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Topological roots of four dimensional gravity

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The connection between Einstein Gravity and the Euler characteristic is reviewed and a generalization is considered. In this way, a four dimensional action principle, gauge invariant under the conformal group, $SO(4,2)$, is regarded as a possible generalization of Einstein theory. This theory is defined by a gauged Wess-Zumino-Witten form. The theory is briefly discussed and is shown how usual Einstein theory plus the cosmological constant term arise.

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1. Motivation and philosophy

A satisfactory description of nature is always accompanied by a reduced number of assumptions. The main difficulty to reduce the number of assumptions is that most of the times they are difficult to identify, and even after identifying them it would be far from obvious how, in a sensible way, relax them. Of course, these kind of considerations are relevant when there is at hand a theory that has been proved to be physically sensible and self contained; something that for the gravitational field, as described by the Einstein field equations (the convention $\hbar = c = 1$ is used),

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.1}$$

has been confirmed by a series of theoretical and experimental achievements. It comprises the advances in the comprehension of the gravitational field during the last century, beginning with the generalization of the weak equivalence principle, the obtention of mechanically stable vacuum solutions and its interpretation as black holes, the realization that the weak field limit not only implies a Newtonian like attraction but also the quadrupolar, gravitational radiation (for a review of the above subjects see [1]), the understanding of the gravitational energy (for discussion and references see [2] and [3]) its positivity and the stability of Minkowski space-time (see [4] and references therein). This construction, of course, has the nature on its side, supported by the experimental success associated with the description of the primordial nucleosynthesis, the binary pulsar and of the solar system tests [5].

All this evidence, indeed suggests that the identification of the minimal set of assumptions that implies (1.1), is a physically relevant question. Luckily, mathematicians think about uniqueness faster than physicists, and Vermeil (1917), Weyl (1922) and Cartan (1922) showed (see [6] and references therein) that it is possible to single out the left hand side of (1.1), in every dimension, by asking

- A rank two, symmetric tensor,
- covariant divergenceless,
- any derivative is at most of second order and the tensor is linear in them.

While the first two assumptions are motivated by what should appear at the right hand side of the Einstein tensor, and in fact are trivial if one begins with an action principle instead of with field equations, the third is not so. As was pointed out by Lovelock (1971) [6] it is possible to relax linearity to quasi-linearity in the second derivatives (for a discussion of quasi-linearity in this context see [7]). Remarkably, this relaxation still implies that in four dimensions the only possibility are the Einstein field equations, while, in higher dimensions, gives rise to the Lovelock series [6].

2. Interlude. Recasting Lovelock theories

A nice pattern that governs the Lovelock series is given by the generalization of the relation between the Hilbert action and a two dimensional topological invariant. The Hilbert action is a non-trivial functional for the metric in all dimensions higher than two, while in two dimensions it becomes a boundary term known as the Euler density:

$$S_H = \frac{c^3}{16\pi G}\zeta_4(M), \qquad \chi_2(M) = \frac{1}{4\pi}\zeta_2(M), \qquad \zeta_D(M) = \int_M R\sqrt{|g|}d^D x. \tag{2.1}$$

The Euler characteristic, χ, (the integral of the Euler density) is a number associated to a family of manifolds that can be related by homotopies see [8]. It exist in all dimensions, however it can be related with differentiable, geometrical features of the manifold only if it is even dimensional, in which case is given by

$$\chi_4 = \beta_4 \int \delta^{\mu\nu\lambda\rho}_{\alpha\beta\gamma\delta} R^{\alpha\beta}{}_{\mu\nu} R^{\lambda\delta}{}_{\lambda\rho}\sqrt{|g|}d^4x, \qquad \chi_6 = \beta_6 \int \delta^{\mu\nu\lambda\rho\eta\tau}_{\alpha\beta\gamma\delta\sigma\zeta} R^{\alpha\beta}{}_{\mu\nu} R^{\lambda\delta}{}_{\lambda\rho} R^{\sigma\zeta}{}_{\eta\tau}\sqrt{|g|}d^6x$$

where $\beta_D = \frac{2}{D!DVOL(S^D)}$ and $VOL(S^D)$ is the volume of the D-dimensional sphere. The pattern in any dimension should be obvious from the above expression.

