Automated calculations for multi-leg processes

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The search for signals of new physics at the forthcoming LHC experiments involves the analysis of final states characterised by a high number of hadronic jets or identified particles. Precise theoretical predictions for these processes require the computation of scattering amplitudes with a large number of external particles and beyond leading order in perturbation theory. The complexity of a calculation grows with the number of internal loops as well as with the number of external legs. Automatisation of at least next-to-leading order calculations for LHC processes is therefore a timely task. I will discuss various approaches.

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1. Introduction

The Large Hadron Collider (LHC) is expected to start next year. The physics program of the two main experiments ATLAS and CMS aims at the discovery of the last missing particle predicted by the Standard Model – the Higgs boson – and at the search for signals of new physics beyond the Standard Model. Common to these searches is the fact that the signal events have to be digged out from a bulk of background events. The background events are due to Standard Model processes, mostly QCD processes which sometimes are accompanied by additional electro-weak bosons. Common to these searches is further the fact, that the final state is characterised by a high number of hadronic jets or identified particles. A rough theoretical understanding of scattering processes at hadron colliders is sketched in fig. 1. The initial protons do not take part as a single entity in the scattering process, instead the basic constituents of the proton – partons like quarks and gluons – enter the hard scattering process. The probability of finding a specific parton inside the proton is described by parton distribution functions (pdf's). The parton distribution functions are non-perturbative objects and therefore at present cannot be calculated from theory. However, they are universal and can be measured and extracted from one experiment and then used as input data for other experiments. In particular, the HERA experiment at DESY contributed significantly to our knowledge of the parton distribution functions. As a technical detail, the parton distribution functions depend on a scale. In simplified terms, they are measured at a scale Q_0, but used as input data at a different scale Q_1. In this context it is worth to note that the variation with the scale of the parton distribution functions can be calculated within perturbation theory.

The hard scattering process can be formulated entirely in terms of the fundamental fields (quarks, gluons, ...) of the Standard Model. It is calculable in perturbation theory. Attached to the hard scattering process are parton showers, where the partons of the hard scattering event radiate off additional collinear or soft partons. From this additional radiation originates the observed hadronic jets, i.e. bunches of particles moving in the same direction. The parton shower increases significantly the number of partons in the event. In principle, the parton shower is governed to a large extend by perturbation theory. But due to our limited computational abilities we are forced to replace the full matrix elements by approximations. These approximations are based on the observation, that matrix elements are enhanced in soft and collinear regions.

After the parton shower, the partons convert to hadrons. Like for the parton distribution functions this is non-perturbative physics. In practise, models like the string model or the cluster model are used to mimic the formation of hadrons. If unstable hadrons are formed, they subsequently decay.

This simplified picture of an event has to be completed with additional complications due to the underlying event, multiple interactions and pile-ups. As the hard scattering breaks up the initial protons, the proton remnants are not colour-neutral. The underlying event describes the interactions of the proton remnants. Usually the underlying event will produce activity in the detector along the beam pipe. However, it can happen that more than one pair of partons of the initial protons undergo a hard scattering. This goes under the name of multiple interactions. Finally, there is the possibility that more than one proton-proton scattering takes place in a single bunch crossing. These are pile-up events.

As can be seen, the theoretical description of a single event is rather involved and requires
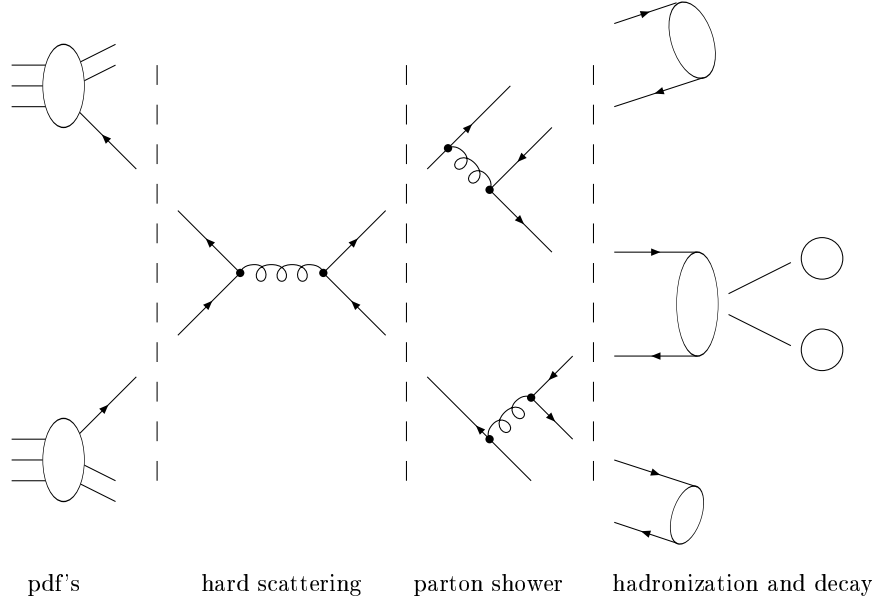
Figure 1: A schematic and simplified description of an event in hadron-hadron collisions.

in particular knowledge on non-perturbative physics. As this knowledge is not available, we have to content ourselves to models and approximations. This will limit the accuracy and precision of theoretical predictions. Fortunately, for a specific class of observables we can do better: This is the class of infrared-safe observables. Infrared safeness implies that the value of an observable does not change, if additional soft or collinear particles are added to an event:

$$\lim_{r \text{ partons unresolved}} O_{n+r}(p_1,...,p_{n+r}) = O_n\left(p'_1,...,p'_n\right) \tag{1.1}$$

Here, $O_n(p_1,...,p_n)$ denotes the value of the observable for an event with particles with four-momenta p_1, ..., p_n. Infrared-safe observables depend only mildly on showering and hadronisation and can be calculated reliably in perturbation theory. At hadron colliders we need of course the additional non-perturbative information on the parton distribution functions, but these quantities have been measured and are available. The master formula for the calculation of an observable is given by

$$\langle O \rangle = \sum_{a,b} \int dx_1\, f_a(x_1) \int dx_2\, f_b(x_2)\ \ \frac{1}{2K(s)}\ \ \frac{1}{(2J_1+1)} \frac{1}{(2J_2+1)} \frac{1}{n_1 n_2}$$
$$\times \sum_n \int d\phi_{n-2} O(p_1,...,p_n)\ \ |\mathscr{A}_n|^2. \tag{1.2}$$

The various ingredients of this formula are: The parton distribution functions, i.e. the probability of finding a parton i with momentum fraction x inside the parent hadron h are denoted by $f_i(x)$. $2K(s)$ is the flux factor, equal to two times the center-of-mass energy squared of the two incoming partons. $1/(2J_1+1)/(2J_2+1)/n_1/n_2$ corresponds to an averaging over the spins and colour degrees of
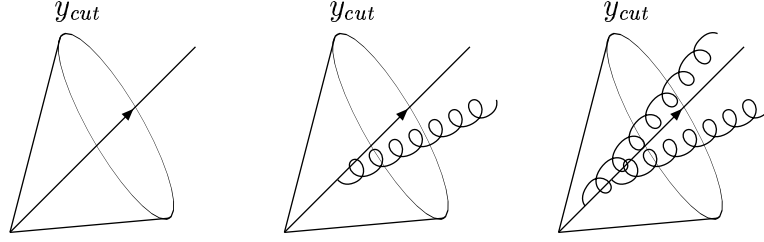
Figure 2: Modelling of jets in perturbation theory. At leading-order a jet is modelled by a single parton, at next-to-leading order either by one or two partons. At next-to-next-to-leading order a jet is modelled by one, two or three partons.

freedom for the initial particles. The second sum is over the number of final-state particles, the integral is over the phase space of $(n-2)$ final-state particles. The most important ingredient of this formula is the matrix element squared $|\mathscr{A}_n|^2$ for n particles, $(n-2)$ in the final state, two in the initial state. As can be seen from this formula, each event is weighted by the matrix element squared.

At high energies the strong coupling is small and the matrix element can be calculated in perturbation theory. For an observable, whose leading-order contribution in perturbation theory is given by an *n*-parton amplitude, the following expansions are relevant for the calculation of the next-to-leading order (NLO) and next-to-next-to-leading order (NNLO) predictions:

$$|\mathscr{A}_n|^2 = \underbrace{\left|\mathscr{A}_n^{(0)}\right|^2}_{\text{Born}} + \underbrace{2\,\mathrm{Re}\left(\mathscr{A}_n^{(0)\,*}\mathscr{A}_n^{(1)}\right)}_{\text{one-loop}} + \underbrace{2\,\mathrm{Re}\left(\mathscr{A}_n^{(0)\,*}\mathscr{A}_n^{(2)}\right) + \left|\mathscr{A}_n^{(1)}\right|^2}_{\text{two-loop and loop-loop}},$$

$$|\mathscr{A}_{n+1}|^2 = \underbrace{\left|\mathscr{A}_{n+1}^{(0)}\right|^2}_{\text{single emission}} + \underbrace{2\,\mathrm{Re}\left(\mathscr{A}_{n+1}^{(0)\,*}\mathscr{A}_{n+1}^{(1)}\right)}_{\text{loop+single emission}},$$

$$|\mathscr{A}_{n+2}|^2 = \underbrace{\left|\mathscr{A}_{n+2}^{(0)}\right|^2}_{\text{double emission}}. \qquad (1.3)$$

In this formulae $\mathscr{A}_n^{(l)}$ denotes an amplitude with *n* external particles and *l* loops. At leading-order only the Born amplitude $\mathscr{A}_n^{(0)}$ contributes. At next-to-leading order we have in addition the contributions from the one-loop amplitude $\mathscr{A}_n^{(1)}$ and the single emission contribution, which is given by the Born amplitudes $\mathscr{A}_{n+1}^{(0)}$ with one additional parton. These two contribution are of the same order with respect to the power counting of the coupling. As far as the phase space for the final-state particles is concerned, these two contributions live on different phase spaces of different dimensions, since the single emission contribution has one additional particle in the final state. At next-to-next-to-leading order we have to include in addition the contributions from the two-loop amplitude and the one-loop amplitude squared, the one-loop amplitudes with a single additional emission and the Born amplitudes with two additional emissions. Up to next-to-next-to-leading
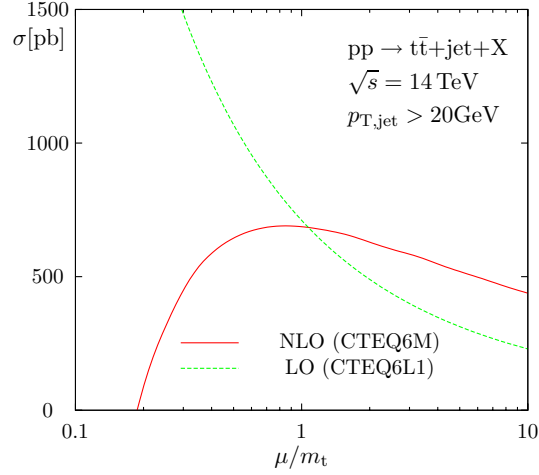
Figure 3: Scale-dependence of the cross-section for the process $pp \rightarrow t\bar{t} + \text{jet}$. The leading-order prediction shows a strong scale-dependence. The next-to-leading order prediction reduces significantly the scale-dependence. The plot is taken from [1].

order we include the radiation of up to two additional particles. Therefore jets are modelled by one, two or three partons. This is shown in fig. 2.

