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Precise Numerical Evaluation of the Scalar One-Loop Integrals with the Infrared Divergence

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We present a new approach for obtaining very precise integration results for infrared vertex and box diagrams, where the integration is carried out directly without performing any analytic integration of Feynman parameters. Using an appropriate numerical integration routine with an extrapolation method, together with a multi-precision library, we have obtained integration results which agree with the analytic results to 10 digits even for such a very small photon mass as 10^{-150} GeV in the infrared vertex diagram.

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1. Introduction

In the field of the particle physics more and more precise computation is required for the progress of the accuracy of the future collider experiments. For this reason, the computation of one- and higher loop corrections is mandatory.

General solution has been known for one-loop integrals[1] and there are several excellent programs developed in this decade such as `FF`[2], `FormCalc-LoopTools`[3], `XLOOPS-Ginac`[4]. In parallel, there have been continual efforts to carry out loop integrals by fully or semi-fully numerical methods. In early 1990 Fujimoto *et al.*[5, 6, 7, 8] have proposed several excellent methods and have shown numerical results. In early 2000 Kurihara *et al.*[9] have developed the numerical contour method and have shown numerical results.

Since 2003 we have been developing a method which enables us to carry out the loop integrals in a completely numerical way. In this numerical method, to prevent the integral diverging, we put $i\varepsilon$ in the denominator of the integrand. Here ε is a real positive constant and it is not necessary to be infinitesimal. Thanks to this $i\varepsilon$, the denominator does not vanish and we carry out the numerical result of $I(\varepsilon)$ for a given ε. Calculating a sequence of $I(\varepsilon_l)$ varying $\varepsilon_l = \varepsilon_0 * (const.)^{-l}$ (l = 0,1,2, $\cdots$) and extrapolating them, we can get the final result of the integration in the limit of $\varepsilon \to 0$. So far in this method we have calculated several one-loop and two-loop integrals and reported the

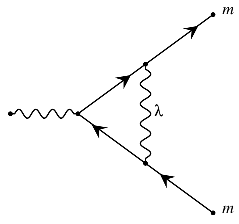

Figure 1: One-loop vertex diagram

numerical results in [10, 11, 12, 13]. In these demonstrations it has been clearly shown that the extrapolation method is very efficient for not only one-loop but also two-loop diagrams.

In the numerical calculation of the loop integrals, it is also very important to confirm whether the method can handle the infrared singularities. In [12], we have calculated the loop integrals with the infrared singularities with a photon mass of up to 10^{-15} GeV. In the calculation we used the double precision arithmetic for the photon mass of up to 10^{-6} GeV. However for the photon mass of below 10^{-8} GeV the double precision is not enough and we use the quadruple precision arithmetic. In the quadruple precision calculation the relative error is given the order of 10^{-7}.

In this paper, we propose a new approach for our numerical method. In this approach we include a precision control technique in addition to the extrapolation method for the loop integral. We show that a precision control is mandatory to get the stable results of one-loop diagrams which have infrared singularities with smaller photon mass below 10^{-30} GeV.

The layout of this paper is as follows. In § 2 we give formulae of the loop integrals we consider in this paper. In § 3 we give an explanation of a new approach of our numerical method. Numerical results are shown in § 4. Finally in § 5 we will summary this paper.

2. Loop Integrals

2.1 One-Loop Vertex

For one-loop vertex diagram in Fig. 1, the loop integral we consider in this paper is

$$I = \int_0^1 dx \int_0^{1-x} dy \frac{1}{D}, \tag{2.1}$$

where

$$D = -xys + (x+y)^2 m^2 + (1-x-y)\lambda^2. \tag{2.2}$$

Here s denotes squared central mass energy and m and λ are a mass of external particles and a fictitious photon mass respectively. We introduce λ so as to regularize the infrared singularities.

Replacing s by $s+i\varepsilon$ in (2.2), the real part of the integral (2.1) becomes

$$\Re = \int_S \frac{D}{D^2+\varepsilon^2 x^2 y^2} dxdy, \tag{2.3}$$

and the imaginary part becomes

$$\Im = \int_S \frac{\varepsilon xy}{D^2+\varepsilon^2 x^2 y^2} dxdy. \tag{2.4}$$

Here S is the triangular region as $0 < x$, $0 < y$ and $x+y < 1$. Corresponding analytic formulae for one-loop vertex diagrams are given in [5, 6, 8].

2.2 One-Loop Box

For one-loop box diagram in Fig. 2, the loop integral we consider in this paper is

$$I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \frac{1}{D^2}, \tag{2.5}$$

where

$$D = -xys - tz(1-x-y-z) + (x+y)\lambda^2 + (1-x-y-z)(1-x-y)m^2 + z(1-x-y)M^2. \tag{2.6}$$

In (2.6) s denotes squared central mass energy and m and M are external masses of particles.

