_e > H^2/m^2$, which is quite natural in the model of inflation assumed. Thus, the amount of the gravitons produced at inflation may be enough to constitute the dark matter of the Universe.

Let us estimate the amplitude of the gravitational waves assuming that they comprise all of the dark matter in the halo of our Galaxy. The energy density in non-relativistic gravitational waves is of order $M_{Pl}^2 m_2^2 h_{ij}^2$. Equating this to the local halo density one gets

$$\langle h_{ij} \rangle \sim 10^{-10} \left(\frac{2 \cdot 10^{-4} \mathrm{Hz}}{m_2} \right). \tag{5.31}$$

At frequencies $10^{-6} \div 10^{-5}$ Hz this value is well above the expected sensitivity of the LISA detector [48]. Note that in the close frequency range $10^{-9} \div 10^{-7}$ Hz there exists a much lower bound [49] on the stochastic background of the gravitational waves coming from the timing of the millisecond pulsars [50, 51], which is at the level of $\Omega_g < 10^{-9}$. Thus, it is possible that the massive graviton as a candidate for the dark matter can be ruled out by the reanalysis of the already existing data on the pulsar timing.

The relic graviton abundance depends on both the specific inflationary model and the details of the (unknown) UV completion of massive gravity. Therefore, in general, massive gravitons may not comprise the whole of the dark matter in the Universe. In that case the exclusion of the graviton as the dark matter candidate does not necessarily rile out the model of massive gravity and massive gravitational waves may still be present at a lower level. These gravitational waves differ from the conventional stochastic gravitational wave background in that they are monochromatic with the frequency equal to the graviton mass. It is important that the expected LISA sensitivity allows to detect the presence of such gravitational waves at a significantly lower level than in Eq. (5.31).

6. Summary and outlook

Summarizing the above discussion we arrive at the following conclusions:

- Existing attempts to give graviton a mass in a Lorentz-invariant way suffer from severe problems (of which the strong coupling is the most harmless one). It is not clear at the moment whether one can deal with the strong coupling efficiently and construct a phenomenologically viable Lorentz-invariant model of massive graviton.

- The breaking of Lorentz invariance introduces enough freedom to circumvent these problems. Namely, one arrives at a variety of models which at the linear level can be parameterized by 5 graviton mass parameters. Some regions in the space of these parameters lead to a consistent low-energy effective theories with massive gravitons.

- A particular class of models, the ones possessing the residual symmetry (4.2), has a number of attractive features. The two transverse traceless polarizations of the graviton are the only propagating degrees of freedom in these models. The cosmological evolution has an attractor point possessing the additional dilatation symmetry (5.20). In this point the two contributions of the Goldstone fields into the energy-momentum tensor have the form of the cosmological constant and of matter with the equation of state which depends on the parameters of the model. The graviton masses go to finite constants during the expansion of the Universe.

- In the models possessing symmetries (4.2) and (5.20) the non-zero mass of the graviton coexists with the unmodified Newtonian potential – the possibility which is due to the violation of the Lorentz invariance. This allows for relatively large masses of the graviton $m_2 \lesssim (10^{14}\text{cm})^{-1}$ to be phenomenologically acceptable. Massive gravitons can be produced in sufficient amount in the early Universe and are a new candidate for the dark matter.

- The relic massive gravitons produce an easily identifiable monochromatic signal in the gravitational wave detectors. Among those LISA has a large potential to probe the presence of massive gravitational waves and to rule out graviton as a dark matter candidate.

At the same time, there remain quite a number of open questions which require further study. Here are some of them:

- Modern cosmological observations are becoming more and more precise. In order to be in accord with these observations any alternative theory of gravity has to successfully address several issues, one of which is the structure formation. The first stage of this process, the linear growth of perturbations, is straightforward to study in the massive gravity model. The model of massive gravity considered in Sect. 5 pass this test [52]. However, the ability of the model to reproduce other cosmological data remains to be tested.

- When solving for the linear perturbations in sect. 5 we have seen that there appears an integration constant $\psi_0(x^i)$. We have set this constant to zero since we were interested in the gravitational fields of the massive bodies which are not related to this constant. In the cosmological context this constant is also present. Presumably it is determined dynamically and is driven to zero at the inflationary stage, but this remains to be demonstrated.

- An interesting special case $\gamma = 1/3$ deserves attention from another perspective. In this case both contributions ρ_1 and ρ_2 have the vacuum equation of state $w = -1$. As a result, the acceleration rate of the late de Sitter phase is a dynamical quantity, determined by the initial conditions in the Goldstone sector rather than by parameters of the action. This is similar to the situation in the unimodular gravity [53] where the cosmological constant is also a constant of integration. Thus, the massive gravity models may shed light on the cosmological problem.

- It is worth studying in more detail the theories which do not possess the dilatation symmetry (5.20). In these models the gravitational potential is modified at large distances from the source. It is possible in principle that these models may provide an explanation of the galactic flat rotation curves alternative to the dark matter.

- Finally, having the full non-linear action one can study non-linear solutions of the model, in particular, black holes. The latter are of particular interest in view of the expected progress in their experimental observations [54]. SOme progress has been already achieved in this direction [55].

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