Despite the condensed notation used, is possible to note that the terms that can be added to the Lovelock Lagrangian increase in complexity with the dimension. The tensor $\delta^{\alpha_1\alpha_2\dots\alpha_n}_{\beta_1\beta_2\dots\beta_n}$ denotes the generalized Kronecker delta, and it corresponds to the determinant

$$\delta^{\alpha_1\dots\alpha_n}_{\beta_1\dots\beta_n} = \begin{vmatrix} \delta^{\alpha_1}_{\beta_1} & .. & \delta^{\alpha_1}_{\beta_n} \\ : & :: & : \\ \delta^{\alpha_n}_{\beta_1} & .. & \delta^{\alpha_n}_{\beta_n} \end{vmatrix}. \tag{2.2}$$

Thus, it is straightforward, but tedious, to explicitly write down any Lovelock term, for instance the cubic one is proportional to [9]

$$\begin{aligned} &2R^{\alpha\beta\gamma\delta}R_{\gamma\delta\lambda\nu}R^{\lambda\nu}{}_{\alpha\beta} + 8R^{\alpha\beta}{}_{\gamma\delta}R^{\gamma\lambda}{}_{\beta\nu}R^{\delta\nu}{}_{\alpha\lambda} + 24R^{\alpha\beta\gamma\delta}R_{\gamma\delta\beta\nu}R^{\nu}_{\alpha} \\ &- 3RR^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} + 24R^{\alpha\beta\gamma\delta}R_{\alpha\gamma}R_{\beta\delta} + 16R^{\alpha\beta}R_{\beta\gamma}R^{\gamma}_{\alpha} - 12RR^{\alpha\beta}R_{\alpha\beta} + R^3. \end{aligned} \tag{2.3}$$

One equation is better than one thousand words, so the previous one is enough to be convinced that a change in the notation is necessary to gain insight into the Lovelock theory. To this end it is necessary to introduce the vielbein, e^a_μ, an isomorphism between the coordinate tangent space and the non-coordinate one[1]. This isomorphism allows to transform geometrical quantities from one space to the other, it is particularly interesting since it is defined by the relation $e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu}$.Using this isomorphism, the curvature two-form $R^{ab} \equiv \frac{1}{2}R^{ab}{}_{\mu\nu}dx^\mu \wedge dx^\nu \equiv \frac{1}{2}e^a_\alpha e^b_\beta R^{\alpha\beta}{}_{\mu\nu}dx^\mu \wedge dx^\nu$,and the torsion two-form $T^a \equiv \frac{1}{2}T^a_{\mu\nu}dx^\mu \wedge dx^\nu \equiv \frac{1}{2}e^a_\gamma T^\gamma_{\mu\nu}dx^\mu \wedge dx^\nu$make their appearance. They are related by means of the spin connection, $\omega^{ab} \equiv \omega^{ab}_\mu dx^\mu$, through the identities: $T^a \equiv de^a + \omega^a_b \wedge e^b \equiv De^a$, $R^{ab} \equiv d\omega^{ab} + \omega^a_c \wedge \omega^{cb}$, $DT^a = R^{ab} \wedge e_b$. Furthermore, using the convention that the wedge product is assumed between forms, the six dimensional Euler density turns out to be

[1]A nice and pedagogical discussion of the non-coordinate tangent space can be found in the last pages of chapter seven, on Riemannian geometry, of [10].