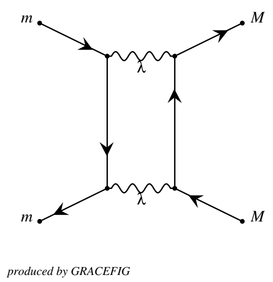

Figure 2: One-loop box diagram

When we replace s by $s+i\varepsilon$ in (2.5), the real part of the integral becomes

$$\Re = \int_S \frac{D^2+\varepsilon^2 x^2 y^2}{(D^2+\varepsilon^2 x^2 y^2)^2} dxdydz, \tag{2.7}$$

and the imaginary part becomes

$$\Im = \int_S \frac{2\varepsilon xyD}{(D^2+\varepsilon^2 x^2 y^2)^2} dxdydz. \tag{2.8}$$

Here S is the tetrahedron region as $0 < x$, $0 < y$, $0 < z$ and $x+y+z < 1$. Corresponding analytic formulae for one-loop box diagrams are given in [6]. When we put $M = \mu$, the scalar integral for one-loop box diagram becomes

$$I_{analytic} = -\frac{1}{s}\frac{1}{\sqrt{(-t+\mu^2-m^2)^2-4m^2t}} \times \ln\left(-\frac{s}{\lambda^2}\right)\ln\frac{(-t+\mu^2+m^2+\sqrt{(-t+\mu^2-m^2)^2-4m^2t})^2}{4m^2\mu^2}. \quad (2.9)$$

3. Numerical Method

The basic idea of our numerical method is the combination of an efficient multi-dimensional integration routine and an extrapolation method. Using this method we have shown several numerical results of loop integrals and that they show good agreements with analytical results. In the calculation so far we have used double precision arithmetic or quadruple precision arithmetic if necessary for one-loop and two-loop integrals. However for the infrared divergent diagram even with the quadruple precision arithmetic it becomes harder to get a result with an enough accuracy as the value of the λ becomes much smaller such as 10^{-15} GeV[12]. Therefore in our new approach we include the concept of a precision control.

In the following subsections, we give a very simple explanation of the multi-dimensional integration routine, an extrapolation method and a multi-precision library we used in our new approach respectively.

3.1 Multi-dimensional Integration Routine

We use `DQAGE` routine in the multi-dimensional integration. It is included in the package `QUADPACK` [14] and it is a globally adaptive integration routine. Though it is a routine for one-dimensional integration, we use it iteratedly [15].

3.2 Extrapolation Method

Although there are several ways to accelerate the conversion of the sequences. Of them, the ε algorithm by P.Wynn[16] is recommended as the best all-purpose method for slowly converging sequences. It is an implementation of Shanks' transformation [17] and is represented in a recursive formulae.

3.3 Multi-Precision Library

When we know correctly whether the precision arithmetic used in the calculation is sufficient or not, we can select the necessary precision arithmetic and the results become reliable. J.Fujimoto *et. al* have discussed the importance of a precision control in [18]. We use `HMlib`[19] as the multi-precision library in our new approach because it gives an information of the lost-bits during the calculation and we can guarantee the precision of the results. Taking the calculation of one-loop vertex diagram as an example, we show the information of the lost-bits supplied by `HMlib` in Table 1.

Table 1: In P-precision presentation implemented in `HMlib`, a sign bit is 1 bit and an exponent bit is 15 bits and a mantissa is $(32 \times P - 16)$ bits. When $P = 4$ (quadruple precision), the mantissa becomes 112 bits. In the Table when λ is 10^{-23} GeV, the maximum number of lost-bits is the same as the number of bits of the mantissa.

λ [GeV]	Average lost-bits	Maximum lost-bits
10^{-21}	88	92
10^{-22}	98	102
10^{-23}	108	112

Table 2: Numerical results of one-loop vertex diagram with $\sqrt{s} = 500$ GeV, $t = -150^2$ GeV^2, $m = m_e = 0.5 \times 10^{-3}$ GeV. The upper is the result of Real part and the lower is one of Imaginary part.

λ [GeV]	Numerical Results	Precision	Analytical Results [quadruple precision]	Agreement
10^{-30}	-0.150899286980769753D-01 $\pm$ 0.771D-26	8	-0.150899286980482291519117078568077 98E-01	12
	0.189229839615898822D-02 $\pm$ 0.124D-25	8	0.189229839615525389345336677808320 51E-02	12
10^{-60}	-0.303593952562854951D-01 $\pm$ 0.178D-16	16	-0.303593952562260154223293070559586 45E-01	12
	0.362840665514227622D-02 $\pm$ 0.896D-13	16	0.362840665513496544845097795014605 48E-02	11
10^{-80}	-0.405390396284235075D-01 $\pm$ 0.580D-15	16	-0.405390396283445396026077065220588 89E-01	11
	0.478581216125478532D-02 $\pm$ 0.401D-12	16	0.478581216112143981844938539818790 66E-02	10
10^{-120}	-0.608983283726997427D-01 $\pm$ 0.556D-15	32	-0.608983283725815879631645054542597 61E-01	11
	0.710062317325786663D-02 $\pm$ 0.415D-12	32	0.710062317309438855844620029427161 00E-02	10
10^{-150}	-0.761677949309069452D-01 $\pm$ 0.931D-15	32	-0.761677949307593742335821045162221 89E-01	11
	0.883673143185801414D-02 $\pm$ 0.260D-13	32	0.883673143207410011344381145073642 54E-02	10
10^{-160}	-0.812576170752810666D-01 $\pm$ 0.549D-10	32	-0.812576183472699269937578080161273 25E-01	7
	0.941543418501223556D-02 $\pm$ 0.109D-11	32	0.941543432496722442714069050305533 99E-02	7