Why are the computations of higher-order corrections necessary ? The answer is that we want to achieve a certain precision. This is illustrated by a simple example. Let us consider a pure QCD process with three hard partons in the final state. The leading-order prediction is proportional to α_s^3. It is a well-known fact that the numerical value of the strong coupling depends on an arbitrary scale μ_{ren}, at leading order the formula reads

$$\alpha_s = \frac{4\pi}{\beta_0 \ln \frac{\mu_{ren}^2}{\Lambda^2}}, \quad \beta_0 = 11 - \frac{2}{3}N_f, \quad \Lambda \approx 165 \text{ MeV} \quad \text{for } N_f = 5 \text{ light flavours.} \quad (1.4)$$

As the choice of the scale is arbitrary, and since α_s enters the leading-order prediction to the third power, this introduces a strong scale-dependence of the theoretical prediction and therefore a large uncertainty. The inclusion of higher-order corrections reduces this uncertainty. This is shown for the example of $t\bar{t} + \text{jet}$ production at the LHC in fig. 3. This is an example where the leading-order prediction is proportional to α_s^3. Shown in fig. 3 is the variation of the theoretical prediction with the scale μ. Apart from the scale entering the formula for the strong coupling, which is called the renormalisation scale there is in addition also a scale, at which the parton distribution functions are evaluated. The latter is called the factorisation scale. In the plot both scale have been identified,

$$\mu = \mu_{ren} = \mu_{fact}. \quad (1.5)$$

As can be seen, the inclusion of the next-to-leading order correction reduces significantly the scale-dependence and clearly motivates the quest for the computation of higher-order corrections.

What are the objectives for LHC physics ? Let us now look more closely where precision calculations are needed for LHC physics. There are three important points to mention:

- In order to predict absolute rates to a good precision, we have to know the input parameters of a theoretical calculation to a high precision. The most important input parameters are the

	process	relevant for
1.	$pp \to VV + \text{jet}$	$t\bar{t}H$, new physics
2.	$pp \to H + 2\,\text{jets}$	Higgs production by vector boson fusion
3.	$pp \to t\bar{t}b\bar{b}$	$t\bar{t}H$
4.	$pp \to t\bar{t} + 2\,\text{jets}$	$t\bar{t}H$
5.	$pp \to VVb\bar{b}$	$VBF \to H \to VV$, $t\bar{t}H$, new physics
6.	$pp \to VV + 2\,\text{jets}$	$VBF \to H \to VV$
7.	$pp \to V + 3\,\text{jets}$	various new physics signatures
8.	$pp \to VVV$	SUSY

Table 1: The experimenter's wish list at Les Houches 2005 [2]. *V* denotes either one of the electro-weak bosons: photon, *Z*-boson or *W*-boson.

strong coupling α_s and the parton distribution functions. These parameters can be extracted from other experiments, for example the strong coupling can be extracted from the process $e^+e^- \to 3$ jets at LEP, the pdf's are measured at HERA. The experimental precision has to be matched with the corresponding precision of the theoretical calculation, requiring the inclusion of the NNLO corrections. This in turn requires a calculation for $e^+e^- \to 3$ jets at NNLO and the computation of the three-loop splitting functions for the evolution of the pdf's.

- Standard candles at the LHC: For a few standard hard pp processes like the production of *W*, *Z*, top, Higgs or jets we would like to have a precise prediction from theory, again requiring a NNLO computation. Higgs production is of clear interest for the search of the last undiscovered particle predicted by the Standard Model. The other standard processes are useful to measure fundamental quantities like the *W*- or top-mass to a better precision or can be used by the experimentalists to understand their detector.

- Finally one would like to have reliable predictions for multi-particle final states that occur at high rates and form background to new physics. This implies next-to-leading order calculations. Examples for such processes have been collected in the experimenter's wish list at Les Houches workshop in 2005 and are shown in table 1.

What is the state-of-the-art for multi-leg processes ? It is clear that the complexity of a calculation increases with the order in perturbation theory as well as with the number of external legs. Therefore we expect that if we go up in the order of perturbation theory the available calculations are restricted to fewer final state particles. Let us first consider leading order calculations. At this order there are techniques for the automated calculation of the (leading order) matrix elements [3–10] as well as techniques for the efficient integration over phase space [11, 12]. Several computer programs like Madgraph/Madevent [13–15], Sherpa/Amegic++ [16], Helac/Phegas [17, 18], Comphep [19], Grace [20] or Alpgen [21] are available, which implement these techniques and can be used to obtain leading order predictions for processes with a rather high number of final state particles.

n	2	3	4	5	6	7	8
diagrams	4	25	220	2485	34300	559405	10525900

Table 2: Number of Feynman diagrams contributing to $gg \to ng$ at tree level.

If we now consider the next order in perturbation theory, we observe that there many NLO calculation for $2 \to 2$ processes at hadron colliders, but only a few for $2 \to 3$ processes. Fully differential numerical programs for $2 \to 3$ processes exist for example for $pp \to 3$ jets [22–24], $pp \to V + 2$ jets [25–29], $pp \to t\bar{t}H$ [30, 31], $pp \to H + 2$ jets [32–34], $pp \to t\bar{t} +$ jet [1] and $pp \to ZZZ$ [35]. Of comparable complexity is the NLO calculation for the production of two vector bosons in vector-boson-fusion [36–38]. On the other side there aren't at present any NLO programs for the LHC with more than 3 hard particles in the final state. NLO programs with four particles in the final state are available for electron-positron annihilation for $e^+e^- \to 4$ fermions [39, 40] and $e^+e^- \to 4$ jets [41–44]. As can be seen, for NLO computations our capabilities are already much more restricted with respect to the number of final state particles. For processes with only two or three final state particles tools which help with the automatisation of the computation like the combination FeynArts, FormCalc, Looptools [45–48] or the Grace package [49] are available.

Finally, we consider NNLO calculations. Here, fully differential predictions for the LHC are available only for a few selected $2 \to 1$ and $2 \to 2$ processes, like Drell-Yan [50], W-production [51] or Higgs production [52–58]. For electron-positron annihilation NNLO predictions are available for $e^+e^- \to 2$ jets [59–61] and the thrust distribution [62].

What are the bottle-necks ? As can be seen from this summary of the state-of-the-art, higher-order corrections are limited to processes with not too many particles in the final state. Let us now look into the difficulties, which prohibit a straight-forward automatisation of the computation of higher order corrections for processes with many particles in the final state. These difficulties can be grouped into three categories:

- Length: Perturbative calculations lead to expressions with a huge number of terms.

- Integrals: At one-loop and beyond, the occurring integrals cannot be simply looked up in an integral table.

- Divergences: At NLO and beyond, infrared divergences occur in intermediate stages, if massless particles are involved.

The first complication – lengthy expressions – affects already leading-order computations. I will discuss methods to handle this problem in section 2. The second and the third item occur for the first time in NLO computations. They are discussed in section 3 and section 4, respectively.

2. Managing lengthy expressions

It is a well-known fact that the complexity of a calculation based on Feynman diagrams growth factorially with the number of external particles. As an example consider the all-gluon amplitude at tree level. Table 2 shows the number of diagrams contributing to the process $gg \to ng$ at tree

level. All diagrams involve three- and four-gluon vertices. The Feynman rules for these vertices blow-up the corresponding expressions even further:
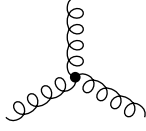
$$= g f^{abc} \left[(k_3 - k_2)_\mu g_{\nu\lambda} + (k_1 - k_3)_\nu g_{\lambda\mu} + (k_2 - k_1)_\lambda g_{\mu\nu} \right],$$
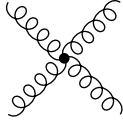
$$\begin{aligned} = -ig^2 \Big[& f^{abe} f^{ecd} \left(g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda} \right) + f^{ace} f^{ebd} \left(g_{\mu\nu} g_{\lambda\rho} - g_{\mu\rho} g_{\lambda\nu} \right) \\ & + f^{ade} f^{ecb} \left(g_{\mu\nu} g_{\lambda\rho} - g_{\mu\lambda} g_{\nu\rho} \right) \Big]. \end{aligned}$$

(2.1)

For the computation of observables we have to square the amplitude and to sum over all spins or helicities. For gluons we replace the polarisation sum by

$$\sum_\lambda \varepsilon_\mu^*(k, \lambda) \varepsilon_\nu(k, \lambda) = -g_{\mu\nu} + \frac{k_\mu n_\nu + n_\mu k_\nu}{kn} - n^2 \frac{k_\mu k_\nu}{(kn)^2}.$$

(2.2)

In this formula, n_μ is an arbitrary four-vector. If the amplitude consists of $O(N)$ terms, squaring the amplitude will produce $O(N^2)$ terms. From this example it is clear, that the number of terms in intermediate expressions of a calculation based on Feynman diagrams growth dramatically with the number of external legs. As already mentioned, this problem occurs already at tree level.

2.1 Computer algebra

As calculations for multi-particle final states tend to involve lengthy intermediate expressions, computer algebra has become an essential tool. In fact, particle physics is and has been a driving force for the development of computer algebra systems. Quite a few computer algebra systems have their roots within the high energy physics community or strong links with them: REDUCE, SCHOONSHIP, MATHEMATICA, FORM [63] or GiNaC [64, 65], to name only a few. In most cases the requirements on a computer-algebra system for computer-intensive symbolic calculations in particle physics can be summarised by:

- The computer algebra system has to provide basic operations like addition, multiplication, sorting, etc..

- Specialised code for the solution of a particular problem is usually written by the user. The computer algebra system has to provide a convenient programming language.

- There is actually no need for a system which knows "more" than the user.

The most widely used computer algebra systems in the community are MATHEMATICA, MAPLE, REDUCE, FORM and GiNaC. The first three are commercial programs, FORM and GiNAC are

non-commercial and freely available[1]. Below there are two small example programs in FORM and GiNaC for the calculation of

$$\text{Tr} \, \not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4 = 4 \, (p_1 \cdot p_2) \, (p_3 \cdot p_4) - 4 \, (p_1 \cdot p_3) \, (p_2 \cdot p_4) + 4 \, (p_1 \cdot p_4) \, (p_2 \cdot p_3). \qquad (2.3)$$

The example in FORM reads:

```
* Example program for FORM

V p1,p2,p3,p4;

L  res = g_(1,p1)*g_(1,p2)*g_(1,p3)*g_(1,p4);

trace4,1;

print;

.end
```

GiNaC is a C++ library, which provides capabilities for symbolic calculations within the C++ programming language. The corresponding example in GiNaC reads:

```
#include <iostream>
#include <ginac/ginac.h>

using namespace std;
using namespace GiNaC;

int main()
{
  varidx mu(symbol("mu"),4), nu(symbol("nu"),4),
         rho(symbol("rho"),4), sigma(symbol("sigma"),4);

  symbol p1("p1"), p2("p2"), p3("p3"), p4("p4");

  ex res = dirac_gamma(mu,1)*dirac_gamma(nu,1)
    *dirac_gamma(rho,1)*dirac_gamma(sigma,1)
    *indexed(p1,mu.toggle_variance())*indexed(p2,nu.toggle_variance())
    *indexed(p3,rho.toggle_variance())*indexed(p4,sigma.toggle_variance());

  res = dirac_trace(res,1);

  res = res.expand();
  res = res.simplify_indexed();

  cout << res << endl;
}
```

Computer algebra is an essential tool, but a brute force application alone will still produce lengthy expressions, which are slow and potentially unstable, when evaluated numerically.