proportional to $\varepsilon_{abcdef} R^{ab} R^{cd} R^{ef}$. With this notation and the torsionless condition, $De^a = 0$, the Lovelock Lagrangians in four, five, six dimensions can be written as

$$
\begin{aligned}
\mathscr{L}_4 &= \varepsilon_{abcd}\left(\alpha_0 e^a e^b e^c e^d + \alpha_1 e^a e^b R^{cd}\right), \\
\mathscr{L}_5 &= \varepsilon_{abcde}\left(\alpha_0 e^a e^b e^c e^d e^e + \alpha_1 e^a e^b R^{cd} e^e + \alpha_2 e^a R^{bc} R^{de}\right), \\
\mathscr{L}_6 &= \varepsilon_{abcdef}\left(\alpha_0 e^a e^b e^c e^d e^e + \alpha_1 e^a e^b R^{cd} e^e + \alpha_2 e^a R^{bc} R^{de}\right) e^f.
\end{aligned}
$$

Where the α*'s* are dimensionful arbitrary coupling constants: α_0 is proportional to the cosmological constant, α_1 is related with the Newton constant while the remaining coupling constants are related to the strength of its accompanying Lovelock term. This implies that the most general Lovelock Lagrangian has $\left[\frac{D+1}{2}\right]$ coupling constants, something that would ruin any possible interpretation of it as a fundamental theory, and enlarge the cosmological constant problem: the value of the $\left[\frac{D+1}{2}\right]$ coupling constants is not protected by any symmetry argument.

3. A different proposal

The relevant phenomenology induced by the dimensional continuation of the two dimensional Euler density, and its Lovelock generalization (for a very nice example see [11]), makes the consideration of the four dimensional physics induced by the next non-trivial case, namely the six dimensional Euler characteristic, something interesting to be considered. This relation has been elaborated in a variety of ways [12, 13, 14, 15].

The perspective followed in [14, 15] is given by the observation that any pair of invariant polynomials (as the Euler density itself) satisfy the Chern-Weil theorem[2]:

$$dP(\mathscr{F}) = 0, \qquad P(\mathscr{F}) - P(\overline{\mathscr{F}}) = dTP(\mathscr{A}, \overline{\mathscr{A}}), \tag{3.1}$$

where $TP(\mathscr{A}, \overline{\mathscr{A}})$ is defined by equation (3.1) up to a closed form. The gauge invariant, globally-defined expression for $TP(\mathscr{A}, \overline{\mathscr{A}})$ stands for the transgression form. In 5 dimensions, the transgression takes the form

$$TP_5(\mathscr{A}, \overline{\mathscr{A}}) = 3\int_0^1 dt\ \left\langle \left(\mathscr{A} - \overline{\mathscr{A}}\right) \mathscr{F}_t^{n-1} \right\rangle, \tag{3.2}$$

where $\mathscr{F}_t \equiv d\mathscr{A}_t + \mathscr{A}_t \mathscr{A}_t$, $\mathscr{A}_t \equiv \mathscr{A}(1-t) + \overline{\mathscr{A}}t$. Thus, the transgression form is uniquely determined by the Chern-Weil theorem. The only way which a five dimensional form would define a four dimensional Lagrangian is if it is closed. From (3.1) it is clear that this would be the case

[2]From here the curvature two-form R^{ab} will be used to refer the four dimensional curvature, while the F^{AB} will denote the six dimensional one. Thus $A, B = 0, ..., 5$, $a, b = 0, ..., 3$ and the relation between the six dimensional connections and curvatures, and the four dimensional ones are given by $\mathscr{A} = \frac{1}{2}\omega^{ab} J_{ab} + c^a J_{a5} + b^a J_{a4} + \Phi J_{45}$, $\mathscr{F} = \frac{1}{2}(R^{ab} + c^a c^b - b^a b^b) J_{ab} + [Db^a + c^a \Phi] J_{a4} + [Dc^a + b^a \Phi] J_{a5} + [d\Phi - b_a c^a] J_{45}$.The six dimensional Euler density is denoted by $P(\mathscr{F}) = \langle \mathscr{F}\mathscr{F}\mathscr{F} \rangle = \frac{1}{6}\varepsilon_{ABCDEF} F^{AB} F^{CD} F^{EF}$, where the definition of the Lie algebra valued form $\mathscr{F} = \frac{1}{2} J_{AB} F^{AB}$ and of the invariant tensor $\langle J_{AB} J_{CD} J_{EF} \rangle = \varepsilon_{ABCDEF}$ was used. The generators J_{AB} would correspond to any of the real forms of $SO(6)$, however the discussion will be remitted to the $SO(4,2)$ case with signature $\eta_{AB} = (-,+,+,+,+,-)$.