4. Numerical Results

4.1 One-Loop Vertex

The numerical results are shown in Table 2. Results are compared to analytic results evaluated by the formulae in [8]. As an example, in Table 3 we show the parameters we used in the calculation which are `key` and `limit` used in `DQAGE`, starting ε and ending ε for an extrapolation method.

4.2 One-Loop Box

The numerical results of one-loop box diagram shown are shown in Table 4. The results are compared to the analytic results evaluated by (2.9) in a quadruple precision. As an example, in Table 5 we show the parameters we used in the calculation which are `key` and `limit` used in `DQAGE`, starting ε and ending ε for an extrapolation method.

4.3 Elapsed Time

In Table 6 and 7, the elapsed times required in the several typical calculations are shown.

Table 3: Parameters used in `DQAGE` and in an extrapolation method for the one-loop vertex diagram with $\sqrt{s} = 500$ GeV, $m = m_e = 0.5 \times 10^{-3}$ GeV, $\lambda = 10^{-150}$ GeV. The upper is the parameters for the Real part and the lower is ones for Imaginary part.

key_x key_y	$limit_x$ $limit_y$	ε_{start}	ε_{end}
1 1	600 600	0.36572620D+04	0.28485158D+03
1 1	600 600	0.36572620D+04	0.28485158D+03

Table 4: Numerical results of the loop integral of one-loop box diagram with $\sqrt{s} = 500$ GeV, t = -$150^2 GeV^2$, m = 0.5 $\times 10^{-5}$ GeV, M = 150 GeV. This is results of Real part.

λ [GeV]	Numerical Results	Precision	Numerical Results [quadruple precision]	Agreement
10^{-15}	-0.1927861102670278D-06 $\pm$ 0.314D-14	4,4,2	-0.19278611224396411606895771777195708E-06	8
10^{-20}	-0.2472486348234972D-06 $\pm$ 0.586D-15	4,4,2	-0.24724863525991758999379683385554222E-06	9
10^{-25}	-0.3017111253761463D-06 $\pm$ 0.111D-13	4,4,2	-0.30171115827587106391863594993913029E-06	7
10^{-30}	-0.3562028882831722D-06 $\pm$ 0.867D-10	4,4,2	-0.35617368129182453784347506602271837E-06	4

Table 5: Parameters used in `DQAGE` and in an extrapolation method for the one-loop box diagram with $\sqrt{s} = 500$ GeV, t = -$150^2 GeV^2$, $m = 0.5 \times 10^{-3}$ GeV, M = 150 GeV and $\lambda = 10^{-25}$ GeV.

key_x key_y key_z	$limit_x$ $limit_y$ $limit_z$	ε_{start}	ε_{end}
1 1 1	400 400 100	0.36572620D+04	0.28485158D+03

Table 6: Elapsed time required in the calculation of the one-loop vertex diagram

λ [GeV]	Real Part	Imaginary Part	Precision	CPU
10^{-30}	1.8days	1.8days	8,8	Xeon 3.06GHz
10^{-150}	6.8days	6.7days	32,32	Xeon 3.06GHz

Table 7: Elapsed time required in the calculation of the one-loop box diagram

λ [GeV]	Real Part	Precision	CPU
10^{-15}	1.5days	4,4,2	Opteron 2.2GHz
10^{-25}	3.0days	4,4,2	Opteron 2.2GHz

5. Summary

We have shown that a new numerical approach presented in this paper is applicable to the scalar one-loop vertex and box diagram with infrared singularities. The new approach consists of an integration routine, an extrapolation method and a precision control. Several numerical results are shown and they agree with analytic ones. All the calculation of the one-loop vertex diagram are carried out in at least quadruple precision arithmetic and some are in 8-, 16- and 32-precision arithmetic. From these demonstrations, it is confirmed that a precision control plays an important role in the calculation of the loop integrals with the infrared singularities. Although higher precision arithmetic costs a lot of CPU time, however, it supplies the information of the lost-bits occurred during the calculation and this is extremely valuable in getting the reliable results. To reduce CPU time we will be able to use the parallel computing technique.

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