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[1] FORM is available at http://www.nikhef.nl/˜form, GiNaC is available at http://www.ginac.de.

2.2 Quantum number management

In order to keep the size of intermediate expressions under control, a divide-and-conquer strategy has been proven useful: One divides the quantity to be calculated into smaller pieces and calculates the small pieces separately. This approach is also called "quantum number management". One first observes that it is not necessary to square the amplitude and sum over the spins and helicities analytically. It is sufficient to do this numerically. This avoids obtaining $O(N^2)$ terms from an expression with $O(N)$ terms. The individual amplitudes have to be calculated in a helicity or spin basis. This is discussed in section 2.2.1. The second observation is related to the fact, that individual helicity amplitudes can be decomposed into smaller gauge-invariant pieces, called partial amplitudes. This is discussed in section 2.2.2. These partial amplitudes can be calculated for a given helicity configuration without reference to Feynman diagrams, as discussed in section 2.2.3. Finally, for specific helicity configurations, compact formulæ are known, which are summarised in section 2.2.4. Supersymmetric identities provide relations between amplitudes with different particle contents, therefore only some of them need to be calculated. This is discussed in section 2.2.5.

2.2.1 Helicity amplitudes

The computation of helicity amplitudes requires the choice of a helicity (or spin) basis. This is straightforward for massless fermions. The two-component Weyl spinors provide a convenient basis:

$$|p\pm\rangle = \frac{1}{2}\left(1\pm\gamma_5\right)u(p). \tag{2.4}$$

In the literature there are different notations for Weyl spinors. Apart from the bra-ket-notation there is the notation with dotted and un-dotted indices: The relation between the two notations is the following:

$$\begin{aligned}|p+\rangle = p_B, \quad \langle p+| = p_{\dot{A}},\\ |p-\rangle = p^{\dot{B}}, \quad \langle p-| = p^A.\end{aligned} \tag{2.5}$$

Spinor products are defined as

$$\langle pq\rangle = \langle p-|q+\rangle, \quad [pq] = \langle p+|q-\rangle, \tag{2.6}$$

and take value in the complex numbers. It was a major break-through, when it was realised that also gluon polarisation vectors can be expressed in terms of two-component Weyl spinors [66–71]. The polarisation vectors of external gluons can be chosen as

$$\varepsilon_\mu^+(k,q) = \frac{\langle q-|\gamma_\mu|k-\rangle}{\sqrt{2}\langle q-|k+\rangle}, \quad \varepsilon_\mu^-(k,q) = \frac{\langle q+|\gamma_\mu|k+\rangle}{\sqrt{2}\langle k+|q-\rangle}, \tag{2.7}$$

where k is the momentum of the gluon and q is an arbitrary light-like reference momentum. The dependence on the arbitrary reference momentum q will drop out in gauge invariant quantities. The polarisation sum is that of an light-like axial gauge:

$$\sum_{\lambda=\pm}\varepsilon_\mu^\lambda(k,q)\left(\varepsilon_\nu^\lambda(k,q)\right)^* = -g_{\mu\nu} + \frac{k_\mu q_\nu + k_\nu q_\mu}{k\cdot q}. \tag{2.8}$$

Changing the reference momentum will give a term proportional to the momentum of the gluon:

$$\varepsilon_\mu^+(k,q_1) - \varepsilon_\mu^+(k,q_2) = \sqrt{2}\frac{\langle q_1 q_2\rangle}{\langle q_1 k\rangle\langle kq_2\rangle}k_\mu. \tag{2.9}$$

For massive fermions we can take the spinors as [72, 73]

$$u(\pm) = \frac{1}{\langle p^\flat \mp |q\pm\rangle}(p\!\!\!/ + m)|q\pm\rangle, \quad \bar{u}(\pm) = \frac{1}{\langle q\mp |p^\flat\pm\rangle}\langle q\mp|(p\!\!\!/ + m),$$

$$v(\pm) = \frac{1}{\langle p^\flat \mp |q\pm\rangle}(p\!\!\!/ - m)|q\pm\rangle, \quad \bar{v}(\pm) = \frac{1}{\langle q\mp |p^\flat\pm\rangle}\langle q\mp|(p\!\!\!/ - m). \tag{2.10}$$

Here $p^\flat$ is a light-like four vector obtained through

$$p^\flat = p - \frac{p^2}{2p\cdot q}q. \tag{2.11}$$

q denotes again an arbitrary light-like reference momentum and is related to the quantisation axis of the spin for the massive fermion. In contrast to the gluon case individual amplitudes with label $+$ or $-$ will depend on the choice of the reference momentum q.

2.2.2 Colour decomposition

Throughout this article I use the normalisation

$$\text{Tr}\, T^a T^b = \frac{1}{2}\delta^{ab} \tag{2.12}$$

for the colour matrices. Amplitudes in QCD may be decomposed into group-theoretical factors (carrying the colour structures) multiplied by kinematic functions called partial amplitudes [74–78]. These partial amplitudes do not contain any colour information and are gauge-invariant objects. In the pure gluonic case tree level amplitudes with n external gluons may be written in the form

$$\mathcal{A}_n(1,2,...,n) = g^{n-2}\sum_{\sigma\in S_n/Z_n} 2\,\text{Tr}(T^{a_{\sigma(1)}}...T^{a_{\sigma(n)}})A_n(\sigma(1),...,\sigma(n)), \tag{2.13}$$

where the sum is over all non-cyclic permutations of the external gluon legs. The quantities $A_n(\sigma(1),...,\sigma(n))$, called the partial amplitudes, contain the kinematic information. They are colour-ordered, e.g. only diagrams with a particular cyclic ordering of the gluons contribute. The colour decomposition is obtained by replacing the structure constants f^{abc} by

$$if^{abc} = 2\left[\text{Tr}\left(T^a T^b T^c\right) - \text{Tr}\left(T^b T^a T^c\right)\right] \tag{2.14}$$

which follows from $\left[T^a, T^b\right] = if^{abc}T^c$. The resulting traces and strings of colour matrices can be further simplified with the help of the Fierz identity :

$$T_{ij}^a T_{kl}^a = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right). \tag{2.15}$$

The colour decomposition for a tree amplitude with a pair of quarks is

$$\mathcal{A}_{n+2}(q,1,2,...,n,\bar{q}) = g^n\sum_{S_n}(T^{a_{\sigma(1)}}...T^{a_{\sigma(n)}})_{i_q j_{\bar{q}}}A_{n+2}(q,\sigma(1),\sigma(2),...,\sigma(n),\bar{q}). \tag{2.16}$$

where the sum is over all permutations of the gluon legs. In general, the colour factors are combinations of open

$$(T^{a_1}...T^{a_n})_{i_q j_{\bar{q}}} \tag{2.17}$$

and closed strings

$$\mathrm{Tr}\left(T^{b_1}...T^{b_m}\right) \tag{2.18}$$

of colour matrices. These building blocks form a basis in colour space. The choice of the basis for the colour structures is not unique, and several proposals for bases can be found in the literature [79–81]. A second useful basis is the colour-flow basis: This basis is obtained by replacing every contraction over an index in the adjoint representation by two contractions over indices i and j in the fundamental representation:

$$V^a E^a = V^a \delta^{ab} E^b = V^a \left(2 T^a_{ij} T^b_{ji}\right) E^b = \left(\sqrt{2} T^a_{ij} V^a\right) \left(\sqrt{2} T^b_{ji} E^b\right). \tag{2.19}$$

In this representation the colour decomposition of the pure gluon amplitude is given by

$$\mathscr{A}_n(1,2,...,n) = \left(\frac{g}{\sqrt{2}}\right)^{n-2} \sum_{\sigma \in S_n/Z_n} \delta_{i_{\sigma_1} j_{\sigma_2}} \delta_{i_{\sigma_2} j_{\sigma_3}}...\delta_{i_{\sigma_n} j_{\sigma_1}} A_n(\sigma_1,...,\sigma_n). \tag{2.20}$$

2.2.3 Off-shell recurrence relations

Berends-Giele type recurrence relations [3, 82] build partial amplitudes from smaller building blocks, usually called colour-ordered off-shell currents. Off-shell currents are objects with n on-shell legs and one additional leg off-shell. Momentum conservation is satisfied. It should be noted that off-shell currents are not gauge-invariant objects. Recurrence relations relate off-shell currents with n legs to off-shell currents with fewer legs. The recursion starts with $n = 1$:

$$J^{\mu}(k_1) = \varepsilon^{\mu}(k_1, q). \tag{2.21}$$

ε^{μ} is the polarisation vector of the gluon and q an arbitrary light-like reference momentum. The recursive relation states that in the pure-gluon off-shell current a gluon couples to other gluons only via the three- or four-gluon vertices :

$$J^{\mu}(k_1^{\lambda_1},...,k_n^{\lambda_n}) = \frac{-i}{K_{1,n}^2} \left[\sum_{j=1}^{n-1} V_3^{\mu\nu\rho}(-K_{1,n}, K_{1,j}, K_{j+1,n}) J_{\nu}(k_1^{\lambda_1},...,k_j^{\lambda_j}) J_{\rho}(k_{j+1}^{\lambda_{j+1}},...,k_n^{\lambda_n}) \right.$$
$$\left. + \sum_{j=1}^{n-2} \sum_{l=j+1}^{n-1} V_4^{\mu\nu\rho\sigma} J_{\nu}(k_1^{\lambda_1},...,k_j^{\lambda_j}) J_{\rho}(k_{j+1}^{\lambda_{j+1}},...,k_l^{\lambda_l}) J_{\sigma}(k_{l+1}^{\lambda_{l+1}},...,k_n^{\lambda_n}) \right], \tag{2.22}$$

where

$$K_{i,j} = k_i + k_{i+1} + ... + k_j \tag{2.23}$$

and V_3 and V_4 are the colour-ordered three-gluon and four-gluon vertices
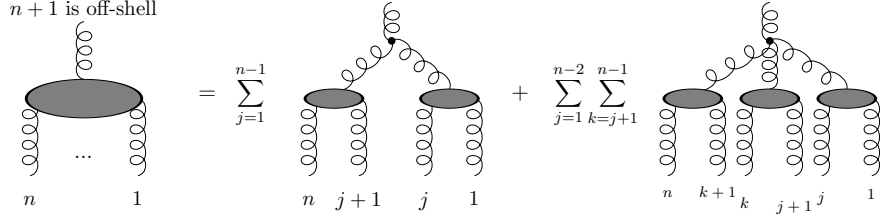
Figure 4: Off-shell recurrence relation: In an off-shell current particle $n+1$ is kept off-shell. This allows to express an off-shell current with n on-shell legs in terms of currents with fewer legs.

$$
\begin{aligned}
V_3^{\mu\nu\rho}(k_1,k_2,k_3) &= i\left[g^{\mu\nu}\left(k_1^\rho - k_2^\rho\right) + g^{\nu\rho}\left(k_2^\mu - k_3^\mu\right) + g^{\rho\mu}\left(k_3^\nu - k_1^\nu\right)\right], \\
V_4^{\mu\nu\rho\sigma} &= i\left(2g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right).
\end{aligned}
\tag{2.24}
$$

The recurrence relation is shown pictorially in fig. 4. The gluon current J_μ is conserved:

$$
\left(\sum_{i=1}^n k_i^\mu\right) J_\mu = 0.
\tag{2.25}
$$

From an off-shell current one easily recovers the on-shell amplitude by removing the extra propagator, taking the leg $(n+1)$ on-shell and contracting with the appropriate polarisation vector.