if $\bar{\mathscr{A}} = h^{-1}\mathscr{A}h + h^{-1}dh \equiv \mathscr{A}^h$. In this case the transgression form defines a four dimensional Lagrangian given by a gauged Wess-Zumino-Witten (**WZW**) term, invariant under the transformations

$$\mathscr{A} \to g^{-1}\mathscr{A}g + g^{-1}dg, \qquad h \to g^{-1}hg. \tag{3.3}$$

A hint of the connection between this object and four dimensional gravity is given through the relation

$$S(\mathscr{A}_0) = -\kappa TP(\mathscr{A}_0, \mathscr{A}_0^{h_0}) = \kappa \sinh\theta_0 \frac{3}{2} \int_{M^4} \varepsilon_{abcd} b^a b^b \left(R^{cd} - \frac{1}{2}\mu b^c b^d \right), \tag{3.4}$$

$$\mathscr{A}_0 = \frac{1}{2}\omega^{ab}J_{ab} + b^a J_{a4}, \;\; h_0 = e^{\theta_0 J_{45}}, \qquad \mu = \frac{2+\cosh\theta_0}{3}, \tag{3.5}$$

where the constant group element, h_0, is chosen to break the $SO(4,2)$ symmetry down to $SO(3,1) \times SO(1,1)$, and the $SO(1,1)$ part is further broken by setting $c^a = 0$. The action principle is defined up to an overall multiplicative constant, $-\kappa$.

The canonical form of the Hilbert action is recovered from (3.4) when a parameter with dimensions of length, l, is used to write it in terms of a dimensionless vielbein $e^a_\mu = l b^a_\mu$, that is related with the spacetime metric through the usual relation, $\eta_{ab} e^a_\mu e^b_\nu = g_{\mu\nu}$, and the torsionless field equation, $De^a = 0$, is used:

$$S(g_{\mu\nu}) = \frac{1}{16\pi G} \int_{M^4} (R - 2\Lambda) \sqrt{|g|} d^4x \tag{3.6}$$

$$\Lambda = \frac{2+\cosh\theta_0}{2l^2} \qquad \frac{1}{16\pi G} = \kappa \sinh\theta_0 \frac{6}{l^2} \tag{3.7}$$

Note that these relations set the Cosmological constant to $\Lambda = \frac{2+\cosh\theta_0}{192\pi G\kappa \sinh\theta_0}$.

4. Conclusions

Indeed, the arguments given in this note, in favor of the four dimensional theory defined by the gauged WZW, should be refined. In [15, 16] is discussed that when the same ansatz is replaced in the field equations of the full gauged WZW theory, the Einstein field equations arise as a degenerated sector of it [17, 18, 19, 20]. Which implies that the open sets in the phase space of the gWZW theory have a completely different dynamical behavior.

Moreover, it is necessary to set to zero a large number of fields to recover Einstein gravity. If the theory here introduced is physically meaningful, or not, also depends in what is the role of that fields, and how they couple to gravity. It is particularly interesting to note that the matter, h fields, and the geometrical part described by $\mathscr{A}$ are intrinsically related. If the configuration $h = 1$ is considered, the field equations associated to the connection are trivially satisfied, while the field equations associated to h determine the connection. These and other considerations are currently under research.

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