2.2.4 Parke-Taylor formulæ

The partial amplitudes have for specific helicity combinations remarkably simple analytic formula or vanish altogether. For the all-gluon tree amplitude one finds

$$
\begin{aligned}
A_n(1^+,2^+,...,n^+) &= 0, \\
A_n(1^+,2^+,...,j^-,...,n^+) &= 0, \\
A_n(1^+,2^+,...,j^-,...,k^-,...,n^+) &= i\left(\sqrt{2}\right)^{n-2}\frac{\langle jk\rangle^4}{\langle 12\rangle...\langle n1\rangle}.
\end{aligned}
\tag{2.26}
$$

The partial amplitudes where all gluons have positive helicities, or where all gluons except one have positive helicities vanish. The first non-vanishing result is obtained for the n-gluon amplitude with $n-2$ gluons of positive helicity and 2 gluons of negative helicity. It is given by a remarkable simple formula. Note that this formula holds for all n. An amplitude with $n-2$ gluons of positive helicity and 2 gluons of negative helicity is called a maximal-helicity violating amplitude (MHV amplitude). Obviously, we find similar formulæ if we exchange all positive and negative helicities:

$$
\begin{aligned}
A_n(1^-,2^-,...,n^-) &= 0, \\
A_n(1^-,2^-,...,j^+,...,n^-) &= 0, \\
A_n(1^-,2^-,...,j^+,...,k^+,...,n^-) &= i\left(\sqrt{2}\right)^{n-2}\frac{[kj]^4}{[1n][n(n-1)]...[21]}.
\end{aligned}
\tag{2.27}
$$

These formulæ have been conjectured by Parke and Taylor [83] and have been proven by Berends and Giele [3].

2.2.5 Supersymmetric relations

After removing the colour factors, QCD at tree-level may be viewed as an effective supersymmetric theory [84–88], where the quarks and the gluons form a super-multiplet (a $N = 1$ vector super-multiple). Let us denote by

$$Q_{\text{SUSY}} = \theta \left(q^A Q_A + \bar{q}_{\dot{A}} \bar{Q}^{\dot{A}} \right) \tag{2.28}$$

the SUSY generators contracted with two-component Weyl spinors q^A and $\bar{q}_{\dot{A}}$ and multiplied by a Grassmann number θ. In an unbroken supersymmetric theory, the supercharge annihilates the vacuum, and therefore

$$\langle 0 | [Q_{\text{SUSY}}, \Phi_1 \Phi_2 ... \Phi_n] | 0 \rangle = \sum_{i=1}^{n} \langle 0 | \Phi_1 ... [Q_{\text{SUSY}}, \Phi_i] ... \Phi_n | 0 \rangle = 0 \tag{2.29}$$

where the field Φ_i denotes either a gauge boson g or a fermion Λ. The commutators are given by

$$\left[Q_{\text{SUSY}}, g^{\pm}(k) \right] = \Gamma^{\pm}(k, q) \Lambda^{\pm}(k), \quad \left[Q_{\text{SUSY}}, \Lambda^{\pm}(k) \right] = \Gamma^{\mp}(k, q) g^{\pm}(k), \tag{2.30}$$

with

$$\Gamma^{\pm}(k, q) = \theta \langle q \pm | k \mp \rangle. \tag{2.31}$$

Let us now consider

$$
\begin{aligned}
0 &= \left\langle 0 \left| \left[Q, \Lambda_1^+ g_2^+ ... g_j^- ... g_{n-1}^+ g_n^- \right] \right| 0 \right\rangle \\
&= \Gamma^-(p_1, q) A_n(g_1^+, g_2^+, ..., g_j^-, ..., g_{n-1}^+, g_n^-) - \Gamma^-(p_j, q) A_n(\Lambda_1^+, g_2^+, ..., \Lambda_j^-, ..., g_{n-1}^+, g_n^-) \\
&\quad - \Gamma^-(p_n, q) A_n(\Lambda_1^+, g_2^+, ..., g_j^-, ..., g_{n-1}^+, \Lambda_n^-).
\end{aligned}
\tag{2.32}
$$

If one further sets the reference momentum equal to $q = p_j$ and uses the expression for the maximally helicity violating gluon amplitudes one obtains the expression for an amplitude with a pair of quarks:

$$A_n(q_1^+, g_2^+, ..., g_j^-, ..., g_{n-1}^+, \bar{q}_n^-) = i \left(\sqrt{2} \right)^{n-2} \frac{\langle j1 \rangle \langle jn \rangle^3}{\langle 12 \rangle \langle 23 \rangle ... \langle n1 \rangle}. \tag{2.33}$$

2.3 New developments: Twistor methods

In the previous section we saw that for the all-gluon tree amplitude there are remarkable simple formulæ if almost all gluons have one helicity and not more than two gluons have the opposite helicity. Of course we are also interested in the case, where more than two gluons have the opposite helicity. MHV vertices, discussed in section 2.3.1, provide an answer to this question and tell us how the complexity increases with the number of opposite helicity gluons. In addition this construction led to new recursion relation, which no longer require that one external leg is kept off-shell. The building blocks of these on-shell recursion relations are gauge-invariant amplitudes. This is discussed in section 2.3.2.

14
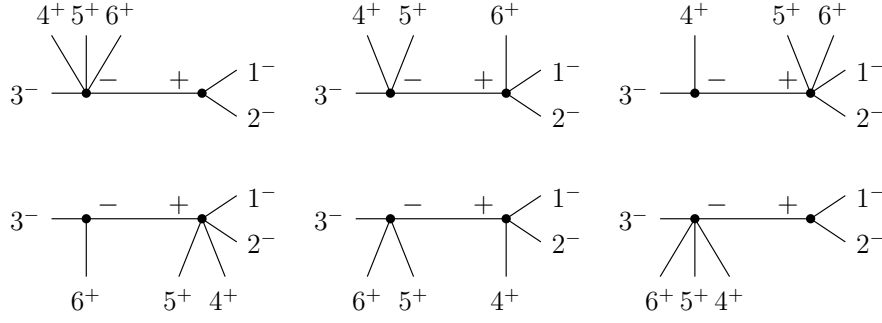
Figure 5: MHV diagrams contributing to the tree-level six-gluon amplitude $A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$.

2.3.1 MHV vertices

As an alternative to usual Feynman graphs, tree amplitudes in Yang-Mills theory can be constructed from tree graphs in which the vertices are tree level MHV scattering amplitudes, continued off shell in a particular fashion [89]. The basic building blocks are the MHV amplitudes, which serve as new vertices:

$$V_n(1^+, ..., j^-, ..., k^-, ..., n^+) = i\left(\sqrt{2}\right)^{n-2} \frac{\langle jk\rangle^4}{\langle 12\rangle...\langle n1\rangle}. \tag{2.34}$$

Each MHV vertex has exactly two lines carrying negative helicity and at least one line carrying positive helicity. Each internal line has a positive helicity label on one side and a negative helicity label on the other side. The propagator for each internal line is the propagator of a scalar particle:
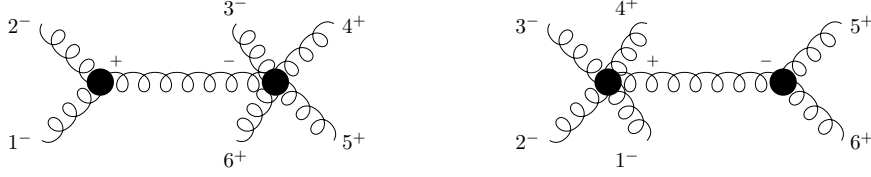
$$\frac{i}{k^2} \tag{2.35}$$

The expression (2.34) for the MHV vertices involves spinors corresponding to massless on-shell momenta $k_j^2 = 0$. Therefore we have state what this light-like four-vector should be for every internal line meeting a MHV vertex. As in eq. (2.11) the light-like four-vector can be taken as [90]

$$k^\flat = k - \frac{k^2}{2k\cdot q}q, \tag{2.36}$$

where k is the momentum flowing through the internal line and q is a fixed light-like reference momentum. Let us now consider an example. The amplitude $A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ has three gluons of positive helicity and three gluons of negative helicity and is one of the first non-trivial amplitudes, which are non-zero and which are not MHV amplitudes. Fig. 5 shows the six MHV diagrams contributing to this amplitude. The first diagram yields

$$\left[i\sqrt{2}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 2\left(-k_{12}^\flat\right)\rangle\langle\left(-k_{12}^\flat\right)1\rangle}\right] \frac{i}{k_{12}^2} \left[i\left(\sqrt{2}\right)^3 \frac{\langle 3k_{12}^\flat\rangle^4}{\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 6k_{12}^\flat\rangle\langle k_{12}^\flat 3\rangle}\right], \tag{2.37}$$

15

Figure 6: Diagrams contributing to the tree-level six-gluon amplitude $A_6(1^-,2^-,3^-,4^+,5^+,6^+)$ in the on-shell recursive approach. The vertices are on-shell amplitudes.

and similar expressions are obtained for the five other diagrams. In this expression, $k_{12} = k_1 + k_2$ is the momentum flowing through the internal line and $k_{12}^\flat$ is the projection onto a light-like four-vector as in eq. (2.36). We recall from table 2 that a brute force approach would require the calculation of 220 Feynman diagrams. Restricting ourselves to a partial amplitude with a fixed cyclic order reduces this number to 36 diagrams. In the approach based on MHV vertices there are only six diagrams. In the next subsection we will discuss a method which reduces the number of diagrams even further.

2.3.2 On-shell recursion relations

Britto, Cachazo and Feng [91] gave a recursion relation for the calculation of the n-gluon amplitude which involves only on-shell amplitudes. To describe this method it is best not to view the partial amplitude A_n as a function of the four-momenta k_j^μ, but to replace each four-vector by a pair of two-component Weyl spinors. In detail this is done as follows: Each four-vector K_μ has a bi-spinor representation, given by

$$K_{A\dot{B}} = K_\mu \sigma^\mu_{A\dot{B}}, \quad K_\mu = \frac{1}{2} K_{A\dot{B}} \bar{\sigma}_\mu^{\dot{B}A}. \tag{2.38}$$

For light-like vectors this bi-spinor representation factorises into a dyad of Weyl spinors:

$$k_\mu k^\mu = 0 \Leftrightarrow k_{A\dot{B}} = k_A k_{\dot{B}}. \tag{2.39}$$

The equations (2.38) and (2.39) allow us to convert any light-like four-vector into a dyad of Weyl spinors and vice versa. Therefore the partial amplitude A_n, being originally a function of the momenta k_j and helicities λ_j, can equally be viewed as a function of the Weyl spinors k_A^j, $k_{\dot{B}}^j$ and the helicities λ_j:

$$A_n(k_1^{\lambda_1},...,k_n^{\lambda_n}) = A_n(k_A^1, k_{\dot{B}}^1, \lambda_1,...,k_A^n, k_{\dot{B}}^n, \lambda_n). \tag{2.40}$$

Note that for an arbitrary pair of Weyl spinors, the corresponding four-vector will in general be complex-valued. If $(\lambda_n, \lambda_1) \neq (+,-)$ we have the following recurrence relation:

$$A_n\left(k_A^1, k_{\dot{B}}^1, \lambda_1,...,k_A^n, k_{\dot{B}}^n, \lambda_n\right) = \tag{2.41}$$

$$\sum_{j=3}^{n-1} \sum_{\lambda=\pm} A_j\left(\hat{k}_A^1, k_{\dot{B}}^1, \lambda_1, k_A^2, k_{\dot{B}}^2, \lambda_2,...,k_A^{j-1}, k_{\dot{B}}^{j-1}, \lambda_{j-1}, i\hat{K}_A, i\hat{K}_{\dot{B}}, -\lambda\right)$$

$$\times \frac{i}{K_{1,j-1}^2} A_{n-j+2}\left(\hat{K}_A, \hat{K}_{\dot{B}}, \lambda, k_A^j, k_{\dot{B}}^j, \lambda_j,...,k_A^{n-1}, k_{\dot{B}}^{n-1}, \lambda_{n-1}, k_A^n, \hat{k}_{\dot{B}}^n, \lambda_n\right).$$

16

method	diagrams
brute force approach	220
colour-ordered amplitudes	36
MHV vertices	6
on-shell recursion	2

Table 3: The number of diagrams contributing to the colour-ordered six-gluon amplitude $A_6(1^-,2^-,3^-,4^+,5^+,6^+)$ using various methods.

If $(\lambda_n,\lambda_1) = (+,-)$ we can always cyclic permute the arguments, such that $(\lambda_n,\lambda_1) \neq (+,-)$. This is possible, since on-shell amplitudes, where all gluons have the same helicity, vanish. In eq. (2.41) the shifted spinors $\hat{k}_A^1$, $\hat{k}_{\dot{B}}^n$, $\hat{K}_A$ and $\hat{K}_{\dot{B}}$ are given by

$$\hat{k}_A^1 = k_A^1 - z k_A^n, \quad \hat{K}_A = \frac{K_{A\dot{B}} k_1^{\dot{B}}}{\sqrt{\langle 1+|K|n+\rangle}},$$

$$\hat{k}_{\dot{B}}^n = k_{\dot{B}}^n + z k_{\dot{B}}^1, \quad \hat{K}_{\dot{B}} = \frac{k_n^A K_{A\dot{B}}}{\sqrt{\langle 1+|K|n+\rangle}}, \tag{2.42}$$

where

$$K_{A\dot{B}} = \sum_{l=1}^{j-1} k_A^l k_{\dot{B}}^l, \quad K_{1,j-1}^2 = \det K_{A\dot{B}}, \quad \text{and} \quad z = \frac{K_{1,j-1}^2}{\langle 1+|K|n+\rangle}. \tag{2.43}$$

Let us again consider as an example the amplitude $A_6(1^-,2^-,3^-,4^+,5^+,6^+)$. The diagrams contributing in the on-shell approach are shown in fig. 6. One obtains as a result for the amplitude

$$A_6(1^-,2^-,3^-,4^+,5^+,6^+) =$$
$$4i \left[\frac{\langle 6+|1+2|3+\rangle^3}{[61][12]\langle 34\rangle\langle 45\rangle s_{126}\langle 2+|1+6|5+\rangle} + \frac{\langle 4+|5+6|1+\rangle^3}{[23][34]\langle 56\rangle\langle 61\rangle s_{156}\langle 2+|1+6|5+\rangle} \right]. \tag{2.44}$$

Note that there only two diagrams, which need to be calculated. Table 3 shows a comparison of the number of diagrams contributing to the colour-ordered six-gluon amplitude in the various approaches. The performance of a numerical implementation of these new methods have been investigated in [92, 93].

2.3.3 Proof of the on-shell recursion relations

For the proof [94–99] of the on-shell recursion relation one considers the function

$$A(z) = A_n \left(\hat{k}_A^1, k_{\dot{B}}^1, \lambda_1, ..., k_A^n, \hat{k}_{\dot{B}}^n, \lambda_n \right) \tag{2.45}$$

of one variable z, where the z-dependence enters through

$$\hat{k}_A^1 = k_A^1 - z k_A^n, \quad \hat{k}_{\dot{B}}^n = k_{\dot{B}}^n + z k_{\dot{B}}^1. \tag{2.46}$$

Note that for all values of z, this is an on-shell amplitude. However, the four-momenta of particles 1 and n are in general complex. The function $A(z)$ is a rational function of z, which has

only simple poles in z. This follows from the Feynman rules and the factorisation properties of amplitudes. Therefore, if $A(z)$ vanishes for $z \to \infty$, $A(z)$ is given by Cauchy's theorem as the sum over its residues. This is just the right hand side of the recursion relation. The essential ingredient for the proof is the vanishing of $A(z)$ at $z \to \infty$. If $(\lambda_1, \lambda_n) = (+, -)$ it can be shown that each individual Feynman diagram vanishes for $z \to \infty$. Consider the flow of the z-dependence in a particular diagram The most dangerous contribution comes from a path, where all vertices are three-gluon-vertices. For a path made of n propagators we have $n + 1$ vertices and the product of propagators and vertices behaves therefore like z for large z. This statement remains true for a path containing only one vertex and no propagators. The polarisation vectors for the helicity combination $(\lambda_1, \lambda_n) = (+, -)$ contribute a factor $1/z^2$, therefore the complete diagram behaves like $1/z$ and vanishes therefore for $z \to \infty$.

3. Calculating loop amplitudes

The second bottle-neck for higher-order computations are loop integrals. I first discuss one-loop integrals in section 3.1. These are relevant for multi-leg NLO calculations, like the processes listed in table 1. In section 3.2 I discuss techniques for two-loop amplitudes and beyond.

3.1 Automated computation of one-loop amplitudes

The simplest, but most important loop integrals are the one-loop integrals. We have a good understanding of these integrals and I will present the main results in this section. An important result is that any scalar integral with more than four external legs can be reduced to scalar integrals with no more than 4 external legs. Therefore the set of basic one-loop integrals is rather limited. I will discuss this reduction in section 3.1.1. For one-loop tensor integrals we can use the Passarino-Veltman method, which reduces any tensor integral to a combination of scalar integrals. Improvements of the Passarino-Veltman algorithm are discussed in section 3.1.2. Section 3.1.3 is devoted to methods, which avoid Feynman diagrams.

3.1.1 Reduction to integrals with no more than four external legs

In this section we discuss the reduction of scalar integrals with more than four external legs to a basic set of scalar one-, two-, three- and four-point functions. It is a long known fact, that higher point scalar integrals can be expressed in terms of this basic set [100, 101], however the practical implementation within dimensional regularisation was only worked out recently [102–106]. The one-loop n-point functions with $n \geq 5$ are always UV-finite, but they may have IR-divergences. Let us first assume that there are no IR-divergences. Then the integral is finite and can be performed in four dimensions. In a space of four dimensions we can have no more than four linearly independent vectors, therefore it comes to no surprise that in an one-loop integral with five or more propagators, one propagator can be expressed through the remaining ones. This is the basic idea for the reduction of the higher point scalar integrals. With slight modifications it can be generalised to dimensional regularisation. I will discuss the method for massless one-loop integrals

$$I_n = e^{\varepsilon \gamma_E} \mu^{2\varepsilon} (-1)^n \int \frac{d^D k}{i \pi^{\frac{D}{2}}} \frac{1}{k^2 (k - p_1)^2 ... (k - p_1 - ... p_{n-1})^2}. \tag{3.1}$$

With the notation

$$q_i = \sum_{j=1}^{i} p_j \tag{3.2}$$

one can associate two matrices S and G to the integral in eq. (3.1). The entries of the $n \times n$ kinematical matrix S are given by

$$S_{ij} = (q_i - q_j)^2, \tag{3.3}$$

and the entries of the $(n-1) \times (n-1)$ Gram matrix are defined by

$$G_{ij} = 2q_i q_j. \tag{3.4}$$

For the reduction one distinguishes three different cases: Scalar pentagons (i.e. scalar five-point functions), scalar hexagons (scalar six-point functions) and scalar integrals with more than six propagators.

Let us start with the pentagon. A five-point function in $D = 4 - 2\varepsilon$ dimensions can be expressed as a sum of four-point functions, where one propagator is removed, plus a five-point function in $6 - 2\varepsilon$ dimensions [102]. Since the $(6 - 2\varepsilon)$-dimensional pentagon is finite and comes with an extra factor of ε in front, it does not contribute at $O(\varepsilon^0)$. In detail we have

$$I_5 = -2\varepsilon B I_5^{6-2\varepsilon} - \sum_{i=1}^{5} b_i I_4^{(i)} = -\sum_{i=1}^{5} b_i I_4^{(i)} + O(\varepsilon), \tag{3.5}$$

where $I_5^{6-2\varepsilon}$ denotes the $(6 - 2\varepsilon)$-dimensional pentagon and $I_4^{(i)}$ denotes the four-point function, which is obtained from the pentagon by removing propagator i. The coefficients B and b_i are obtained from the kinematical matrix S_{ij} as follows:

$$b_i = \sum_{j} \left(S^{-1} \right)_{ij}, \quad B = \sum_{i} b_i. \tag{3.6}$$

The six-point function can be expressed as a sum of five-point functions [103] without any correction of $O(\varepsilon)$

$$I_6 = -\sum_{i=1}^{6} b_i I_5^{(i)}, \quad b_i = \sum_{j} \left(S^{-1} \right)_{ij}, \tag{3.7}$$

where the coefficients b_i are again related to the kinematical matrix S_{ij}. For the seven-point function and beyond we can again express the n-point function as a sum over $(n-1)$-point functions [106]:

$$I_n = -\sum_{i=1}^{n} r_i I_{n-1}^{(i)}. \tag{3.8}$$

In contrast to eq. (3.7), the decomposition in eq. (3.8) is no longer unique. A possible set of coefficients r_i can be obtained from the singular value decomposition of the Gram matrix

$$G_{ij} = \sum_{k=1}^{4} U_{ik} w_k \left(V^T \right)_{kj}. \tag{3.9}$$

as follows [107]

$$r_i = \frac{V_{i5}}{W_5}, \quad 1 \le i \le n-1, \quad r_n = -\sum_{j=1}^{n-1} r_j, \quad W_5 = \frac{1}{2} \sum_{j=1}^{n-1} G_{jj} V_{j5}. \tag{3.10}$$

19

3.1.2 Improvements of the Passarino-Veltman algorithm

We now consider the reduction of tensor loop integrals (e.g. integrals, where the loop momentum appears in the numerator) to a set of scalar loop integrals (e.g. integrals, where the numerator is independent of the loop momentum). For one-loop integrals a systematic algorithm has been first worked out by Passarino and Veltman [108]. Consider the following three-point integral

$$I_3^{\mu\nu} = \int \frac{d^D k}{i\pi^{D/2}} \frac{k^\mu k^\nu}{k^2 (k-p_1)^2 (k-p_1-p_2)^2}, \tag{3.11}$$

where p_1 and p_2 denote the external momenta. The reduction technique according to Passarino and Veltman consists in writing $I_3^{\mu\nu}$ in the most general form in terms of form factors times external momenta and/or the metric tensor. In our example above we would write

$$I_3^{\mu\nu} = p_1^\mu p_1^\nu C_{21} + p_2^\mu p_2^\nu C_{22} + \{p_1^\mu, p_2^\nu\} C_{23} + g^{\mu\nu} C_{24}, \tag{3.12}$$

where $\{p_1^\mu, p_2^\nu\} = p_1^\mu p_2^\nu + p_2^\mu p_1^\nu$. One then solves for the form factors C_{21}, C_{22}, C_{23} and C_{24} by first contracting both sides with the external momenta $p_1^\mu p_1^\nu$, $p_2^\mu p_2^\nu$, $\{p_1^\mu, p_2^\nu\}$ and the metric tensor $g^{\mu\nu}$. On the left-hand side the resulting scalar products between the loop momentum k^μ and the external momenta are rewritten in terms of the propagators, as for example

$$2p_1 \cdot k = k^2 - (k-p_1)^2 + p_1^2. \tag{3.13}$$

The first two terms of the right-hand side above cancel propagators, whereas the last term does not involve the loop momentum anymore. The remaining step is to solve for the form-factors C_{2i} by inverting the matrix which one obtains on the right-hand side of equation (3.12). Due to this step Gram determinants usually appear in the denominator of the final expression. In the example above we would encounter the Gram determinant of the triangle

$$\Delta_3 = 4 \begin{vmatrix} p_1^2 & p_1 \cdot p_2 \\ p_1 \cdot p_2 & p_2^2 \end{vmatrix}. \tag{3.14}$$

One drawback of this algorithm is closely related to these determinants : In a phase space region where p_1 becomes collinear to p_2, the Gram determinant will tend to zero, and the form factors will take large values, with possible large cancellations among them. This makes it difficult to set up a stable numerical program for automated evaluation of tensor loop integrals. Quite some effort went therefore into improvements and alternatives, which avoid these instabilities. They are centered around the following ideas:

- The use of a different reduction scheme in critical regions, based on an expansion around the small invariants [109–112].

- Choosing a different set of basic integrals. The new set can be non-minimal. Quite often the new basic integrals correspond to scalar integrals in higher dimensions. [107, 113–118].

- Direct numerical integration [119–124].

- The use of spinor techniques, which avoid to a certain extent the occurrence of Gram determinants [125–130].
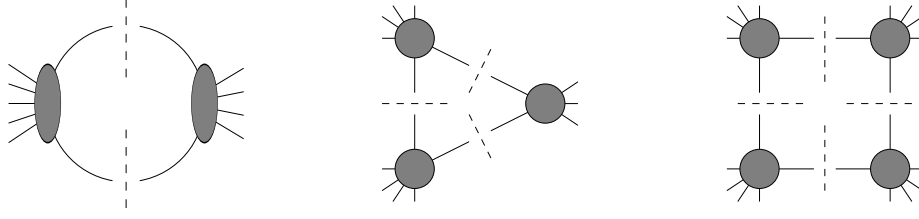
Figure 7: Double, triple and quadruple cuts.

3.1.3 Methods based on unitarity

We know that the final answer for a one-loop amplitude in a massless gauge theory can be written as

$$A_n^{(1)} = \sum_i c_i I_2^{(i)} + \sum_{i,j} c_{ij} I_3^{(ij)} + \sum_{i,j,k} c_{ijk} I_4^{(ijk)} + R. \qquad (3.15)$$

I_2, I_3 and I_4 are the scalar bubble, triangle and box integral functions. In a massive theory we would have in addition also scalar one-point functions. In a massless theory these functions are zero within dimensional regularisation. Note that there are no integral functions with more than four internal propagators. These higher-point functions can always be reduced to the set above, as we have seen in section 3.1.1. R is called the rational term. The set of all occurring integral functions

$$\mathscr{F} = \{I_2^{(i)}, I_3^{(ij)}, I_4^{(ijk)}\} \qquad (3.16)$$

is rather easily obtained from pinching in all possible ways internal propagators in all occurring diagrams. We can assume that we know this set in advance. To compute the amplitude requires therefore the determination of the coefficients c_i, $c_{i,j}$, $c_{i,j,k}$ and the rational term R. In eq. (3.15) any ε-dependence of the coefficients has been removed. The original ε-dependent parts of the coefficients, which are multiplied with poles from the integral functions are collected in the rational term R. Within the unitarity-based methods one calculates the coefficients and the rational term without resorting to a Feynman diagram calculation.

The original formulation of the method [131, 132] is based on the observation that the basic integral functions contain logarithms and dilogarithms, which develop imaginary parts in certain regions of phase space, for example

$$\operatorname{Im} \ln \left(\frac{-s - i0}{-t - i0} \right) = -\pi \left[\theta(s) - \theta(t) \right],$$

$$\operatorname{Im} \operatorname{Li}_2 \left(1 - \frac{(-s - i0)}{(-t - i0)} \right) = -\ln \left(1 - \frac{s}{t} \right) \operatorname{Im} \ln \left(\frac{-s - i0}{-t - i0} \right). \qquad (3.17)$$

Knowing the imaginary parts, one can reconstruct uniquely the corresponding integral functions. In general there will be imaginary parts corresponding to different channels (e.g. to the different possibilities to cut a one-loop diagram into two parts). The imaginary part in one channel of a

one-loop amplitude can be obtained via unitarity from a phase space integral over two tree-level amplitudes. With the help of the Cutkosky rules we have

$$\text{Im}\, A^{(1)} = \text{Im} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k_1^2} \frac{1}{k_2^2} A_L^{(0)} A_R^{(0)}. \tag{3.18}$$

$A^{(1)}$ is the one-loop amplitude under consideration, $A_L^{(0)}$ and $A_R^{(0)}$ are tree-level amplitudes appearing on the left and right side of the cut in a given channel, as shown in the first picture of fig. 7. Lifting eq. (3.18) one obtains

$$A^{(1)} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k_1^2} \frac{1}{k_2^2} A_L^{(0)} A_R^{(0)} + \text{cut free pieces}, \tag{3.19}$$

where "cut free pieces" denote contributions which do not develop an imaginary part in this particular channel. By evaluating the cut, one determines the coefficients of the integral functions, which have an imaginary part in this channel. Iterating over all possible cuts, one finds all coefficients. One advantage of a cut-based calculation is that one starts with tree amplitudes on both sides of the cut, which are already sums of Feynman diagrams. Therefore cancellations and simplifications, which usually occur between various diagrams, can already be performed before we start the calculation of the loop amplitude. This technique was used in the calculation of the one-loop amplitudes for $e^+ e^- \to 4$ partons [133, 134]. The rational part R can be obtained by calculating higher order terms in ε within the cut-based method. At one-loop order an arbitrary scale $\mu^{2\varepsilon}$ is introduced in order to keep the coupling dimensionless. In a massless theory the factor $\mu^{2\varepsilon}$ is always accompanied by some kinematical invariant $s^{-\varepsilon}$ for dimensional reasons. If we write symbolically

$$A^{loop} = \frac{1}{\varepsilon^2} c_2 \left(\frac{s_2}{\mu^2} \right)^{-\varepsilon} + \frac{1}{\varepsilon} c_1 \left(\frac{s_1}{\mu^2} \right)^{-\varepsilon} + c_0 \left(\frac{s_0}{\mu^2} \right)^{-\varepsilon}, \tag{3.20}$$

the cut-free pieces $c_0(s_0/\mu^2)^{-\varepsilon}$ can be detected at order ε:

$$c_0 \left(\frac{s_0}{\mu^2} \right)^{-\varepsilon} = c_0 - \varepsilon c_0 \ln \left(\frac{s_0}{\mu^2} \right) + O(\varepsilon^2). \tag{3.21}$$

With the advent of the new methods based on twistors, significant improvements were added to the unitarity-based technique [135–152]. Apart from the two-particle cut discussed above, one can also consider triple or quadruple cuts as shown in fig. 7. A particular nice result follows from quadruple cuts [135]: The coefficients of the box integral functions are given as a product of four tree amplitudes, summed over the two solutions of the on-shell conditions

$$l_1^2 = l_2^2 = l_3^2 = l_4^2 = 0. \tag{3.22}$$

Unitarity-based methods contributed significantly to the calculation of the one-loop six-gluon amplitude [131, 132, 136–140, 143, 145–147, 153–157] and the one-loop six-photon amplitude [152, 158–161].

3.2 Two-loop amplitudes and beyond

For NNLO calculation two-loop amplitudes are required. The relevant two-loop integrals are far from trivial. In this section I review a few techniques which can used for the calculation of two-loop integrals. The Mellin-Barnes transformation is discussed in section 3.2.1. Multiple polylogarithms together with their algebraic properties are introduced in section 3.2.2. Section 3.2.3 is devoted to sector decomposition. Not treated in detail, but equally important are methods which reduce the work-load: Integration-by-parts identities [162] have a long-standing tradition. In addition there are the reduction algorithms of Tarasov [163, 164] and Laporta [165], which can be used to relate all tensor integrals to a set of basic scalar integrals. With the help of these techniques many two-loop amplitudes have been calculated: Bhabha scattering [166], $pp \rightarrow 2$ jets [167–173], $e^+e^- \rightarrow 3$ jets [174–176] and Higgs production [177, 178]. In addition, the three-loop splitting functions, required for the evolution of parton distribution functions at NNLO, have been calculated [179, 180].

3.2.1 The Mellin-Barnes transformation

The Mellin-Barnes transformation can be used to transform any Feynman parameter integral into a particular simple form, such that the integrals over the Feynman parameters can be performed:

$$\int_0^1 \left(\prod_{j=1}^n dx_j x_j^{v_j-1} \right) \delta \left(1 - \sum_{i=1}^n x_i \right) = \frac{\prod_{j=1}^n \Gamma(v_j)}{\Gamma(v_1 + ... + v_n)}. \tag{3.23}$$

The Mellin-Barnes transformation reads

$$(A_1 + A_2 + ... + A_n)^{-c} = \frac{1}{\Gamma(c)} \frac{1}{(2\pi i)^{n-1}} \int_{-i\infty}^{i\infty} d\sigma_1 ... \int_{-i\infty}^{i\infty} d\sigma_{n-1} \tag{3.24}$$

$$\times \Gamma(-\sigma_1)...\Gamma(-\sigma_{n-1})\Gamma(\sigma_1 + ... + \sigma_{n-1} + c) A_1^{\sigma_1}...A_{n-1}^{\sigma_{n-1}}A_n^{-\sigma_1 - ... - \sigma_{n-1} - c}$$

Each contour is such that the poles of $\Gamma(-\sigma)$ are to the right and the poles of $\Gamma(\sigma + c)$ are to the left. This transformation can be used to convert any Feynman parameter integral to the form of eq. (3.23). Therefore we exchange the Feynman parameter integrals against multiple complex contour integrals. As this transformation converts sums into products it is the "inverse" of Feynman parametrisation. The contour integrals are then performed by closing the contour at infinity and summing up all residues which lie inside the contour. Here it is useful to know the residues of the Gamma function:

$$\text{res} \left(\Gamma(\sigma + a), \sigma = -a - n \right) = \frac{(-1)^n}{n!}, \quad \text{res} \left(\Gamma(-\sigma + a), \sigma = a + n \right) = -\frac{(-1)^n}{n!}. \tag{3.25}$$

Therefore we obtain (multiple) sum over residues. Techniques to manipulate these sums are discussed in the next section. In particular simple cases the contour integrals can be performed in closed form with the help of two lemmas of Barnes. Barnes first lemma states that

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\sigma \Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(c-\sigma)\Gamma(d-\sigma) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)},$$

$$\tag{3.26}$$

if none of the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$ coincides with the ones from $\Gamma(c-\sigma)\Gamma(d-\sigma)$. Barnes second lemma reads

$$
\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty}d\sigma\frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(c+\sigma)\Gamma(d-\sigma)\Gamma(e-\sigma)}{\Gamma(a+b+c+d+e+\sigma)}
$$
$$
= \frac{\Gamma(a+d)\Gamma(b+d)\Gamma(c+d)\Gamma(a+e)\Gamma(b+e)\Gamma(c+e)}{\Gamma(a+b+d+e)\Gamma(a+c+d+e)\Gamma(b+c+d+e)}. \tag{3.27}
$$

Although the Mellin-Barnes transformation has been known for a long time, the method has seen a revival in applications in recent years [181–196].

3.2.2 Multiple polylogarithms

The multiple polylogarithms are defined by

$$
\text{Li}_{m_1,\dots,m_k}(x_1,\dots,x_k) = \sum_{i_1>i_2>\dots>i_k>0}\frac{x_1^{i_1}}{i_1^{m_1}}\cdots\frac{x_k^{i_k}}{i_k^{m_k}}. \tag{3.28}
$$

The multiple polylogarithms are generalisations of the classical polylogarithms $\text{Li}_n(x)$ whose most prominent examples are

$$
\text{Li}_1(x) = \sum_{i_1=1}^{\infty}\frac{x^{i_1}}{i_1} = -\ln(1-x), \quad \text{Li}_2(x) = \sum_{i_1=1}^{\infty}\frac{x^{i_1}}{i_1^2}, \tag{3.29}
$$

as well as Nielsen's generalised polylogarithms

$$
S_{n,p}(x) = \text{Li}_{n+1,1,\dots,1}(x,\underbrace{1,\dots,1}_{p-1}), \tag{3.30}
$$

and the harmonic polylogarithms

$$
H_{m_1,\dots,m_k}(x) = \text{Li}_{m_1,\dots,m_k}(x,\underbrace{1,\dots,1}_{k-1}). \tag{3.31}
$$

Multiple polylogarithms and the closely related harmonic sums have been studied extensively in the literature [197–214].

In addition, multiple polylogarithms have an integral representation. To discuss the integral representation it is convenient to introduce for $z_k \neq 0$ the following functions

$$
G(z_1,\dots,z_k;y) = \int\limits_0^y\frac{dt_1}{t_1-z_1}\int\limits_0^{t_1}\frac{dt_2}{t_2-z_2}\cdots\int\limits_0^{t_{k-1}}\frac{dt_k}{t_k-z_k}. \tag{3.32}
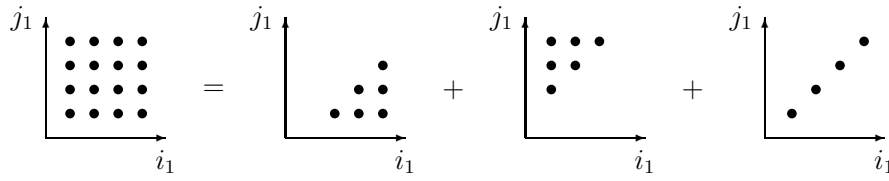$$

In this definition one variable is redundant due to the following scaling relation:

$$
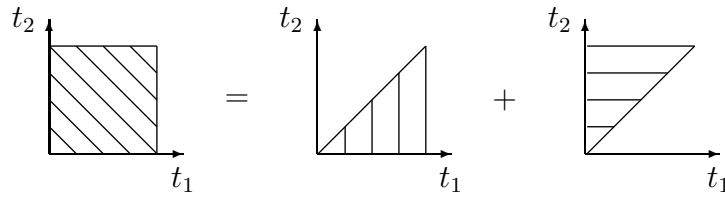G(z_1,\dots,z_k;y) = G(xz_1,\dots,xz_k;xy) \tag{3.33}
$$

To relate the multiple polylogarithms to the functions G it is convenient to introduce the following short-hand notation:

$$
G_{m_1,\dots,m_k}(z_1,\dots,z_k;y) = G(\underbrace{0,\dots,0}_{m_1-1},z_1,\dots,z_{k-1},\underbrace{0\dots,0}_{m_k-1},z_k;y) \tag{3.34}
$$

Figure 8: Sketch of the proof for the quasi-shuffle product of nested sums. The sum over the square is replaced by the sum over the three regions on the r.h.s.

Figure 9: Sketch of the proof for the shuffle product of two iterated integrals. The integral over the square is replaced by two integrals over the upper and lower triangle.

Here, all z_j for $j = 1, ..., k$ are assumed to be non-zero. One then finds

$$\text{Li}_{m_1,...,m_k}(x_1,...,x_k) = (-1)^k G_{m_1,...,m_k}\left(\frac{1}{x_1}, \frac{1}{x_1 x_2}, ..., \frac{1}{x_1...x_k}; 1\right). \tag{3.35}$$

Eq. (3.35) together with (3.34) and (3.32) defines an integral representation for the multiple polylogarithms. Multiple polylogarithms form two (independent) algebras. As a consequence, products of multiple polylogarithms can again be expressed as sums of single polylogarithms. To give an example, one has

$$\text{Li}_{m_1}(x_1)\text{Li}_{m_2}(x_2) = \text{Li}_{m_1,m_2}(x_1,x_2) + \text{Li}_{m_2,m_1}(x_2,x_1) + \text{Li}_{m_1+m_2}(x_1 x_2),$$
$$G(z_1;y)G(z_2;y) = G(z_1,z_2;y) + G(z_2,z_1;y). \tag{3.36}$$

The first relation is based on the representation in terms of nested sums. A sketch of the proof is shown in fig. 8. The second relation is based on the integral representation. The corresponding sketch of the proof is shown in fig. 9. These algebraic properties allow automated manipulations of expressions related to multiple polylogarithms by computer algebra programs. Several programs are available [202, 215–218].

Up to now we treated multiple polylogarithms from an algebraic point of view. Equally important are the analytical properties, which are needed for an efficient numerical evaluation. As an example I first discuss the numerical evaluation of the dilogarithm [219]:

$$\text{Li}_2(x) = -\int_0^x dt \frac{\ln(1-t)}{t} = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \tag{3.37}$$

25
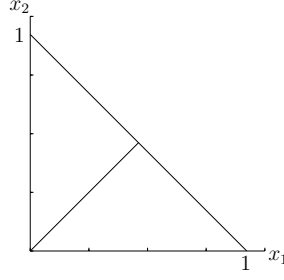
Figure 10: Sector decomposition splits the integration over the large triangle $x_1 + x_2 < 1$ into the sectors $x_1 > x_2$ and $x_2 > x_1$.

The power series expansion can be evaluated numerically, provided $|x| < 1$. Using the functional equations

$$\mathrm{Li}_2(x) = -\mathrm{Li}_2\left(\frac{1}{x}\right) - \frac{\pi^2}{6} - \frac{1}{2}\left(\ln(-x)\right)^2,$$

$$\mathrm{Li}_2(x) = -\mathrm{Li}_2(1-x) + \frac{\pi^2}{6} - \ln(x)\ln(1-x). \tag{3.38}$$

any argument of the dilogarithm can be mapped into the region $|x| \leq 1$ and $-1 \leq \mathrm{Re}(x) \leq 1/2$. The numerical computation can be accelerated by using an expansion in $[-\ln(1-x)]$ and the Bernoulli numbers B_i:

$$\mathrm{Li}_2(x) = \sum_{i=0}^{\infty} \frac{B_i}{(i+1)!}\left(-\ln(1-x)\right)^{i+1}. \tag{3.39}$$

The generalisation for the numerical evaluation of multiple polylogarithms has been worked out in [211].

3.2.3 Sector decomposition

In this section I will discuss an algorithm, which allows to compute numerically the coefficients of the Laurent expansion for a multi-loop integral for a given kinematical configuration of external momenta. The major challenge such an algorithm has to face is the disentanglement of overlapping singularities. An example for an overlapping singularity is given by

$$\int d^3x \, \delta\left(1 - \sum_{i=1}^{3} x_i\right) \frac{x_1^{-1-\varepsilon} x_2^{-1-\varepsilon}}{x_1 + x_2}. \tag{3.40}$$

The term $1/(x_1 + x_2)$ is an overlapping singularity. Sector decomposition [220–222] is a convenient tool to disentangle overlapping singularities. For the example in eq. (3.40) one splits the integration region into two sectors $x_1 > x_2$ and $x_1 < x_2$. This is shown in fig. 10 In the first sector one rescales x_2 as $x_2' = x_2/x_1$, while in the second sector one rescales $x_1' = x_1/x_2$. By applying the sector decomposition iteratively, one finally arrives at a form where all singularities are factorised
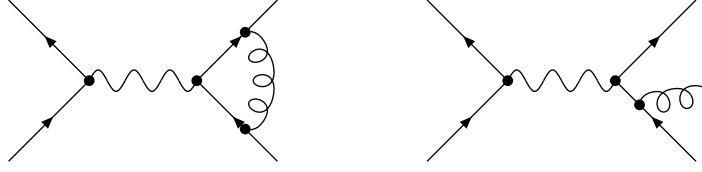
Figure 11: Cancellation of infrared divergences between virtual corrections and real corrections.

explicitly in terms of factors of Feynman parameters like $x_j^{-1-\varepsilon}$. Subtractions of the form

$$\int_0^1 dx_j \, x_j^{-1-\varepsilon} f(x_j) = -\frac{1}{\varepsilon} f(0) + \int_0^1 dx_j \, x_j^{-1-\varepsilon} \left[f(x_j) - f(0) \right] \tag{3.41}$$

for each j, where $\lim_{x_j \to 0} f(x_j)$ is finite by construction, allow to extract all poles and lead to integrals which are finite and can be integrated numerically.

4. Cancellation of infrared divergences

Infrared divergences occur at next-to-leading order and beyond. At NLO real and virtual corrections contribute. The virtual corrections contain the loop integrals and can have, in addition to ultraviolet divergences, infrared divergences. If loop amplitudes are calculated in dimensional regularisation, the IR divergences manifest themselves as explicit poles in the dimensional regularisation parameter $\varepsilon = 2 - D/2$. These poles cancel with similar poles arising from amplitudes with additional partons but less internal loops, when integrated over phase space regions where two (or more) partons become "close" to each other. This is illustrated in fig. 11. In general, the Kinoshita-Lee-Nauenberg theorem [223, 224] guarantees that any infrared-safe observable, when summed over all states degenerate according to some resolution criteria, will be finite. However, the cancellation occurs only after the integration over the unresolved phase space has been performed and prevents thus a naive Monte Carlo approach for a fully exclusive calculation. It is therefore necessary to cancel first analytically all infrared divergences and to use Monte Carlo methods only after this step has been performed.

4.1 Infrared divergences at NLO

At NLO, general methods to circumvent this problem are known. This is possible due to the universality of the singular behaviour of the amplitudes in soft and collinear limits. Examples are the phase-space slicing method [225–227] and the subtraction method [228–232]. It is worth to examine a simple NLO example in detail to understand the basic concepts. We consider the NLO corrections to $\gamma^* \to 2$ jets. The real corrections are given by the matrix element for $\gamma^* \to q g \bar{q}$ and read, up to colour and coupling factors

$$|\mathscr{A}_3|^2 = 8(1-\varepsilon)\left[\frac{2}{x_1 x_2} - \frac{2}{x_1} - \frac{2}{x_2} + (1-\varepsilon)\frac{x_2}{x_1} + (1-\varepsilon)\frac{x_1}{x_2} - 2\varepsilon \right], \tag{4.1}$$

where $x_1 = s_{12}/s_{123}$ and $x_2 = s_{23}/s_{123}$. This term is integrated over the three particle phase space. Singularities occur at the boundaries of the integration region at $x_1 = 0$ and $x_2 = 0$. Within the subtraction method one subtracts a suitable approximation term $d\sigma^A$ from the real corrections $d\sigma^R$. This approximation term must have the same singularity structure as the real corrections. If in addition the approximation term is simple enough, such that it can be integrated analytically over a one-parton subspace, then the result can be added back to the virtual corrections $d\sigma^V$.

$$\sigma^{NLO} = \int_{n+1} d\sigma^R + \int_n d\sigma^V = \int_{n+1} \left(d\sigma^R - d\sigma^A \right) + \int_n \left(d\sigma^V + \int_1 d\sigma^A \right). \tag{4.2}$$

Since by definition $d\sigma^A$ has the same singular behaviour as $d\sigma^R$, $d\sigma^A$ acts as a local counter-term and the combination $(d\sigma^R - d\sigma^A)$ is integrable and can be evaluated numerically. Secondly, the analytic integration of $d\sigma^A$ over the one-parton subspace will yield the explicit poles in ε needed to cancel the corresponding poles in $d\sigma^V$. For the example discussed above the approximation term can be taken as a sum of two dipole subtraction terms:

$$
\begin{aligned}
&\left| \mathscr{A}_2(p_1', p_3') \right|^2 \frac{1}{s_{123}} \left[\frac{2}{x_1(x_1 + x_2)} - \frac{2}{x_1} + (1-\varepsilon)\frac{x_2}{x_1} \right] \\
&+ \left| \mathscr{A}_2(p_1'', p_3'') \right|^2 \frac{1}{s_{123}} \left[\frac{2}{x_2(x_1 + x_2)} - \frac{2}{x_2} + (1-\varepsilon)\frac{x_1}{x_2} \right]
\end{aligned}
\tag{4.3}
$$

The momenta p_1', p_3', p_1'' and p_3'' are linear combinations of the original momenta p_1, p_2 and p_3. The first term is an approximation for $x_1 \to 0$, whereas the second term is an approximation for $x_2 \to 0$. Note that the soft singularity is shared between the two dipole terms and that in general the Born amplitudes $\mathscr{A}_2$ are evaluated with different momenta. Antenna subtraction [233, 234] allows to reduce the number of subtraction terms needed and interpolates smoothly between the $x_1 \to 0$ and $x_2 \to 0$ regions. Within antenna subtraction one could take for our example as approximation

$$
\left| \mathscr{A}_2(p_1''', p_3''') \right|^2 \frac{1}{s_{123}} \left[\frac{2}{x_1 x_2} - \frac{2}{x_1} - \frac{2}{x_2} + (1-\varepsilon)\frac{x_2}{x_1} + (1-\varepsilon)\frac{x_1}{x_2} \right]. \tag{4.4}
$$

Again, the Born amplitude $\mathscr{A}_2$ is evaluated with momenta p_1''' and p_2''', which are linear combinations of the original momenta p_1, p_2 and p_3. As a rule of thumb, one antenna subtraction term equals two dipol subtraction terms.

The dipole subtraction scheme has been worked out for general NLO calculations. The matrix element corresponding to the approximation term $d\sigma^A$ is given as a sum over dipoles:

$$
\sum_{pairs\ i,j} \sum_{k \neq i,j} \mathscr{D}_{ij,k}. \tag{4.5}
$$

Each dipole contribution has the following form:

$$
\mathscr{D}_{ij,k} = \mathscr{A}_{n+2}^{(0)*} \left(p_1, ..., \tilde{p}_{(ij)}, ..., \tilde{p}_k, ... \right) \frac{(-\mathbf{T}_k \cdot \mathbf{T}_{ij})}{\mathbf{T}_{ij}^2} \frac{V_{ij,k}}{2p_i \cdot p_j} \mathscr{A}_{n+2}^{(0)} \left(p_1, ..., \tilde{p}_{(ij)}, ..., \tilde{p}_k, ... \right). \tag{4.6}
$$

Here $\mathbf{T}_i$ denotes the colour charge operator for parton i and $V_{ij,k}$ is a matrix in the spin space of the emitter parton (ij). In general, the operators $\mathbf{T}_i$ lead to colour correlations, while the $V_{ij,k}$'s lead to spin correlations. The generation of all subtraction terms can be automated [81].

4.2 Infrared divergences at NNLO

The following terms contribute at NNLO:

$$d\sigma_{n+2}^{(0)} = \left(\mathscr{A}_{n+2}^{(0)} {}^* \mathscr{A}_{n+2}^{(0)} \right) d\phi_{n+2},$$

$$d\sigma_{n+1}^{(1)} = \left(\mathscr{A}_{n+1}^{(0)} {}^* \mathscr{A}_{n+1}^{(1)} + \mathscr{A}_{n+1}^{(1)} {}^* \mathscr{A}_{n+1}^{(0)} \right) d\phi_{n+1},$$

$$d\sigma_n^{(2)} = \left(\mathscr{A}_n^{(0)} {}^* \mathscr{A}_n^{(2)} + \mathscr{A}_n^{(2)} {}^* \mathscr{A}_n^{(0)} + \mathscr{A}_n^{(1)} {}^* \mathscr{A}_n^{(1)} \right) d\phi_n, \tag{4.7}$$

where $\mathscr{A}_n^{(l)}$ denotes an amplitude with n external partons and l loops. $d\phi_n$ is the phase space measure for n partons. We would like to construct a numerical program for an arbitrary infrared safe observable O. Several options for the cancellation of infrared divergences have been discussed [235–253]. Among those, the subtraction method – well-known from NLO computations and sector decomposition are the most promising candidates. Let us look again at the subtraction method. To render the individual contributions finite, one adds and subtracts suitable pieces:

$$\langle O \rangle_n^{NNLO} = \int O_{n+2} \, d\sigma_{n+2}^{(0)} - O_{n+1} \circ d\alpha_{n+1}^{(0,1)} - O_n \circ d\alpha_n^{(0,2)}$$

$$+ \int O_{n+1} \, d\sigma_{n+1}^{(1)} + O_{n+1} \circ d\alpha_{n+1}^{(0,1)} - O_n \circ d\alpha_n^{(1,1)}$$

$$+ \int O_n \, d\sigma_n^{(2)} + O_n \circ d\alpha_n^{(0,2)} + O_n \circ d\alpha_n^{(1,1)}.$$

Here $d\alpha_{n+1}^{(0,1)}$ is a subtraction term for single unresolved configurations of Born amplitudes. This term is already known from NLO calculations. The term $d\alpha_n^{(0,2)}$ is a subtraction term for double unresolved configurations. Finally, $d\alpha_n^{(1,1)}$ is a subtraction term for single unresolved configurations involving one-loop amplitudes.

To construct these terms the universal factorisation properties of QCD amplitudes in unresolved limits are essential. QCD amplitudes factorise if they are decomposed into primitive amplitudes. Primitive amplitudes are defined by a fixed cyclic ordering of the QCD partons, a definite routing of the external fermion lines through the diagram and the particle content circulating in the loop. One-loop amplitudes factorise in single unresolved limits as [131, 236, 254–258]

$$A_n^{(1)} = \text{Sing}^{(0,1)} \cdot A_{n-1}^{(1)} + \text{Sing}^{(1,1)} \cdot A_{n-1}^{(0)}. \tag{4.8}$$

Tree amplitudes factorise in the double unresolved limits as [235, 259–264]

$$A_n^{(0)} = \text{Sing}^{(0,2)} \cdot A_{n-2}^{(0)}. \tag{4.9}$$

In addition, the pole structure of two-loop amplitudes is known [265–267]. The subtraction terms have to interpolate between the various singular limits. Spin-averaged antenna subtraction terms have been worked out in [60, 245] and first results at NNLO based on the subtraction method are available for $e^+e^- \to 2$ jets [60, 61] and the thrust distribution [62].

5. Summary

Precision calculations for multi-parton processes will play an important role for the physics program at the LHC. I discussed various approaches how to overcome the bottle-necks we face

technically: the length of expressions, the calculation of loop integrals and the occurrence of infrared divergences. Computer algebra, a divide-and-conquer approach and new developments based on on-shell recursion relations help us to keep the length of expressions manageable. Sophisticated techniques related to Mellin-Barnes transformations, shuffle algebras or sector decomposition allow us to compute multi-loop integrals. Urgently needed are also automated computations of one-loop amplitudes. I discussed improvements of the Passarino-Veltman algorithm and new ideas base on generalised unitarity. As far as the cancellation of infrared divergences is concerned, there are systematic methods at NLO. Several methods, which address the issue at NNLO have been proposed and first numerical results for observables based on the use of these methods have emerged.

Acknowledgements

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